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Exact penalization of terminal constraints for optimal control problems

Martin Gugat[1,*,†] and Enrique Zuazua[2]

[1] *Friedrich-Alexander-Universität Erlangen-Nürnberg, Lehrstuhl 2 für Angewandte Mathematik, 91058 Erlangen, Germany*
[2] *Departamento de Matemáticas, Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain*

SUMMARY

We study optimal control problems for linear systems with prescribed initial and terminal states. We analyze the exact penalization of the terminal constraints. We show that for systems that are exactly controllable, the norm-minimal exact control can be computed as the solution of an optimization problem without terminal constraint but with a nonsmooth penalization of the end conditions in the objective function, if the penalty parameter is sufficiently large. We describe the application of the method for hyperbolic and parabolic systems of partial differential equations, considering the wave and heat equations as particular examples.

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1. INTRODUCTION

The notion of *exact controllability* plays an essential role for the understanding of control systems. A system is said to be exactly controllable if, in a given control time, starting from an initial state with a certain regularity and given a final target state, the system can be steered exactly from one to another with controls of a given regularity.

Similarly, in constrained optimization, the notion of *exact penalization* plays a fundamental role. Given a constrained optimization problem, a penalty function is said to be exact, if for a sufficiently large penalty parameter the minimizers of the sum of the original objective function and the penalty term also solve the original constrained optimization problem. In this case, the solution of a constrained optimization problem is equivalent to the solution of an unconstrained optimization problem. In general, this only works with a non-smooth penalty term.

Typically, the lower bound for the successful penalty parameters is given by the norm of the multipliers corresponding to the penalized constraint. Thus, the existence of a finite bound for the penalty parameter is connected with the regularity of the multipliers. Due to this fact, in state-constrained optimal control problems for partial differential equation (PDE) with pointwise space-time state constraints, often, exact penalization is impossible on account of the lack of regularity of the multipliers.

In this paper, we show that the situation is different for terminal constraints that are given by end conditions of the type that appear in exact controllability problems. In these terminal constraints, the state at the given terminal time T (that is the terminal state) is prescribed exactly. We consider systems that are exactly controllable and replace the end conditions that we regard as state constraints

*Correspondence to: Martin Gugat, Friedrich-Alexander-Universität Erlangen-Nürnberg, Lehrstuhl 2 für Angewandte Mathematik, Cauerstr. 11, 91058 Erlangen, Germany.
†E-mail: gugat@am.uni-erlangen.de

in the optimal control problems by a non-smooth penalty term. We show that in this situation, exact penalization is possible. In other words, we can characterize optimal exact controls as solutions of problems without end conditions and with non-smooth objective functions.

Hence, exact controllability implies exact penalizability. In the context of exact null-controllability (when the final target is the trivial null state), the lower bound for the successful penalty parameters is given by the product of the norm of initial state and the square of the exact controllability constant from the definition of the exact controllability. In the Hilbert-space case, this constant makes sure that the norm-minimal exact control is given by a bounded linear operator.

For conservative systems, the penalization in the standard energy-norm suffices to guarantee the efficiency of the method. The particular case of the 1D wave equation was treated in [1]. If the initial states are more regular, then the penalization with weaker norms suffices. This is related with the fact that, for time-reversible systems, the optimal control inherits the stronger regularity of the data to be controlled [2].

To illustrate our approach, we continue this Introduction with a motivating example in the finite-dimensional context. But our results apply for infinite-dimensional systems too.

Note that the exact penalization leads to optimal control problems with objective functions that contain non-smooth norm-terms depending on the terminal state. Recently, there have been several studies of optimization problems with non-differentiable objective functions where the non-differentiable term depends on the control (for example, [3], [4]). Also, in [5] and [6], an objective function of this type is considered. In this case, the L^1-norm of the control is a part of the objective function that leads to sparsity of the optimal controls.

