Random ordinary differential equations and their numerical approximation

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joint work with Yusuke Asai, Lars Grüne, Arnulf Jentzen and others

Differential equations with noise

Langevin, Smoluchowsky etc used "noisy" differential equations to model the particle dynamics of Brownian motion in the early 1900s:

$$\frac{dx}{dt} = f(t,x) + g(t,x)\,\eta_t, \qquad \text{noise:} \quad \eta_t$$

• Gaussian white noise: many mathematical problems finally resolved by Itô in the 1940s with the introduction of stochastic calculus

$\Longrightarrow$ **Stochastic differential equations** (SODEs)

$$dX_t = f(t,X_t) + g(t,X_t)\,dW_t, \qquad \text{Wiener process:} \quad W_t$$

- More regular noise $\Longrightarrow$ **Random ODEs**

Let $f : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ be smooth and let ζ_t be an m-dimensional stochastic process with Hölder continuous sample paths.

A random ordinary differential equation (RODE) on $\mathbb{R}^d$

$$\boxed{\frac{dx}{dt} = f(\zeta_t, x)}$$

is pathwise an ordinary differential equation (ODE) on $\mathbb{R}^d$

$$\frac{dx}{dt} = F_\omega(t, x) := f(\zeta_t(\omega), x), \qquad \omega \in \Omega.$$

Example of a RODE: $\dfrac{dx}{dt} = -x + W_t$

The solutions of RODEs have continuously differentiable sample paths and can be handled pathwise using deterministic calculus.

Example of an SODE: $dX_t = -X_t\,dt + dW_t$

The sample paths of the solutions of SODEs are continuous, but nowhere differentiable

$\Longrightarrow$ SODEs require **Ito stochastic calculus**

Remark SODEs are really stochastic integral equations : their representation as differential equations is only symbolic.

- **SODEs can be rewritten as RODEs**

$$dX_t = f(X_t)\,dt + dW_t \qquad \Leftrightarrow \qquad \frac{dz}{dt} = f(z + O_t) + O_t$$

with an Ornstein-Uhlenbeck process O_t satisfying the linear SODE

$$dO_t = -O_t\,dt + dW_t \qquad \text{and} \quad z(t) = X_t - O_t$$

$$\begin{aligned} \Rightarrow \; z(t) = X_t - O_t &= X_0 - O_0 + \int_0^t [f(X_s) + O_s]\; ds \\ &= z(0) + \int_0^t [f(z(s) + O_s) + O_s]\; ds \end{aligned}$$

is pathwise differentiable by continuity and the fundamental theorem of calculus. (Doss, Sussmann (1970s), Imkeller. Leder, Schmalfuss (2000s)).

- **RODEs driven by Itô processes rewritten as SODEs**

Suppose that the stochastic process ζ_t in a RODE

$$\frac{dx}{dt} = f(x, \zeta_t)$$

is an **Itô process**, i.e., the solution of an SODE,

$$d\zeta_t = a(\zeta_t)\, dt + b(\zeta_t)\, dW_t.$$

These combine to give an SODE

$$d \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} f(X_t, Y_t) \\ a(Y_t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ b(Y_t) \end{pmatrix} dW_t$$

Numerical schemes for RODEs

Since the paths of the stochastic process ζ are often at most Hölder continuous, the mapping $t \mapsto F_\omega(t,x) := f(\zeta_t(\omega),x)$ is usually only continuous but not differentiable — no matter how smooth the function f.

$\star$ Classical numerical schemes for ODEs such as Runge-Kutta schemes do not achieve their usual order when applied to RODEs.

$\star$ The solution of a RODE $t \mapsto x(t,\omega)$ is at most continuously differentiable

$\Rightarrow$ **no Taylor expansion!**

Simple numerical schemes for RODES (Grüne & Kloeden (2001))

Traditional numerical methods for ODEs attain at best a low convergence order when applied to RODEs.

