How to use IpOpt and AMPL to solve time optimal control problems

In this short tutorial we explain how to use IpOpt in order to solve time optimal control problems. We refer to [1, 4, 5] for a survey on numerical methods in optimal control and how to implement them efficiently according to the context.

Solving Problem (P0)

- IpOpt is an interior-point optimization routine (see [6]).
  Basically, it numerically solves optimisation problems of the form:

$$\left|\begin{array}{ll} \min & f(x) \\ & x \in \mathbb{R}^n\,, \\ & g(x) \leqslant 0\,, \\ & h(x) = 0\,. \end{array}\right. \tag{P0}$$

  IpOpt can be used together with Matlab, FreeFem++... and can be used directly in cpp codes.

- AMPL is an automatic differentiation and the modelling language (see [2]). The interest of using AMPL together with IpOpt is that, the gradient of the cost function and the constraints is automatically generated. Solving problems with IpOpt and AMPL can be made online through the NEOS solvers.

Let us write in AMPL language the problem (P0).

```
var x {i in 1..n}                          # The variable of the problem
minimize cost : f(x);                       # The cost function to be minimized
subject to inequality_constraints : g(x)<=0; # The inequality constraints
subject to equality_constraints   : h(x)=0;  # The equality constraints
option solver ipopt;                        # Set IpOpt as solver
# Set options for IpOpt, such as the maximal number of iterations...
option ipopt_options "max_iter=20000 linear_solver=mumps halt_on_ampl_error yes";
solve;                                      # Solve the problem
printf : "# cost = %24.16e\n", cost;        # Display the cost value
printf : "# Data\n";
printf {i in 0..n} : "%24.16e\n", x[i];     # Display the optimal values of x
end;                                        # Quit AMPL
```

Solving a time optimal control problem with constraints

Let us now turn to a time optimal control problem.
Given a dynamical system,

$$\dot{y} = f(y, u), \tag{1a}$$

some initial condition $\mathrm{y}^0 \in \mathbb{R}^n$ and some terminal condition $\mathrm{y}^1 \in \mathbb{R}^n$ and some bound $M > 0$, the aim is to find the minimal time $T \geqslant 0$ such that there exist a control $u \in L^\infty(0,T)^m$ such that,

$$|u_i(t)| \leqslant M \qquad (t \in (0,T) \text{ a.e.}\,,\ i \in \{1,\dots,m\})\,, \tag{1b}$$

and the solution y of (1a) satisfies

$$y(T) = \mathrm{y}^1 \qquad \text{and} \qquad g(y(t)) \leqslant 0 \quad (t \in (0,T))\,, \tag{1c}$$

where g is given and define constraints of the state variable y. Of course to be able to solve this system, one needs to have $g(\mathrm{y}^0) \leqslant 0$ and $g(\mathrm{y}^1) \leqslant 0$.

The optimal control problem (1), can be recast as an optimisation problem under constraints similar to (P0). More precisely, it is

$$\begin{array}{ll} \min & T \\ \left| \right. & u \in L^\infty(0,T)^M\,, \\ & \dot{y} = f(y,u)\,, \quad y(0) = \mathrm{y}^0\,, \\ & y(T) = \mathrm{y}^1\,, \\ & g(y(t)) \leqslant 0 \qquad (t \in (0,T))\,. \end{array}$$

In order to handle numerically, this problem, we will use a time discretization. Let us explain it with the explicit Euler method. But any other time discretization can be used.

Fix some parameter $N_t \in \mathbb{N}^*$ (the number of time steps) and for $T > 0$ given, define y_i the estimation of $y(iT/N_t)$ for $i \in \{0,\dots,N_t\}$. The explicit Euler scheme, gives the relation,

$$y_{i+1} = y_i + \frac{T}{N_t} f(y_i, u_i)\,,$$

with $u_i \simeq u(iT/N_t)$. Then the state and control constraints are replaced by:

$$g(y_i) \leqslant 0 \qquad \text{and} \qquad |u_i| \leqslant M\,.$$

Consequently, in discretized version, we end up with a finite dimensional control problem under constraints, whose `AMPL` version is:

```
# Define the parameters of the problem
param Nt=100; # number of time step discretization points
param M =1;   # bound on the control
param y0=1;   # initial condition
param y1=0;   # final condition

# Define varioables of the problem
var y {i in 0..Nt};
var u {i in 0..Nt} >=M, <=M;  # The control shall be in [-M,M]
var T >=0;                    # The time T shall be nonnegative

# The cost function is the time T
minimize cost: T;
# Set the constraints
subject to y_dyn {i in 0..Nt-1} :                    # y is solution of (1)
        y[i+1]=y[i]+T/Nt*f(y[i],u[i]);
subject to y_init : y[0] =y0;                        # y(0)=y0
subject to y_end  : y[Nt]=y1;                        # y(T)=y1
```

```
subject to state_constraint {i in 1..Nt-1} : g(y[i])<=0; # g(y(t))<=0

# Solve with IpOpt
option solver ipopt;
option ipopt_options "max_iter=20000 linear_solver=mumps halt_on_ampl_error yes";
solve;
# Display solution
printf : "# T = %24.16e\n", T;
printf : "# Nt = %d\n", Nt;
printf : "# Data\n";
printf {i in 0..Nt} : " %24.16e\n", u[i];
printf {i in 0..Nt} : " %24.16e\n", y[i];
end;
```

Application to the constrained heat equation

Now we can turn to the control of the heat equation with nonnegative state constraint.