1.1. A motivating example

To illustrate our approach, we present an example with an ordinary differential equation. Consider the optimal control problem

$$\begin{cases} \min_{u\in L^2(0,1)} & \frac{1}{2}\,\|u\|^2_{L^2(0,1)} \\ \quad\text{subject to} & \\ y'(t)-y(t) = & \exp(t)\,u(t) \\ y(0) = & -1 \\ y(1) = & 0. \end{cases} \tag{1}$$

Then, the unique optimal control is the constant function $u_\kappa(t) = 1$.

In order to avoid the terminal constraint $y(1) = 0$ in the optimal control problem, we consider the penalized problems with a penalty parameter $\gamma \geqslant 0$

$$\begin{cases} \min_{u\in L^2(0,1)} & \frac{1}{2}\,\|u\|^2_{L^2(0,1)} + \gamma\,|y(1)| \\ \quad\text{subject to} & \\ y'(t)-y(t) = & \exp(t)\,u(t) \\ y(0) = & -1. \end{cases} \tag{2}$$

If $\gamma \in [0, \exp(-1))$, the solution is $u_\gamma(t) = \mathrm{e}\gamma$, and if $\gamma \geqslant \exp(-1)$, the solution is $u_\gamma(t) = 1$. This means that for $\gamma \geqslant \exp(-1)$, the solution is independent of γ and equal to the solution of (1). The corresponding optimal value as a function of the penalty parameter γ is

$$v(\gamma) = \frac{1}{2}\,\|u_\gamma\|^2_{L^2(0,1)} + \gamma\,\left|y_\gamma(1)\right|,$$

where y_γ denotes the state generated by u_γ. It is given by

$$v(\gamma) = \begin{cases} \mathrm{e}\,\gamma - \frac{1}{2}\mathrm{e}^2\,\gamma^2 & \text{if } \gamma \in [0, \exp(-1)), \\ \frac{1}{2} & \text{if } \quad \gamma > \exp(-1). \end{cases}$$

In particular, the optimal value is constant for γ sufficiently large. Here, the optimal value is differentiable as a function of γ. The control to state map for our system is given by

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$$y(t) = \exp(t) \left(-1 + \int_0^t u(s)\, ds\right).$$

This means that (2) is equivalent to the unconstrained problem

$$\min_{u \in L^2(0,T)} J_\gamma(u) := \frac{1}{2} \|u\|^2_{L^2(0,1)} + \gamma \mathrm{e} \left| \int_0^1 u(s)\, ds - 1 \right|. \tag{3}$$

For $\gamma \geqslant \exp(-1)$, this is an exact penalization of an optimization problem with the moment equation $\int_0^1 u(s)\, ds = 1$ as equality constraint. For the convenience of the reader, we show that $u_\gamma(t) = 1$ is optimal for $\gamma \geqslant \exp(-1)$. In fact, for all $\delta \in L^2(0,1)$, $\delta \neq 0$ we have

$$\begin{aligned} J_\gamma(u_\gamma + \delta) = \quad & \frac{1}{2} + \int_0^1 \delta(s)\, ds + \frac{1}{2} \|\delta\|^2_{L^2(0,1)} + \gamma \mathrm{e} \left| \int_0^1 \delta(s)\, ds \right| \\ \geqslant \quad & \frac{1}{2} + \frac{1}{2} \|\delta\|^2_{L^2(0,1)} + \gamma \mathrm{e} \left| \int_0^1 \delta(s)\, ds \right| - \left| \int_0^1 \delta(s)\, ds \right| > \frac{1}{2} = J_\gamma(u_\gamma). \end{aligned}$$

In Section 3, we will consider problems of a similar type with a sequence of moment equations as constraints that arise in PDE-constrained optimal control problems.

1.2. *Structure of the paper*

This paper has the following structure: First, we consider problems of L^2-norm minimal control. In Section 2, we consider the penalization by the natural norm in Hilbert state spaces. We apply the result to boundary control problems for wave equations and the heat equation. We also present a result that gives an upper bound for the optimal value of the penalized problems as a function of the penalty parameter. The upper bound is a polynomial of degree two of the penalty parameter and helps to understand how the optimal values increase with the penalty parameter until they saturate to remain constant.