For the RODE with ζ_t pathwise Hölder continuous with exponent $\frac{1}{2}$

$$\frac{dx}{dt} = -x + \zeta_t,$$

the **Euler scheme** with step size Δ_n

$$Y_{n+1} = (1 - \Delta_n)\ Y_n + \zeta_{t_n}\, \Delta_n$$

has pathwise order $\frac{1}{2}$.

However, the **averaged Euler scheme**

$$Y_{n+1} = (1 - \Delta_n)\ Y_n + \int_{t_n}^{t_{n+1}} \zeta_t\, dt,$$

has pathwise order 1 provided the integral is approximated with Riemann sums

$$\int_{t_n}^{t_{n+1}} \zeta_t\, dt \approx \sum_{j=1}^{J_{\Delta_n}} \zeta_{t_n + j\delta}\, \delta$$

with the step size δ satisfying $\delta^{1/2} \approx \Delta_n$ and $\delta \cdot J_{\Delta_n} = \Delta_n$.

More generally, for RODEs with an affine structure

$$\frac{dx}{dt} = g(x) + G(x)\zeta_t,$$

where $g : \mathbb{R}^d \to \mathbb{R}^d$ and $G : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$, the explicit **averaged Euler scheme**

$$Y_{n+1} = Y_n + \left[g\left(Y_n\right) + G\left(Y_n\right)\, I_n\right]\, \Delta_n,$$

where

$$I_n := \frac{1}{\Delta_n} \int_{t_n}^{t_{n+1}} \zeta_s \, ds \approx \frac{1}{\Delta_n} \sum_{j=1}^{J_{\Delta_n}} \zeta_{t_n + j\delta} \, \delta.$$

has order pathwise order 1 provided $\delta^{1/2} \approx \Delta_n$ and $\delta \cdot J_{\Delta_n} = \Delta_n$.

Taylor-like expansions for RODEs (Jentzen & Kloeden (2007))

Replace the process ζ_t by its sample paths $t \mapsto \omega(t)$ with $\omega \in \Omega := C(\mathbb{R}, \mathbb{R}^m)$.

Then the vector field $(\omega, x) \mapsto f(\omega, x)$ has the Taylor expansion

$$f(\omega(s), x(s)) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \partial^\alpha f(\omega_0, x_0) \, (\Delta\omega_s)^{\alpha_1} (\Delta x_s)^{\alpha_2} + R_{k+1}(s)$$

where $\omega_0 := \omega(t_0)$, $x_0 := x(t_0)$ and

$$\Delta\omega_s := \omega(s) - \omega_0, \qquad \Delta x_s := x(s) - x_0,$$

with remainder term $R_{k+1}(s)$ and multi-indices $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$,

$$|\alpha| := \alpha_1 + \alpha_2 \, , \qquad \alpha! := \alpha_1! \, \alpha_2! \, .$$

Substituting this into the integral equation version of the RODE gives

$$\Delta x_t = \underbrace{\sum_{|\alpha|\leq k} \frac{1}{\alpha!}\partial^\alpha f(\omega_0, x_0) \int_{t_0}^{t} (\Delta\omega_s)^{\alpha_1}(\Delta x_s)^{\alpha_2}\, ds}_{\textit{Taylor-like approximation}} + \underbrace{\int_{t_0}^{t} R_{k+1}(s)\, ds}_{\textit{remainder}}$$

Apply this on subintervals $[t_n, t_{n+1}]$ with step size $h_n = t_{n+1} - t_n$ and discard the reminder.

The simplest case for $k = 0$ and $\alpha = (0, 0)$ gives the Euler scheme

$$y_{n+1} = y_n + h_n\, f(\omega(t_n), y_n).$$

BIG PROBLEM

Higher order Taylor-like approximations are implicit in Δx_t !!!

