Long time behaviour of Optimal Control problems and the Turnpike Property

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Turnpike Property in Control Theory

- Let U and X be 2 Banach spaces and $\mathcal{U}$ the set of admissible controls.
- We take into account Λ the **operator** describing the **Control System** considered.
- For any **control** function $u \in \mathcal{U}$, the corresponding **state** is $x \in AC([0, T]; X)$ solution of:

$$\frac{d}{dt}x + \Lambda(x, u) = 0 \qquad \text{in } (0, T).$$

- We define a Lagrangian:

$$\mathcal{L} : X \times U \longmapsto \mathbb{R}$$

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Turnpike Property in Control Theory

The NonStationary **Optimal Control Problem** $(OCP)^T$ consists in **minimizing** the following functional

$$J^T : \mathcal{U} \longmapsto \mathbb{R}$$

$$u \longrightarrow \int_0^T \mathcal{L}(x(t), u(t))dt.$$

The corresponding **optimal couple** will be called (x^T, u^T).

Turnpike Property in Control Theory

- We define

$$M = \{(x,u) \in X \times U \mid \Lambda(x,u) = 0\}.$$

A generic element of M solves the equation:

$$\Lambda(x,u) = 0,$$

which is exactly the stationary version of $\frac{d}{dt}x + \Lambda(x,u) = 0$.

- The **minimization** problem of the **Lagrangian** Function $\mathcal{L}$ on M will be called $(OCP)^s$. We will call $(\overline{x}, \overline{u})$ the optimal pair for $(OCP)^s$.

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Turnpike Property in Control Theory

At this stage, it is natural to look for a **link** between $(OCP)^T$ and $(OCP)^s$. In this seminar, we will assume that both $(OCP)^T$ and $(OCP)^s$ are well posed and admit a **unique minimizer**.

Turnpike Property in Control Theory

A Control System enjoys **Turnpike Property** if there exists $\tau > 0$, independent of the time horizon $T \in (0, +\infty)$, such that:

- in a **long time interval** $[\tau, T - \tau]$, the optimal couple (x^T, u^T) remains **near** to $(\overline{x}, \overline{u})$;
- in a short initial interval $[0, \tau]$ and in a short final interval $[T - \tau, T]$, the optimal couple (x^T, u^T) can be far from $(\overline{x}, \overline{u})$, in order to fulfill the initial-final conditions required by the problem.

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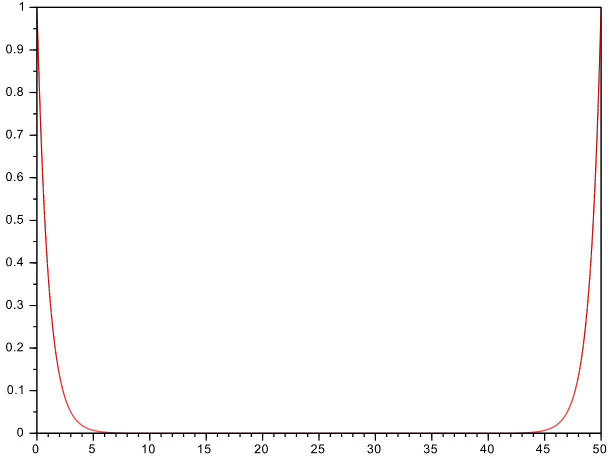
Turnpike Property in Control Theory

Behaviour of $\|x^T - \overline{x}\| + \|u^T - \overline{u}\|$



History

- **Paul Samuelson**, Econometrician, Nobel Prize in Economic Sciences in 1970.
- *"... **if we are planning long-run growth, no matter where we start and where we desire to end up, it will pay in the intermediate stages to get into a growth phase of this kind.** It is exactly like a turnpike paralleled by a network of minor roads. There is a fastest route between any two points; and if origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But **if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.**"*

History

- Turnpike results have been proven all along the second half of the 20th century. Furthermore, it has been applied in many fields of application such as **Econometrics** and **Biology**.
- some results in **Mean Field Games** Theory motivated a new source of investigation of the Turnpike Property in Optimal Control Theory, specifically in an Infinite Dimensional Setting, where the nearness of (x^T, u^T) to $(\overline{x}, \overline{u})$ on $[\tau, T-\tau]$ is required to be exponential.