In Section 3.3, we turn to optimal control problems where the end conditions are replaced by a sequence of moment equations. As examples, we consider optimal control problems for the wave equation with Neumann and Dirichlet boundary control. First, we look at L^2-norm objective functions. We give sufficient conditions for exact penalizability of these end conditions by an l^2-norm penalty term. In Section 3.4, we consider the corresponding problem with a penalization by an l^∞-norm penalty term. This result illustrates how the regularity of the problem data influences the norms for which the penalization is exact. In Section 3.5, we present a result about penalization by an l^1-norm penalty term. This penalization is suitable for L^∞-norm optimal control problems.

In the last section, we consider problems of L^1-norm optimal control and give a sufficient condition for a non-smooth penalization to be exact.

2. PENALIZATION BY STATE SPACE NORMS: OPTIMAL CONTROL WITH ABSTRACT CAUCHY PROBLEMS

In this section, we study our problem in a Hilbert space setting. Let X and U be Hilbert spaces with the inner products $\langle \cdot, \cdot \rangle_X$, $\langle \cdot, \cdot \rangle_U$, respectively, and the corresponding norms $\| \cdot \|_X$, $\| \cdot \|_U$, respectively. Let $T > 0$ be given. The space X contains the current state and the space U is used as a framework for the control functions in $L^2(0, T; U)$.

2.1. *Exact controllability*

As in [7], let $A\colon \mathcal{D}(A) \subset X \to X$ be the generator of a strongly continuous semigroup, and let B denote an admissible control operator. As in [7], Proposition 4.2.5., we consider a control system of the form

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$$\begin{cases} x'(t) + Ax = Bu, \\ x(0) = x_0. \end{cases} \tag{4}$$

For all $u \in L^2(0,T;U)$, the Cauchy problem (4) has a unique solution $x \in C([0,T];X)$ [8]. Moreover, the solution varies continuously with the data in the sense that

$$\|x\|_{L^\infty(0,T;X)} \leqslant C_T \left(\|x_0\|_X + \|u\|_{L^2(0,T;U)}\right).$$

We use the following stability property: For controls u_n and the corresponding states x_n, if $(u_n)_n$ converges weakly in $L^2(0,T;U)$ to u^*, then $(x_n)_n$ weakly* converges in $L^\infty(0,T;X)$, and in addition, $x_n(T)$ weakly converges in X to the state at the time T that is generated by u^*.

Important examples that satisfy this assumption of well-posedness of the Cauchy problem are presented in [7].

Assume that (4) is exactly controllable using L^2-controls in time T, that is, there exists a constant $C_1 > 0$ such that for all initial states $x_0 \in X$ and all final states $x_1 \in X$, there is a control $u \in L^2(0,T;U)$ such that the solution $x \in C([0,T];X)$ of (4) satisfies

$$\begin{cases} x(T) = x_1, \\ \|u\|_{L^2(0,T;U)} \leqslant C_1(\|x_0\|_X + \|x_1\|_X). \end{cases} \tag{5}$$

2.2. *Penalization by state space norms in Hilbert space*

We consider the following optimization problem:

$$\textbf{EP:} \begin{cases} \min_{u\in L^2(0,T;U)} \frac{1}{2}\, \|u\|^2_{L^2(0,T;U)} + \gamma\|x(T)\|_X \\ \text{subject to} \\ x'(t) + Ax = Bu,\ x(0) = x_0. \end{cases}$$

In problem **EP**, the end condition $x(T) = 0$ does not appear. And, for each $\gamma > 0$, Problem **EP** has a unique solution.