Resolution: replace Δx_t inside the integral by a lower order approximation, e.g., for $k = 1$ and $\alpha = (0, 1)$

$$\begin{aligned} x(t_{n+1}) &\approx x(t_n) + h_n\, f(\omega(t_n), x(t_n)) + \partial_x f(\omega(t_n), x(t_n)) \int_{t_n}^{t_{n+1}} \Delta x_s\, ds \\ &\approx x(t_n) + h_n\, f(\omega(t_n), x(t_n)) \\ &\quad + \partial_x f(\omega(t_n), x(t_n)) \int_{t_n}^{t_{n+1}} \underbrace{[(s - t_n) f(\omega(t_n), x(t_n))]}_{\textit{Euler approximation}}\, ds \end{aligned}$$

$\Rightarrow$ $\{(0,0),(0,1)\}$-Taylor numerical scheme

$$y_{n+1} = y_n + h_n\, f(\omega(t_n), y_n) + \frac{1}{2} h_n^2\, f(\omega(t_n), y_n)\, \partial_x f(\omega(t_n), y_n)$$

RODE–Taylor schemes

RODE–Taylor schemes are a family of explicit one–step schemes for RODEs on subintervals $[t_n, t_{n+1}]$ of $[t_0, T]$ with step size, which are derived from the corresponding Taylor–like expansions. Higher order schemes are built up recursively for *sets of multi–indices* of the form

$$\mathcal{A}_K := \left\{\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2 \ : \ |\alpha|_\theta = \theta\alpha_1 + \alpha_2 < K\right\},$$

where $K \in \mathbb{R}_+$ and $\theta \in (0, 1]$ is the Hölder exponent of the noise process.

It is necessary to distinguish two cases, Case A in which the Hölder estimate of the noise also holds for the supremum θ of the admissible exponents itself and Case B when it does not.

Theorem *The local discretisation error for a RODE–Taylor scheme in Case A satisfies*

$$\left| L_h^{(K)}(\hat{t}, \hat{x}) \right| \leq C_K \, h^{K+1}$$

for each $0 \leq h \leq 1$, *where*

$$C_K := \left(e^{\|\omega\|_\theta + 2R_K} \right)^{K+1}.$$

In Case B it satisfies

$$\left| L_h^{(K)}(\hat{t}, \hat{x}) \right| \leq C_K^\varepsilon \cdot h^{K+1-\varepsilon}$$

for $\varepsilon > 0$ *arbitrarily small, where*

$$C_K^\varepsilon := \left(e^{\|\omega\|_{\gamma_\varepsilon} + 2R_K} \right)^{K+1}, \qquad \gamma_\varepsilon := \theta - \frac{\varepsilon}{(k+1)^2}.$$

global error order = local error order -1

The 1.5-RODE–Taylor scheme for a **Wiener process**

$$y_{n+1} = y_n + hf + f_{(1,0)} \int_{t_n}^{t_{n+1}} \Delta\omega_s \, ds + \frac{f_{(2,0)}}{2} \int_{t_n}^{t_{n+1}} (\Delta\omega_s)^2 \, ds + f_{(0,1)} f \frac{h^2}{2}$$

corresponds to the index set $\mathcal{A}_{1.5} = \{(0,0), (1,0), (2,0), (0,1)\}$.

The 1.5-RODE–Taylor scheme for a **fractional Brownian motion** with Hurst exponent $H = \frac{3}{4}$

$$y_{n+1} = y_n + hf + f_{(1,0)} \int_{t_n}^{t_{n+1}} \Delta\omega_s + f_{(0,1)} f \frac{h^2}{2},$$

corresponds to the index set $\mathcal{A}_K = \{(0,0), (1,0), (0,1)\}$.
It omits one of the terms in the above RODE–Taylor scheme for a Wiener process.

• A similar idea is used to construct Taylor expansions for SPDE, see

A. Jentzen and P.E. Kloeden, *Taylor Expansions of Stochastic Partial Differential Equations,* CBMS Lecture series, SIAM, Philadelphia, 2011.

• The schemes obtained above are not optimal, i.e., they may contain more terms than are essential to ensure the given order is attained. Optimal schemes can be obtained by a modified approach, see

A. Jentzen and P.E. Kloeden, Pathwise Taylor schemes for random ordinary differential equations, *BIT* **49** (1) (2009), 113–140.