To this end, we consider the controlled 1D heat equation with Neumann boundary control.

$$\dot{y}(t,x) = \partial_x^2 y(t,x) \qquad (t>0\,,\ x\in(0,1))\,, \tag{2a}$$
$$\partial_x y(t,0) = v_0(t) \qquad (t>0)\,, \tag{2b}$$
$$\partial_x y(t,1) = v_1(t) \qquad (t>0)\,, \tag{2c}$$

To this problem, we add the control constraints,

$$|v_0(t)| \leqslant M \quad \text{and} \quad |v_1(t)| \leqslant M \qquad (t>0 \text{ a.e.})\,,$$

with some $M>0$ given and we add the state constraint,

$$y(t,x) \geqslant 0 \qquad ((t,x)\in\mathbb{R}_+^*\times(0,1) \text{ a.e.})\,.$$

In [3], it has been proved that every positive constant state y^0 can be steered to some over positive constant state y^1 in a large enough time T. Our goal here is to solve this system numerically. Firstly, we will use a space discretization to reduce the system (2). To this end, we define $N\in\mathbb{N}^*$ and for every $n\in\{0,\dots,N\}$, $x_n=\frac{n}{N}$. Based on this discretization of $[0,1]$, we will discretized (2) using centered finite differences. That is to say, given $V(t)=(v_0(t)\,,\ v_1(t))^\top$ and $Y(t)\in\mathbb{R}^{N+1}$, a vector approaching $\big(y(t,x_0),\ \dots,\ y(t,x_N)\big)^\top$, Y is solution of

$$\dot{Y} = AY + BV \qquad (t>0)\,, \tag{3a}$$

with the initial condition

$$Y(0) = \mathrm{y}^0 e_{N+1} \tag{3b}$$

the target state,

$$Y(T) = \mathrm{y}^1 e_{N+1} \tag{3c}$$

the state constraint,

$$Y(t) \geqslant 0 \qquad (t>0 \text{ a.e.}) \tag{3d}$$

and the control constraint,

$$|V_0(t)| \leqslant M \quad \text{and} \quad |V_1(t)| \leqslant M \qquad (t>0 \text{ a.e.})\,, \tag{3e}$$

where we have set $e_{N+1}=(1,\dots,1)^\top\in\mathbb{R}^{N+1}$,

$$A = N^2\begin{pmatrix} 2 & -2 & 0 & \dots & \dots & \dots & 0\\ 1 & -2 & 1 & 0 & \dots & \dots & 0\\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & \dots & \dots & 0 & 1 & -2 & 1\\ 0 & \dots & \dots & \dots & 0 & 2 & -2 \end{pmatrix} \in M_{N+1}(R) \qquad \text{and} \qquad B = 2N\begin{pmatrix} 1 & 0\\ 0 & \vdots\\ \vdots & \vdots\\ \vdots & \vdots\\ \vdots & \vdots\\ \vdots & 0\\ 0 & 1 \end{pmatrix} \in M_{N+1,2}(\mathbb{R})\,.$$