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Optimality System

We take into account $L \in C^2(\mathbb{R}^M, \mathbb{R})$ and $F \in C^2(\mathbb{R}^N, \mathbb{R})$ bounded from below. Moreover, we consider $(A, B) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R})$ and an initial datum $x_0 \in \mathbb{R}^N$. For any control $u \in L^2((0, T); \mathbb{R}^M)$, the corresponding state x fulfills the linear system:

$$\begin{cases} \frac{d}{dt}x(t) + Ax(t) = Bu(t) & \text{a.e. } t \in (0, T) \\ x(0) = x_0. \end{cases}$$

We aim at minimizing the following functional

$$J^T : L^2((0, T); \mathbb{R}^M) \longmapsto \mathbb{R}$$

$$u \longrightarrow \int_0^T [L(u(t)) + F(x(t))]dt.$$

We assume the the problem is well defined and that there exists at least one minimizer.

Optimality System

For any u^T minimizer for J^T, we would like to impose the condition

$$< dJ^T(u^T), v >= 0$$

for any direction $v \in L^2((0,T);\mathbb{R}^M)$. To this purpose, for any such direction v, we consider φ solution of:

$$\begin{cases} \varphi_t(t) + A\varphi(t) = Bv(t) & \text{a.e. } t \in (0,T) \\ \varphi(0) = 0. \end{cases}$$

Optimality System

Then, we define the function:

$$f : \mathbb{R} \longmapsto \mathbb{R}$$

$$\eta \longrightarrow \int_0^T \left[L(u^T(t) + \eta v(t)) + F(x^T(t) + \eta \varphi(t)) \right] dt$$

We observe that f attains its minimum at 0. Therefore, by Differentiable Dependence Lebesgue Theorem and Fermat Theorem,

$$\exists \frac{d}{d\eta} f(0) = \int_0^T \left[\left(L_u(u^T(t), v(t) \right)_{\mathbb{R}^M} + \left(F_x(x^T(t)), \varphi(t) \right)_{\mathbb{R}^N} \right] dt = 0.$$

Optimality System

In order to show the continuous and linear dependence of the Gateaux Differential on v, we define the adjoint state p^T as the solution of the backward problem below:

$$\begin{cases} -p_t^T(t) + A^* p^T(t) = F_x(x^T(t)) & \text{a.e. } t \in (0, T) \\ p^T(T) = 0. \end{cases} \tag{1}$$

Optimality System

Multiplying in L^2 the equation of p^T by φ,

$$\int_0^T \left(F_x(x^T(t)), \varphi(t)\right)_{\mathbb{R}^N} dt = \int_0^T \left(-p_t^T(t) + A^* p^T(t), \varphi(t)\right)_{\mathbb{R}^N} dt =$$

integrating by parts,

$$= \left(p^T(0), \varphi(0)\right) - \left(p^T(T), \varphi(T)\right) + \int_0^T \left(p^T(t), \varphi_t(t) + A\varphi(t)\right) dt =$$

using the equation of φ and the initial-final conditions,

$$= \int_0^T \left(p^T(t), Bv(t)\right)_{\mathbb{R}^N} dt = \int_0^T \left(B^* p^T(t), v(t)\right)_{\mathbb{R}^M} dt.$$

Optimality System

Hence, $\forall v \in L^2((0,T);\mathbb{R}^M)$

$$\int_0^T \left[\Big(L_u(u^T(t)), v(t) \Big)_{\mathbb{R}^M} + \Big(B^* p^T(t), v(t) \Big)_{\mathbb{R}^M} \right] dt = 0.$$