Numerical schemes for RODEs via SODEs (Asai & Kloeden, 2013)

Suppose that the process ζ_t in the RODE is an Itô process, i.e., the solution of an SODE, so

$$\frac{dx}{dt} = f(x, \zeta_t), \qquad d\zeta_t = a(\zeta_t)\, dt + b(\zeta_t)\, dW_t.$$

$\Rightarrow$ <u>use numerical schemes for SODEs</u>

$$d \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} f(X_t, Y_t) \\ a(Y_t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ b(Y_t) \end{pmatrix} dW_t$$

$\star$ stochastic Taylor schemes for SODEs that converge in a strong or weak sense (Kloeden & Platen).

$\star$ they also converge in a pathwise sense (Kloeden & Neuenkirch (2007)).

A numerical scheme is said to **converge strongly** with order γ if

$$\mathbb{E}\left(\left|X_T - X_{N_T}^{(\Delta)}\right|\right) \leq K_T\,\Delta^{\gamma}$$

Theorem *Under classical assumptions an Itô-Taylor scheme of strong order $\gamma > 0$* **converges pathwise** *with order $\gamma - \epsilon$ for all $\epsilon > 0$, i.e.*

$$\sup_{i=0,\ldots,N_T} \left|X_{t_n}(\omega) - X_n^{(\Delta)}(\omega)\right| \leq K_{\epsilon,T}^{\gamma}(\omega) \cdot \Delta^{\gamma-\epsilon}$$

for almost all $\omega \in \Omega$.

P.E. Kloeden and A. Neuenkirch, The pathwise convergence of approximation schemes for stochastic differential equations, *LMS J. Comp. Math.* **10** (2007), 235-253.

$\star$ The classical assumptions require the appropriate partial derivatives of the coefficients to be **globally bounded**.

$\star$ This ensures that all moments exist, so a Borel-Cantelli argument can be used.

$$\Longrightarrow \text{ excludes many interesting applications.}$$

- It is not essential for a pathwise analysis of RODEs.

- A similar result applies by a **localisation argument** with stopping times etc when the derivative are only locally bounded.

Y. Asai and P.E. Kloeden, Numerical schemes for random ODEs via stochastic differential equations *Commun. Appl. Anal.* **17** (2013), no. 3 & 4, 51–528.

A RODE driven by an Ornstein-Uhlenbeck process can be formulated an SODE

$$d\begin{pmatrix} X_t \\ O_t \end{pmatrix} = \begin{pmatrix} f(X_t, O_t) \\ -O_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW_t$$

The Euler-Maruyama scheme

$$X_{n+1} = X_n + f(X_n, O_n)\,\Delta_n$$

is, in fact, the same as the Milstein scheme.

- it thus has pathwise order 1.0.

• Higher order step schemes can be derived using strong Itô-Taylor expansions and require multiple stochastic integral terms to achieve a higher order of strong convergence.

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \sum_{\alpha \in \Lambda_\gamma \setminus \{\emptyset\}} L^\alpha id_{\mathbf{X}}\,(t_n, \mathbf{X}_n)\; I_{\alpha, t_n, t_{n+1}}$$

for hierarchical set Λ_γ of multi-indices, where $\gamma = \frac{1}{2}, 1, \frac{2}{2}, 2, \ldots$.

• For the special case of the RODE-SODE pair with the diffusion coefficient function $\mathbf{G}(\mathbf{X})$ as above, the differential operators reduce to $L^1 id_{\mathbf{X}}^k \equiv 0$ for the components of the RODE.

Hence only the multi-indices in the subset

$$\Lambda_\gamma^0 \;=\; \{\alpha \in \Lambda_\gamma : \text{the last component } j_l = 0\} \cup \{\emptyset\}$$

of Λ_γ appear in these RODE components of the stochastic Taylor expansion.

The corresponding remainder set is

$$\mathcal{B}(\Lambda_\gamma^0) = \{\alpha \in \mathcal{M} \backslash \Lambda_\gamma^0 \ : \ -\alpha \in \Lambda_\gamma^0\}$$

Note that $(j) \in \mathcal{B}(\Lambda_\gamma^0)$ for $j = 1, \ldots, m$, since $(j) \notin \Lambda_\gamma^0$ but $-(j) = \emptyset \in \Lambda_\gamma^0$.