This can be seen as follows: by means of a straightforward application of the Direct Method of the Calculus of Variations. Let $\hat{x}(t)$ denote the solution with null control $u \equiv 0$. Then, for the optimal value $v(\gamma)$ of **EP**, we have the upper bound

$$v(\gamma) \leqslant \gamma\|\hat{x}(T)\|_X,$$

which corresponds to the value of the functional with $u \equiv 0$. Let $(u_n)_n$ denote a minimizing sequence for **EP** and $x_n(t)$ the corresponding solutions of (4). Then we have

$$v(\gamma) = \liminf_{n\to\infty} \frac{1}{2}\, \|u_n\|^2_{L^2(0,T;U)} + \gamma\|x_n(T)\|_X \leqslant \gamma\|\hat{x}(T)\|_X.$$

In particular, the sequence $(u_n)_n$ is bounded in $L^2(0,T;U)$, and the sequence $(x_n(T))_n$ is bounded in X. Hence, there exists a weakly convergent subsequence that, with a slight abuse of notation, we denote again by $((u_n,\ x_n(T)))_n$, the limit being (u^*, x). The well-posedness of the Cauchy problem implies that $\mathrm{x} = x^*(T)$, where x^* is the solution of (4) corresponding to the limit control u^*. Then the sequential weak lower semicontinuity of the objective function implies

$$v(\gamma) = \liminf_{n\to\infty} \frac{1}{2}\, \|u_n\|^2_{L^2(0,T;U)} + \gamma\|x_n(T)\|_X \geqslant \frac{1}{2}\, \|u^*\|^2_{L^2(0,T;U)} + \gamma\|x^*(T)\|_X.$$

On the other hand, we have $\frac{1}{2}\, \|u^*\|^2_{L^2(0,T;U)} + \gamma\|x^*(T)\|_X \geqslant v(\gamma)$, thus

$$v(\gamma) = \frac{1}{2}\, \|u^*\|^2_{L^2(0,T;U)} + \gamma\|x^*(T)\|_X.$$

Hence u^* is a solution of **EP**. The uniqueness follows from the strict convexity of the objective function.

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In the sequel, we use the notation u_γ for the solution of **EP** for $\gamma > 0$. The corresponding state is x_γ. Our goal is to show that, because of the property of exact controllability using L^2–controls of the system, for γ sufficiently large at time T, the state x_γ satisfies the desired end condition

$$x_\gamma(T) = 0.$$

Moreover, the method of penalization with γ large leads to the control of minimal norm that steers the system exactly to the desired target state.

Theorem 1
Assume that the system is exactly controllable using L^2-controls in time T. If

$$\gamma > C_1^2 \, \|x_0\|_X, \tag{6}$$

where C_1 is as in (5), the solution of problem **EP** is independent of γ and solves the optimal control problem **EC** with the terminal constraint $x(T) = 0$, defined as

$$\textbf{EC:} \begin{cases} \min_{u \in L^2(0,T;U)} \frac{1}{2} \, \|u\|^2_{L^2(0,T;U)} \\ \text{subject to} \\ x'(t) + Ax = Bu, \; x(0) = x_0, \\ x(T) = 0. \end{cases}$$

Problem **EC** has a unique solution.

Proof
Similar as for **EP**, an application of the Direct Method of the Calculus of Variations shows that a solution of **EC** exists. The strict convexity of the objective function $\frac{1}{2} \| \cdot \|^2_{L^2(0,T;U)}$ implies that the solution of **EC** is uniquely determined, because for all solutions u_0, u_1 of **EC** also $\frac{u_0+u_1}{2}$ is feasible and must have the same objective value, which is only possible if $u_0 = u_1$.