This integral condition is equivalent to:

$$B^* p^T(t) = -L_u(u^T(t)) \qquad \text{a.e. } t \in (0,T).$$

Hence, an equivalent version of the first order conditions for u^T is the following:

$$\begin{cases} x_t^T(t) + Ax^T(t) = Bu^T(t) & \text{a.e. } t \in (0,T) \\ -p_t^T(t) + A^* p^T(t) = F_x(x^T(t)) & \text{a.e. } t \in (0,T) \\ L_u(u^T(t)) = -B^* p^T(t) & \text{a.e. } t \in [0,T] \\ x^T(0) = x_0 \\ p^T(T) = 0. \end{cases} \qquad (2)$$

Definition of the NonStationary Problem

- We work with a **linear** and **finite dimensional** dynamics. For every **control** function $u \in L^2((0,T);\mathbb{R}^M)$, we take into account the corresponding **state** solution of the Cauchy Problem:

$$\begin{cases} \frac{d}{dt}x(t) + Ax(t) = Bu(t) & \text{a.e. } t \in (0,T) \\ x(0) = x_0, \end{cases}$$

where $(A,B) \in \mathcal{M}(N,N;\mathbb{R}) \times \mathcal{M}(N,M;\mathbb{R})$.

- $(OCP)^T$ is the **minimization** of the functional

$$J^T : L^2((0,T);\mathbb{R}^M) \longmapsto \mathbb{R}$$

$$u \longrightarrow \frac{1}{2}\int_0^T \left(\|u(t)\|^2 + \|Cx(t) - z\|^2\right) dt,$$

where $C \in \mathcal{M}(N,N;\mathbb{R})$.

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Optimality Condition for $(OCP)^T$

- **Existence and Uniqueness** of the minimizer u^T of J^T holds.
- Moreover, **there exists a unique** optimal triple (x^T, p^T, u^T), fulfilling the Optimality System:

$$\begin{cases} \frac{d}{dt}x^T(t) + Ax^T(t) = -BB^*p^T(t) & \forall\ t \in (0,T) \\ -\frac{d}{dt}p^T(t) + A^*p^T(t) = C^*(Cx^T(t) - z) & \forall\ t \in (0,T) \\ x^T(0) = x_0, \qquad p^T(T) = 0. \end{cases}$$

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- On the other hand, we define the vector subspace:

$$M = \left\{(x,u) \in \mathbb{R}^N \times \mathbb{R}^M \mid Ax = Bu\right\}$$

$(OCP)^s$ consists in **minimizing** the following Lagrangian function on M:

$$J^s : M \longmapsto \mathbb{R}$$
$$(x,u) \longrightarrow \frac{1}{2}\left[\|u\|^2 + \|Cx - z\|^2\right].$$

- **Existence and Uniqueness** of the minimizer $(\overline{x}, \overline{u})$ holds.
- Furthermore, **there exists a unique** optimal triple $(\overline{x}, \overline{p}, \overline{u})$, which solves:

$$\begin{cases} A\overline{x} = B\overline{u} \\ A^*\overline{p} = C^*(C\overline{x} - z), \end{cases}$$

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Controllability and Observability hypotheses

1. (A, B) is **Kalman-Controllable**, i.e. for any couple $(x_1, x_2) \in \mathbb{R}^{2N}$ there exists a control u driving the system from the initial state x_1 to the final state x_2. Thanks to Kalman's Theorem, the Kalman Controllability is equivalent to the algebraic condition:

$$\text{rank}[B, AB, A^2 B, \dots, A^{N-1} B] = N.$$

2. the pair (A, C) is **Kalman-Observable**, which actually means that if y solves the system

$$\begin{cases} \frac{d}{dt} y + Ay = 0 & \forall t \in (0, T) \\ Cy(t) = 0 & \forall t \in [0, T], \end{cases}$$

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Turnpike Property

Theorem (Porretta-Zuazua)