Example When $m = 1$ and $\gamma = \frac{3}{2}$, the hierarchical set Λ_γ, $\mathcal{B}(\Lambda_\gamma)$, the reduced set Λ_γ^0 and $\mathcal{B}(\Lambda_\gamma^0)$ are given by

$$\begin{aligned}
\Lambda_{\frac{3}{2}} &= \{\emptyset, (1), (0), (1,1), (1,0), (0,1), (0,0), (1,1,1)\} \\
\Lambda_{\frac{3}{2}}^0 &= \{\emptyset, (0), (1,0), (0,0)\} \\
\mathcal{B}(\Lambda_{\frac{3}{2}}^0) &= \{(1), (1,1,0), (0,1,0), (1,0,0), (0,0,0)\}.
\end{aligned}$$

The RODE-Taylor scheme of order $\gamma = \frac{3}{2}$ is

$$x_{n+1} = x_n + f(x_n, y_n)\Delta_n + b(y_n) f_y(x_n, y_n)\, I_{(1,0),n}$$
$$+ \left(f(x_n, y_n) f_x(x_n, y_n) + a(y_n) f_y(x_n, y_n) + \frac{1}{2} b(y_n)^2 f_{yy}(x_n, y_n) \right) \frac{1}{2}\Delta_n^2,$$

since

$$\Lambda_{\frac{3}{2}}^{0} = \{\emptyset, (0), (1,0), (0,0)\}.$$

This scheme contains the mixed stochastic-deterministic integral

$$I_{(1,0),n} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s dt.$$

which is a correlated with $I_{(1),n} = \int_{t_n}^{t_{n+1}} dW_s = \Delta W_n$. (Both Gaussian).

Multi-step schemes for Itô RODEs

An Euler-Maruyama type linear k-step method for coupled RODE-SODE is given by

$$\sum_{j=0}^{k} \alpha_j \mathbf{X}_{n-j} = h \sum_{j=0}^{k} \beta_j \mathbf{F}_{n-j} + \Delta W \sum_{j=1}^{k} \gamma_j \mathbf{G}_{n-j}.$$

where

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \ \mathbf{F}(\mathbf{X}) = \begin{pmatrix} f(x,y) \\ a(y) \end{pmatrix}, \ \mathbf{G}(\mathbf{X}) = \begin{pmatrix} 0 \\ b(y) \end{pmatrix}$$

Since $\mathbf{G}^1 \equiv 0$ its X-component reduces to

$$\sum_{j=0}^{k} \alpha_j x_{n-j} = h \sum_{j=0}^{k} \beta_j f_{n-j}. \tag{1}$$

Typical examples are the Adams-Bashforth and Adams-Moulton methods. In the scalar case, the x-component gives the

- *RODE-Adams-Bashforth 2 scheme*

$$x_{n+1} = x_n + \left\{ \frac{3}{2} f(x_n, y_n) - \frac{1}{2} f(x_{n-1}, y_{n-1}) \right\} \Delta_n$$

- *RODE-Adams-Moulton-2 scheme*

$$x_{n+1} = x_n + \left\{ \frac{5}{12} f(x_{n+1}, y_{n+1}) + \frac{8}{12} f(x_n, y_n) - \frac{1}{12} f(x_{n-1}, y_{n-1}) \right\} \Delta_n.$$

These schemes both have order $\gamma = 1.0$ convergence and coincide with their counterparts in the deterministic case.

A γ-order k-step SLMM has the general form

$$\begin{aligned}\sum_{j=0}^{k} C_{\emptyset,j} X_{n-j} &= h \sum_{j=0}^{k} C_{(0),j} f(X_{n-j}, \bar{Y}_{n-j}) \\ &+ \sum_{j=1}^{k} \sum_{\alpha \in \Lambda_\gamma^0 \setminus \{\emptyset,(0)\}} L^\alpha id_{\mathbf{X}}^1(X_{n-j}, \bar{Y}_{n-j}) \Big(C_{\alpha,j} I_{\alpha,t_{n-j}} + C_{\alpha,j}^* I_{\alpha-,t_{n-j}} h \Big),\end{aligned}$$