Suppose that $x_\gamma(T) \neq 0$. Then the objective functional of **EP** is differentiable at (u_γ, x_γ) and satisfies the necessary optimality conditions

$$\int_0^T \langle u_\gamma, v \rangle_U \, dt + \gamma \frac{\langle x_\gamma(T), y(T) \rangle_X}{\|x_\gamma(T)\|_X} = 0 \tag{7}$$

for all $v \in L^2(0, T; U)$, where $y(t)$ solves

$$y' + Ay = Bv, \; y(0) = 0.$$

This implies

$$\int_0^T \langle u_\gamma, v \rangle_U \, dt = -\gamma \frac{\langle x_\gamma(T), y(T) \rangle_X}{\|x_\gamma(T)\|_X}.$$

Let u_κ denote the solution of **EC**. Because u_κ is a feasible control for **EP**, evaluating the objective function of **EP** at u_κ yields the inequality

$$\|u_\gamma\|^2_{L^2(0,T;U)} \leqslant \|u_\kappa\|^2_{L^2(0,T;U)}. \tag{8}$$

Because of the exact controllability of the system, in view of (5), this implies the upper bound

$$\|u_\gamma\|_{L^2(0,T;U)} \leqslant C_1 \, \|x_0\|_X. \tag{9}$$

Hence, the linear functionals

$$\varphi_\gamma(v) = \int_0^T \langle u_\gamma, v \rangle_U \, dt,$$

are uniformly bounded with respect to γ

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$$|\varphi_\gamma(v)| \leqslant C_1 \, \|x_0\|_X \, \|v\|_{L^2(0,T;U)}.$$

On the other hand, φ_γ admits the representation

$$\varphi_\gamma(v) = -\gamma \frac{\langle x_\gamma(T), y(T)\rangle_X}{\|x_\gamma(T)\|_X}.$$

Because of the exact controllability of the system, we can choose v in such a way that

$$y(T) = \frac{1}{\|x_\gamma(T)\|_X} \, x_\gamma(T).$$

and

$$\|v\|_{L^2(0,T;U)} \leqslant C_1. \tag{10}$$

This implies

$$\varphi_\gamma(v) = -\gamma,$$

thus

$$|\varphi_\gamma(v)| = \gamma \leqslant C_1^2 \, \|x_0\|_X,$$

which is a contradiction to (6). Hence, we have $x_\gamma(T) = 0$. Therefore x_γ is feasible for **EC**, so

$$\|u_\kappa\|^2_{L^2(0,T;U)} \leqslant \|u_\gamma\|^2_{L^2(0,T;U)}.$$

Because of (8), this implies $\|u_\kappa\|^2_{L^2(0,T;U)} = \|u_\gamma\|^2_{L^2(0,T;U)}$. Hence u_γ is a solution of problem **EC**. Because the solution of **EC** is uniquely determined, this implies $u_\kappa = u_\gamma$. □

2.3. *Why non-smooth penalization?*

In this section, we explain why the exact penalization term $||x(T)||_X$ needs to be non-smooth. In optimization, this fact is well-known. It is illustrated by the following lemma.

Lemma 1
Let a differentiable penalty function $p : X \to [0, \infty)$ be given with $p(0) = 0$ and $p(x) > 0$ for all $x \neq 0$. We consider the following optimization problem:

$$\textbf{DP:} \begin{cases} \min_{u \in L^2(0,T;U)} \frac{1}{2} \, \|u\|^2_{L^2(0,T;U)} + \gamma \; p(x(T)) \\ \text{subject to} \\ x'(t) + Ax = Bu, \; x(0) = x_0. \end{cases}$$

If the solution u_κ of **EC** is not zero, for all $\gamma > 0$, we have $x_\gamma(T) \neq 0$, where x_γ denotes the optimal state for **DP**. This means that the differentiable penalization by p is *not* exact, and this whatever the value of the penalization parameter γ is.