We assume (A, B) is ***Kalman-Controllable*** *and (A, C) is* ***Kalman-Observable****. Then, there exists $(C, \mu) \in (0, +\infty)^2$, such that, for any time horizon $T \in (0, +\infty)$, $\forall t \in [0, T]$:*

$$\|x^T(t) - \overline{x}\| + \|u^T(t) - \overline{u}\| \leq C \left[\|x_0 - \overline{x}\| e^{-\mu t} + \|\overline{p}\| e^{-\mu(T-t)}\right].$$

Idea of the proof - Riccati's Equations

1. **There exists a unique** $\mathcal{E} \in C^1([0,+\infty);\mathcal{M}(N,N;\mathbb{R}))$ solution of:

$$\begin{cases} \mathcal{E}_t + (\mathcal{E}A + A^*\mathcal{E}) + \mathcal{E}BB^*\mathcal{E} = C^*C & \forall t \in (0,+\infty) \\ \mathcal{E}(0) = 0. \end{cases}$$

2. $$\mathcal{E}(T) \underset{T\to+\infty}{\longrightarrow} \widehat{E} \qquad \text{exponentially.}$$

3. **There exists a unique** $\widehat{E}$ positive definite solution of **Algebraic Riccati's Equation**:

$$(\widehat{E}A + A^*\widehat{E}) + \widehat{E}BB^*\widehat{E} = C^*C \qquad \text{(ARE)}.$$

4. The optimal control is a **feedback control**

$$u^T \cong -B^*\mathcal{E}(T-t)x^T.$$

Investigation Plan

After the Finite Dimensional Linear Quadratic Case, we analyse **Turnpike Property** for more general Control problems.

1. Whenever we want to study the Turnpike Property in a more general setting, we encounter some additional difficulties. For instance, the Optimality System becomes NonLinear.
2. A possible approach is the **linearization** of the Optimality System.
3. By employing this approach, Emmanuel Trélat and Enrique Zuazua have proved a "Local Turnpike" for general control problems, under suitable hypotheses such as the **Controllability** of the linearised of the Optimality System.
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Definition of the NonStationary Control Problem

For any **admissible control** u, the correspondiong **state** solves:

$$\frac{d}{dt}x(t) + Ax(t) = Bu(t) \qquad \text{a.e. } t \in (0, T)$$

<u>case state fixed in 0</u>

$$x(0) = x_0$$

<u>case state fixed both in 0 and in T</u>

$$x(0) = x_0 \qquad x(T) = x_1 .$$

We suppose that $(A, B) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R})$ is **Kalman-Controllable**.

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We take into account:

- $F \in C^2(\mathbb{R}^N, \mathbb{R})$ **strongly convex**, i.e. such that there exists $\alpha \in (0, +\infty)$ such that $\alpha I_N \leq F_{xx}$.
- $L \in C^2(\mathbb{R}^M, \mathbb{R})$ **strongly convex** and such that the **second derivative is bounded**. In other words, there exist $\alpha,\ \beta > 0$ such that $\alpha I_M \leq L_{uu} \leq \beta I_M$.
- The stationary control problems is the **minimization** of the funtional:

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- When only the **initial condition** $x(0) = x_0$ is fulfilled, **there exists a unique** minimizer u^T of J^T.
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Optimality Condition for $(OCP)^T$ (2)

- Moreover, if both the **initial conditon** $x(0) = x_0$ and the **final condition** $x(T) = x_1$ are imposed, **there exists a unique** minimizer u^T of J^T.
- In addition, **there exists a unique** optimal triple (x^T, p^T, u^T) fulfilling the Optimality System:

$$\begin{cases} \frac{d}{dt}x^T(t) + Ax^T(t) = Bu^T(t) & \forall\ t \in (0,T) \\ -\frac{d}{dt}p^T(t) + A^*p^T(t) = F_x(x^T(t)) & \forall\ t \in (0,T) \\ x^T(0) = x_0 \qquad x^T(T) = x_1. \end{cases}$$

$$\boxed{L_u(u^T) = -B^*p^T.}$$

Optimality Condition for $(OCP)^T$ (2)