with the consistency conditions

$$\begin{cases} \sum_{j=0}^{k} C_{\emptyset,j} = 0, \qquad \sum_{j=0}^{k} (k-j) C_{\emptyset,j} = \sum_{j=0}^{k} C_{(0),j} \\ C_{\alpha,i} = \sum_{j=0}^{i-1} C_{\emptyset,j} \quad \text{for} \quad i = 1, \ldots, k, \\ C_{\alpha,i}^* = \sum_{j=0}^{i-1} \Big((k-1-j) C_{\emptyset,j} - C_{(0),j} \Big) \quad \text{for} \quad i = 1, \ldots, k. \end{cases}$$

for $\alpha \in \Lambda_\gamma^0 \setminus \{\emptyset, (0)\}$.

• Here $\bar{Y}_{n-j}$ is an approximation of Y_t at t_{n-j} by a scheme of high enough order or Y_{t_n} itself when Y_t can be generated exactly.

• The Taylor expansions for the Itô diffusion components of the SODE and other functions of the solutions still require all of the multi-indices in the original hierarchical set Λ_γ.

It is decoupled from the RODE components of the Taylor scheme.

• The scheme reduces to an explicit scheme when $C_{(0),0} = 0$.

By the order conditions this happens when, e.g., the $C^*_{\alpha,1} = 0$ for all $\alpha \in \Lambda^0_\gamma \backslash \{\emptyset, (0)\}$.

• Y. Asai and P.E. Kloeden, Multi-step methods for random ODEs driven by Itô diffusions, *J. Comput. Appl. Math.* **294** (2016), 210-224.

RODEs with affine noise

A scalar RODE with scalar affine noise has the form

$$\frac{dx}{dt} = f^0(t,x) + f^1(t,x)\,\eta_t^1,$$

written in compact integral equation form

$$x(t) = x(t_0) + \sum_{j=0}^{1} \int_{t_0}^{t} f^j(s,x(s))\,\eta_s^j\,ds$$

with a fictitious "noise" component $\eta_t^0 \equiv 1$

It is an ODE is the noise has continuous sample paths and a Carathéodory DE if they are just measurable.

Since the chain rule for such RODEs is analogous to that for Stratonovich SODE, their Taylor expansions are analogous too.

The *affine-RODE-Taylor scheme of order N* for $N = 1, 2, 3, \ldots$ is defined by

$$x_{n+1} = x_n + \sum_{\alpha \in \mathcal{A}_N \setminus \{\emptyset\}} id_X^{\alpha}(t_n, x_n)\, I_{\alpha, t_n, t_{n+1}} \tag{2}$$

for the hierarchical set of multi-indices

$$\mathcal{A}_N = \{\alpha \in \mathcal{M}_1 : l(\alpha) \leq N\}$$

with the multiple integrals are defined by

$$I_{\alpha, t_n, t_{n+1}} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_l} \cdots \int_{t_n}^{s_2} \zeta^{j_1}(s_1) \cdots \zeta^{j_l}(s_l)\, ds_1 \cdots ds_l$$

and the coefficient function F_α is defined recursively by

$$F^\alpha = \begin{cases} F & : \quad l(\alpha) = 0 \\ L^{j_1} F^{-\alpha} & : \quad l(\alpha) \geq 1. \end{cases}, \tag{3}$$

for the function $F \equiv id_X$, where the partial differential operators L^0, L^1, ..., L^m are defined by

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} f^{0,k} \frac{\partial}{\partial x^k}, \qquad L^1 = \sum_{k=1}^{d} f^{j,k} \frac{\partial}{\partial x^k},$$

Theorem Suppose that noise sample paths are continuous or essentially bounded on bounded time intervals and that all of the derivatives of f^0 and f^1 appearing here exist and are continuous.
Then, the affine-RODE-Taylor scheme (2) has pathwise order of convergence N.