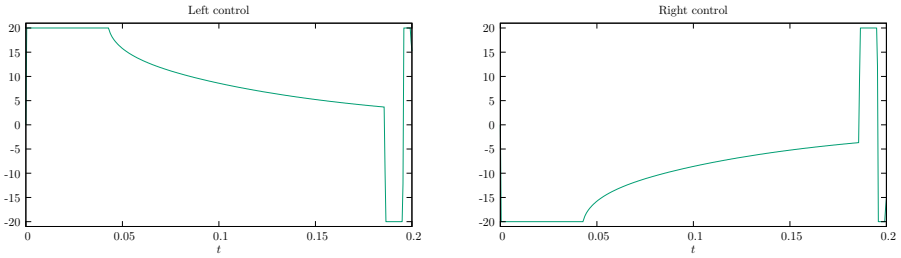
Now we can discretize (3a) using explicit Euler scheme and similarly to the previous example, we obtain an optimisation problem of finite dimension with constraints whose `AMPL` formulation is

```
# Parameters of the problem
param Nx=30;   # Number of space discretisation points
param Nt=300;  # Number of time discretisation points
# One have to check a posteriori that the number of time steps is large enough so that the
# CFL condition, T*Nx^2/Nt <=1/2, is satisfied
param dx=1/Nx; # Space step
param M =20;   # Bound on the controls

# Variables of the system
# i stands for the time index and j for the space index
var y {i in 0..Nt, j in 0..Nx} >=0; # State of the control problem
# Neuman controls in 0 and 1. The controls are in [-M,M].
var v0 {i in 0..Nt} >=-M, <=M;
var v1 {i in 0..Nt} >=-M, <=M;
var T >=0;      # Control time
var dt=T/Nt;    # Time step

# Define the cost function
minimize cost: T;

# Define the constraints
# y is solution of the discretize system
subject to y_dyn {i in 0..Nt-1, j in 1..Nx-1}:
        (y[i+1,j]-y[i,j])*(dx)^2=(y[i,j-1]-2*y[i,j]+y[i,j+1])*dt;
# Neuman boundary conditions in 0 and 1
subject to left_boundary  {i in 1..Nt-1}: y[i,1]-y[i,0]    =v0[i]*dx;
subject to right_boundary {i in 1..Nt-1}: y[i,Nx]-y[i,Nx-1]=v1[i]*dx;
# y(0)=y0 and y(T)=y1
subject to y_init {j in 0..Nx}: y[0,j] =5;
subject to y_end  {j in 0..Nx}: y[Nt,j]=1;

# Solve with IpOpt
option solver ipopt;
option ipopt_options "max_iter=2000 linear_solver=mumps halt_on_ampl_error yes";
solve;

# Write the solution in the file out.txt
printf: " # T = %24.16e\n", T >  out.txt;
printf: " # Nx = %d\n", Nx    >> out.txt;
printf: " # Nt = %d\n", Nt    >> out.txt;
printf: " # Data\n"           >> out.txt;
printf {i in 0..Nt}: " %24.16e\n", v0[i] >> out.txt;
printf {i in 0..Nt}: " %24.16e\n", v1[i] >> out.txt;
printf {i in 0..Nt, j in 0..Nx}: " %24.16e\n", y[i,j] >> out.txt;

end;
```

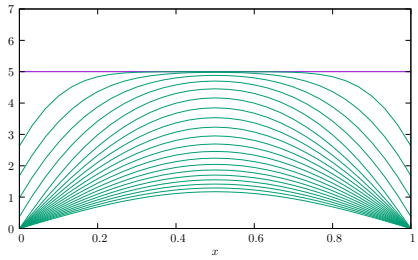
Once the file `out.txt` is written, it can be for instance read by `Scilab` with the following code

```
fid= mopen('out.txt','r');            // Open out.txt
T  = mfscanf(fid,'%s %s %s');         // Read ``# T =''
T  = mfscanf(fid,'%f');               // Read value of T
Nx = mfscanf(fid,'%s %s %s');         // Read ``# Nx =''
Nx = mfscanf(fid,'%d');               // Read value of Nx
Nt = mfscanf(fid,'%s %s %s');         // Read ``# Nt =''
Nt = mfscanf(fid,'%d');               // Read value of Nt
s  = mfscanf(fid,'%s %s'); s=[];      // Read ``# Data''
v0 = mfscanf(Nt+1,fid,'%f'); v0=v0';  // Read the Nt+1 values of v0
v1 = mfscanf(Nt+1,fid,'%f'); v1=v1';  // Read the Nt+1 values of v1
y  = mfscanf((Nt+1)*(Nx+1),fid,'%f'); // Read the (Nt+1)*(Nx+1) values of y
mclose(fid);                          // Close out.txt

y = matrix(y,Nx+1,Nt+1); y=y'; // Reshape y as a matrix, line i is the solution at time i/Nt
x = 0:1/Nx:1; t = 0:T/Nt:T;    // Define the space and time discretisations
printf('time:\t%f\n',T);       // Display the control time
plot(t,[v0;v1]); sleep(2000); clf();              // plot controls and wait 2s
plot2d(x,y(1,:),rect=[0 0 1 10]); sleep(100);     // plot the initial state and wait 0.1s
for i=2:1:Nt,
        plot(x,y(i,:),rect=[0 0 1 10]); sleep(10); // plot the state at each time instants
end
plot(x,y($,:),rect=[0 0 1 10]);                   // plot the final state
```

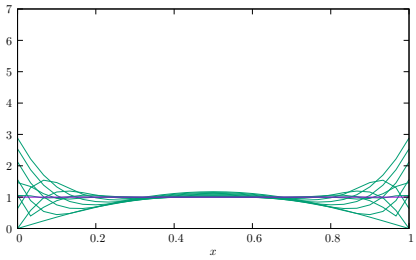
Based on these two codes, we obtain the results displayed on Fig. 1.



(a) Time evolution of the boundary controls.



(b) Evolution of the state between times 0 and 0.1856.



(c) Evolution of the state between times 0.1856 and the final time, 0.2093.

Figure 1: Evolution of the controls and of the state from the initial value (= 5) to the final one (= 1) in the minimal computed time $T \simeq 0.2093$.

Useful links

- `IpOpt` project: https://projects.coin-or.org/Ipopt
- `AMPL` project: http://ampl.com/products/ampl/
- `NEOS` solvers: https://neos-server.org/neos/solvers/

References

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[4] E. Trélat. *Contrôle optimal. Théorie et applications.* Paris: Vuibert, 2005.

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