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Definition of the Stationary Problem

We define the vector subspace

$$M = \left\{ (x,u) \in \mathbb{R}^N \times \mathbb{R}^M \mid Ax = Bu \right\}.$$

$(OCP)^s$ concerns the **minimization** of the lagrangian restricted to M, namely:

$$J^s : M \longmapsto \mathbb{R}$$

$$(x,u) \longrightarrow [F(x) + L(u)].$$

Existence and Uniqueness of the minimizer $(\overline{x}, \overline{u})$ for $(OCP)^s$ holds. Moreover, **there exists a unique** optimal triple $(\overline{x}, \overline{p}, \overline{u})$, which fulfills the first order condition

$$\begin{cases} A\overline{x} = B\overline{u} \\ A^* \overline{p} = F_x(\overline{x}). \end{cases}$$

$$\boxed{L_u(\overline{u}) = -B^* \overline{p}.}$$

"Local Turnpike" Property

Theorem (Trélat-Zuazua)

In the above hypotheses, there exists $\overline{\varepsilon} > 0$*, which enables us to define the following conditons. If the initial end of the state is fixed and the final end is free, we assume:*

$$\|\overline{x} - x_0\| + \|\overline{p}\| \leq \overline{\varepsilon}.$$

If both the initial condition and the final condition of the state are imposed, we suppose:

$$\|\overline{x} - x_0\| + \|\overline{x} - x_1\| \leq \overline{\varepsilon}.$$

Then, the optimal triple (x^T, p^T, u^T) *fulfills* $\forall t \in [0, T]$ *the Turnpike estimate:*

$$\|x^T(t) - \overline{x}\| + \|p^T(t) - \overline{p}\| + \|u^T(t) - \overline{u}\| \leq C\left[e^{-\mu t} + e^{-\mu(T-t)}\right].$$

Idea of the proof

Definition of the **perturbation functions**:

$$\delta x^T : [0, T] \longmapsto \mathbb{R}^N$$

$$t \longrightarrow x^T(t) - \overline{x},$$

$$\delta p^T : [0, T] \longmapsto \mathbb{R}^N$$

$$t \longrightarrow p^T(t) - \overline{p}$$

and

$$\delta u^T : [0, T] \longmapsto \mathbb{R}^N$$

$$t \longrightarrow u^T(t) - \overline{u}.$$

Idea of the proof

First Step

Linearization of the Optimality System.

$$\begin{cases} \frac{d}{dt}\delta x^T(t) = -A\delta x^T(t) - BL_{uu}(\overline{u})^{-1}B^*(\delta p^T(t)) + B\Lambda_2(\delta p^T(t)) \\ \frac{d}{dt}\delta p^T(t) = A^*\delta p^T(t) - F_{xx}(\overline{x})(\delta x^T(t)) - \Lambda_1(\delta x^T(t)) \\ \delta u^T(t) = -L_{uu}(\overline{u})^{-1}B^*(\delta p^T(t)) + \Lambda_2(\delta p^T(t)) \end{cases}$$

Furthermore, we keep in mind that, by the hypotheses, on $(\delta x^T, \delta p^T)$ the following **initial-final conditions** are imposed. If the state is fixed in 0 and left free in T:

$$\delta x^T(0) = x_0 - \overline{x}, \qquad \delta p^T(T) = -\overline{p}. \tag{3}$$

On the other hand, if both ends are fixed:

$$\delta x^T(0) = x_0 - \overline{x}, \qquad \delta x^T(T) = x_1 - \overline{x}. \tag{4}$$

Idea of the proof

We rewrite the **first** 2 **equations** of the above system in a compact form:

$$\frac{d}{dt}\begin{pmatrix}\delta x^T(t)\\ \delta p^T(t)\end{pmatrix} = M\begin{pmatrix}\delta x^T(t)\\ \delta p^T(t)\end{pmatrix} + \begin{pmatrix}B\Lambda_2(\delta p^T(t))\\ -\Lambda_1(\delta x^T(t))\end{pmatrix} \tag{5}$$