RODEs or SODEs? – fluctations in a cell fission model

◇ SODE version:

$$d\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ k_1^1 - k_2(u)v \end{pmatrix} dt + \begin{pmatrix} Du \\ 0 \end{pmatrix} dW_t,$$

where

$$f(u,v) \quad = \quad \frac{k_1^1}{G} - [k_2(u) + k_{wee}]u + k_{25}(u)\left(\frac{v}{G} - u\right)$$

with

$$k_2(u) = k_2' + k_2'' u^2$$

, $\qquad \mathrm{k}_{25}(u) = k_{25}' + k_{25}'' u^2.$

◇ RODE version with Ornstein-Uhlenbeck process:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ k_1^1 - k_2(u)v \end{pmatrix} + \begin{pmatrix} Du \\ 0 \end{pmatrix} O_t,$$

R. Steuer, Effects of stochasticity in models of the cell cycle: from quantized cycle times to noise-induced oscillations, *J. Theoret. Biology* **228** (2004) 293–301.

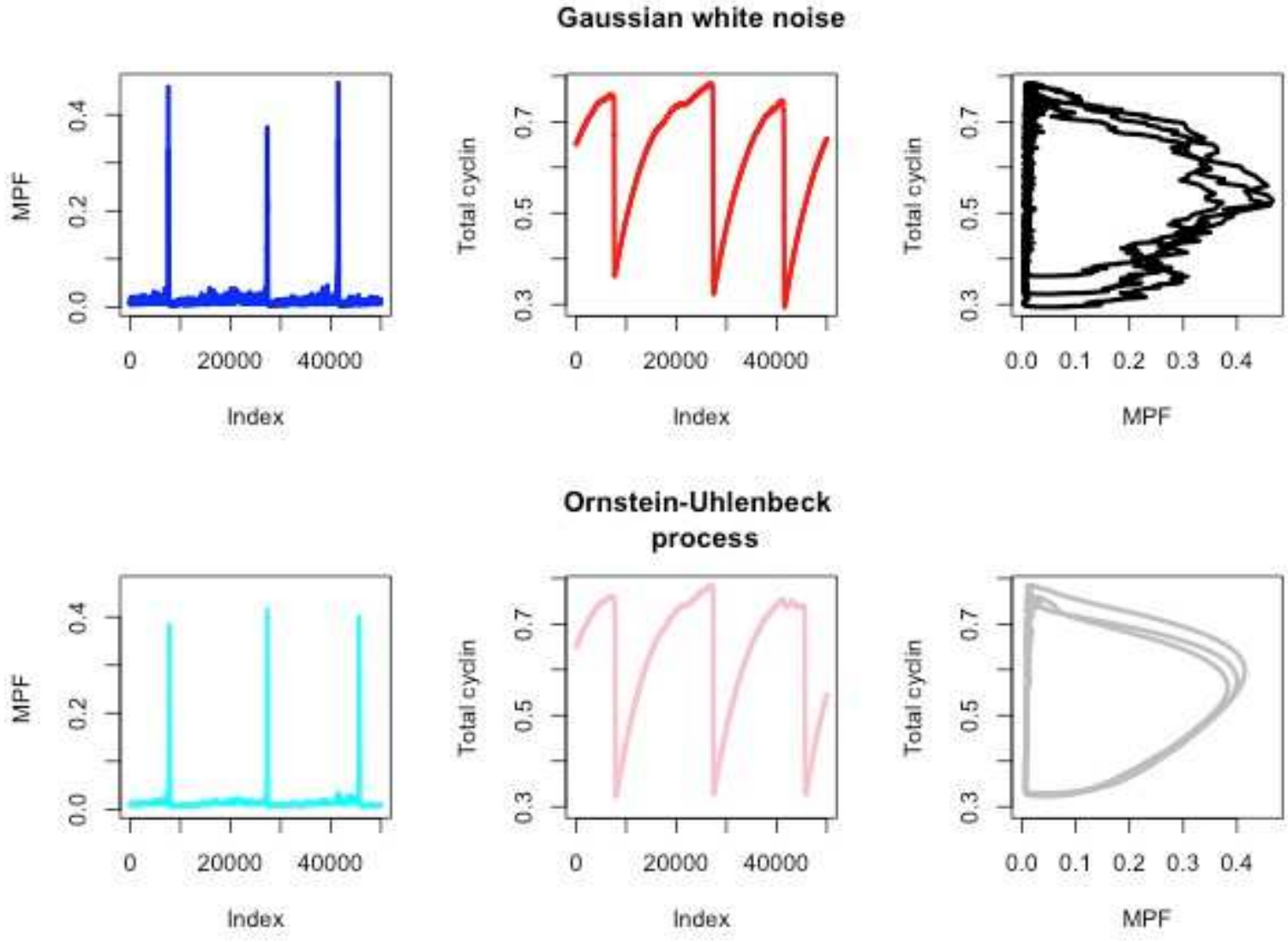

Figure: SODE-Euler-Maruyama and RODE-Euler schemes

Bounded noise

In biological models bounded noise is often more realistic.

It can be introduced by allowing a mass-action parameter to vary randomly within a bounded interval about an idealised value.

Alberto d'Onofrio (editor),

Random Bounded Noises in Physics, Biology, and Engineering,

Birkhäuser, 2013.

Two possibilities, which modify an input noise such as an Ornstein-Uhlenbeck process Y_t, are:

- a positive parameter ζ might be replaced by the bounded stochastic process

$$\zeta(Y_t) := \zeta_0\left(1 - 2\nu\frac{Y_t}{1+Y_t^2}\right), \tag{4}$$

where ζ_0 and ν are positive constants with $\nu \in (0, 1]$.

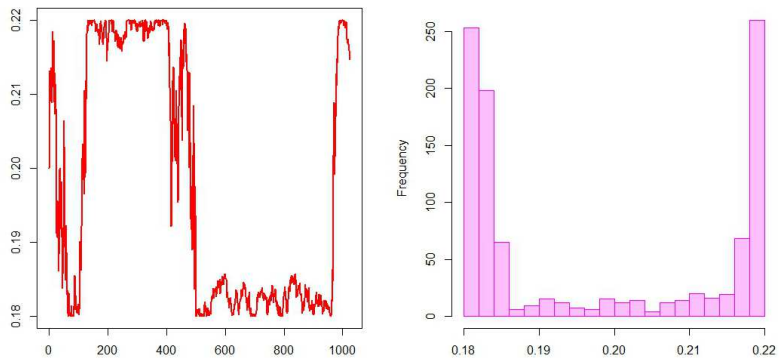

Figure: Switching noise (4) driven by a Wiener process.

The noise here tends to peak around $\zeta_0(1 \pm \nu)$, and is thus suitable for a noisy switching scenario.

- a positive parameter δ might be replaced by the stochastic process

$$\delta(Y_t) := \delta_0\left(1 - \frac{2\nu}{\pi}\arctan Y_t\right), \tag{5}$$

where δ_0 and ν are positive constants with $\nu \in (0, 1]$.

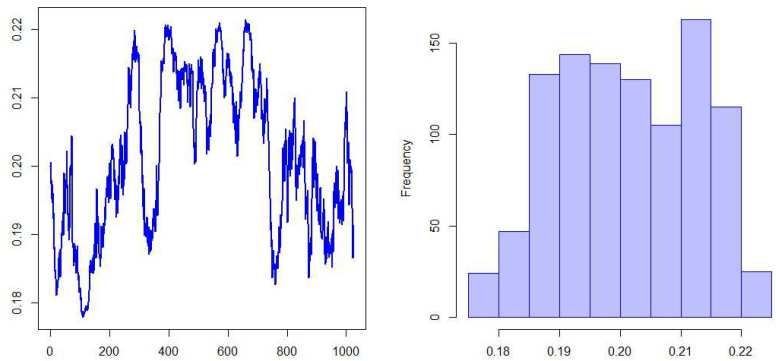

Figure: Centered noise (5) driven by an Ornstein-Uhlenbeck process.

The noise then takes values in the interval $(\delta_0(1-\nu), \delta_0(1+\nu))$ with the probability density taking its maximum at δ_0.

Literature

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