Idea of the proof

By the unique positive definite solution $\widehat{E}_+$ [resp. $\widehat{E}_-$ negative definite] the **Algebraic Riccati's Equation**:

$$\widehat{E}A + A\widehat{E} + \widehat{E}BL_{uu}(\overline{u})^{-1}B^*\widehat{E} - F_{xx}(\overline{x}) = 0 \qquad \text{(ARE)},$$

we define a **transformation**

$$P = \begin{pmatrix} I_N & I_N \\ \widehat{E}_+ & \widehat{E}_- \end{pmatrix}$$

which **diagonalises** the Optimality system in a **contractive** part and an **expanding** one.

Idea of the proof

$$P^{-1}MP = \begin{pmatrix} -A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_+ & 0 \\ 0 & -A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_- \end{pmatrix}$$

where

$$Re(\sigma(-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_+)) \subset (-\infty, 0)$$

and

$$Re(\sigma(-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_-)) \subset (0, +\infty).$$

Idea of the proof

In the **new coordinates**, we can show that, for initial-final data (g_0, h_T) sufficiently small, **existence and uniqueness** holds for the problem:

$$\begin{cases} \frac{d}{dt}g(t) = (-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_+)g(t) + p_1(\Lambda_3(g(t), h(t))) \\ \frac{d}{dt}h(t) = (-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_-)h(t) + p_2(\Lambda_3(g(t), h(t))) \\ g(0) = g_0 \\ h(T) = h_T. \\ \|g(t)\| + \|h(t)\| \leq \rho \qquad \forall t \in [0, T]. \end{cases}$$

Idea of the proof

Second Step

- Thanks to **Brouwer Fixed Point Theorem**, we prove that there exist initial-final data (g_0, h_T), such that the corresponding solution $\begin{pmatrix} g \\ h \end{pmatrix}$ of the previuous system is such that $P\begin{pmatrix} g \\ h \end{pmatrix}$ fulfills the **initial-final conditions** in the **old coordinates**.
- By the **strict convexity** of the functionals J^T and J^s, we deduce that $P\begin{pmatrix} g \\ h \end{pmatrix}$ **coincides** with $\begin{pmatrix} \delta x^T \\ \delta p^T \end{pmatrix}$.
- The property of **contraction-expantion** shown in the new coordinates enables us to prove the **"Local Turnpike" property**.

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Global Turnpike Property

We state the Theorem about Global Turnpike Property, when only the **initial condition** on the state is imposed.

Theorem

Under our assumptions, ***for any initial data*** $x_0 \in \mathbb{R}^N$*, there exists* $(C, \mu) \in (0, +\infty)^2$ *such that the optimal triple* (x^T, p^T, u^T)*,* $\forall t \in [0, T]$*, fulfills the Turnpike estimate:*

$$\|x^T(t) - \overline{x}\| + \|p^T(t) - \overline{p}\| + \|u^T(t) - \overline{u}\| \leq C\left[e^{-\mu t} + e^{-\mu(T-t)}\right].$$

Idea of the proof

First of all, we prove the Convergence of Averages, namely for every $(a, b) \in [0, 1]^2$ such that $a \neq b$:

$$\frac{1}{(b-a)\,T}\int_{aT}^{bT} x^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{x}$$

$$\frac{1}{(b-a)\,T}\int_{aT}^{bT} u^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{u}$$

$$\frac{1}{(b-a)\,T}\int_{aT}^{bT} p^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{p}.$$

Idea of the proof

Moreover, we obtain the existence of a constant C **independent of the time horizon** $T \in (0, +\infty)$ such that:

$$\int_0^T \left[\|x^T(t) - \overline{x}\|^2 + \|u^T(t) - \overline{u}\|^2 \right] dt \leq C \qquad \forall T \in (0, +\infty).$$

Idea of the proof

At this stage, we **put together** the Convergence of Averages and the "Local Turnpike" property.

- Thanks to the **uniform estimate** proven together with the Convergence of Averages and the **Mean Value Theorem for Integrals**, for any $\varepsilon > 0$, there exists T_ε big enough, such that for any time horizon $T \geq T_\varepsilon$:

$$\|x^T(t_1^T) - \overline{x}\|^2 = \frac{1}{T_\varepsilon}\int_0^{T_\varepsilon} \|x^T(s) - \overline{x}\|^2 ds < \frac{\varepsilon^2}{4}$$

  e

$$\|x^T(t_2^T) - \overline{x}\|^2 = \frac{1}{T_\varepsilon}\int_{T-T_\varepsilon}^{T} \|x^T(s) - \overline{x}\|^2 ds < \frac{\varepsilon^2}{4}.$$

- By this result, taking ε small enough, we apply the Theorem about the **"Local Tunrpike" property** in the interval $[t_1^T, t_2^T]$ and we conclude the proof.

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Summary

- In this third part, we study the **Infinite Dimensional Linear Quadratic Case**.
- We build up a functional framework suitable for **Distributed Control of Parabolic Equations**.
- We study the Turnpike Property by a **dynamical approach** as in the 1^{st} part, but **without** employing Infinite Dimensional **Riccati's Theory**.

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Example: Localised Control and Observation

For any control $u \in L^2((0,T); L^2(\omega;\mathbb{R}))$, the corresponding state is the solution of the Cauchy-Dirichelet Problem:

$$\begin{cases} \frac{d}{dt}x - div(A\nabla x) + cx + (b, \nabla x)_{\mathbb{R}^N} = u\chi_{\omega} & \text{in } (0,T)\times\Omega \\ x = 0 \quad \text{in } (0,T)\times\partial\Omega \\ x(0) = x_0 \quad \text{in } \Omega \end{cases}$$

Definition of the NonStationary Problem

$(OCP)^T$ is related to th **minimization** of the functional:

$$J^T : L^2((0,T); L^2(\omega;\mathbb{R})) \longmapsto \mathbb{R}$$
$$u \longrightarrow \frac{1}{2}\int_0^T \left[\int_{\omega} \|u(t,w)\|^2 dw + \int_{\omega_0} \|x(t,w) - z(w)\|^2 dw\right] dt.$$

Definition of the Stationary Problem

We define the closed vector subspace of $H_0^1(\Omega;\mathbb{R}) \times L^2(\omega;\mathbb{R})$:

$$M = \left\{ -div(A\nabla x) + cx + (b, \nabla x)_{\mathbb{R}^N} = u\chi_\omega \right\}.$$

$(OCP)^s$ is the **minimization** of the functional

$$\begin{aligned} J^s : M &\longmapsto \mathbb{R} \\ (x,u) &\longrightarrow \frac{1}{2}\int_{\omega} \|u(w)\|^2 dw + \frac{1}{2}\int_{\omega_0} \|x(w) - z(w)\|^2 dw. \end{aligned}$$

Turnpike Property

Theorem (Porretta-Zuazua)

We assume A uniformly coercive and bounded, b bounded, c bounded and positive. Then, there exists a pair of positive constants (C, μ)*, such that for any time horizon T e* $\forall t \in [0, T]$*:*

$$\|x^T(t) - \overline{x}\|_{L^2(\Omega)} + \|u^T(t) - \overline{u}\|_{L^2(\omega)} \leq C\left[e^{-\mu t} + e^{-\mu(T-t)}\right].$$

Open Problems

- The next challenge could be the proof of a **Global Turnpike Property** for Control Systems defined by **NonLinear Dynamics**, both in the finite dimensional case and the infinite dimensional case.
- To this extent, it is essential to prove a **Global Controllability Property** for the NonLinear Systems considered.

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