Proof
Let $\gamma > 0$ be given. Suppose that $x_\gamma(T) = 0$. Then $p(x_\gamma(T)) = 0$, thus the optimal value of **DP** is given by $\frac{1}{2} \, \|u_\gamma\|^2_{L^2(0,T;U)} > 0$. Because p is differentiable at zero, and this is the global minimum, we have $p'(0) = 0$ and $p(0) = 0$, hence

$$\lim_{x \to 0} \frac{p(x)}{\|x\|} = 0. \tag{11}$$

Choose $\delta \in L^2(0, T; U)$. Because u_γ is the optimal control, we have

$$\frac{1}{2} \, \|u_\gamma + \delta\|^2_{L^2(0,T;U)} + \gamma \; p(x_\delta(T)) \geqslant \frac{1}{2} \, \|u_\gamma\|^2_{L^2(0,T;U)},$$

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where x_δ is the state generated by δ starting with the zero initial state. This implies the inequality

$$\langle u_\gamma,\ \delta\rangle_{L^2(0,T;U)} + \frac{1}{2}\,\|\delta\|^2_{L^2(0,T;U)} + \gamma p(x_\delta(T)) \geqslant 0.$$

Hence, we have

$$p(x_\delta(T)) \geqslant -\frac{1}{\gamma}\langle u_\gamma,\ \delta\rangle_{L^2(0,T;U)} - \frac{1}{2\gamma}\|\delta\|^2_{L^2(0,T;U)}.$$

For $\lambda \in (0, 1]$, choose $\delta(\lambda) = -\lambda u_\gamma \neq 0$. Then we have

$$p(x_{\delta(\lambda)}(T)) \geqslant \frac{1}{\gamma}\lambda\|u_\gamma\|^2_{L^2(0,T;U)} - \frac{1}{2\gamma}\lambda^2\|u_\gamma\|^2_{L^2(0,T;U)}.$$

Because of the linearity of our system, we have $x_{\delta(\lambda)} = \lambda\ x_{\delta(1)}$. Suppose that $x_{\delta(1)}(T) = 0$, then the aforementioned inequality implies $0 \geqslant \|u_\gamma\|^2_{L^2(0,T;U)} - \frac{1}{2}\|u_\gamma\|^2_{L^2(0,T;U)}$, which is a contradiction.

If $x_{\delta(1)}(T) \neq 0$, we have

$$\liminf_{\lambda\to 0} \frac{p(x_{\delta(\lambda)}(T))}{\|x_{\delta(\lambda)}(T)\|_X} \geqslant \frac{\|u_\gamma\|^2_{L^2(0,T;U)}}{\gamma\,\|x_{\delta(1)}(T)\|_X} > 0,$$

which is a contradiction to (11). Hence, we have proved that $x_\gamma(T) \neq 0$. In other words, the differentiable penalization is not exact. □

2.4. *Example 1. The wave equation*

In this section, we consider a system that is governed by the wave equation on a two-dimensional domain. Let $\Omega \subset \mathbb{R}^2$ be a domain with C^2-boundary Γ. Assume that $y_0 \in L^2(\Omega)$, $y_1 \in H^{-1}(\Omega)$. Consider the Dirichlet boundary control problem

$$\textbf{ECD2}\begin{cases} \min_{u\in L^2(\Gamma\times(0,T))} \quad \frac{1}{2}\int_0^T\int_\Gamma |u(t,x)|^2\,dx\,dt \\ \qquad\qquad\text{subject to} \\ y_{tt} - \Delta y = 0 \ \text{ in } \Omega\times(0,T) \\ y(0,\cdot) = y_0,\ y_t(0,\cdot) = y_1 \ \text{ in } \Omega \\ y(t,\cdot) = u(t,\cdot) \ \text{ on } \Gamma\times(0,T) \\ y(T,\cdot) = y_t(T,\cdot) = \ \text{ in } \Omega. \end{cases} \tag{12}$$

For T sufficiently large, the system is exactly controllable using L^2-controls [9]. Such result is well-known, in fact, in a more general setting under the so-called geometric control condition.

Hence, there exists a positive constant $C_1 > 0$ such that the exact control of minimal L^2-norm satisfies

$$\int_0^T\int_\Gamma |u(t,x)|^2\,dx\,dt \leqslant C_1\ \left(\|y_0\|^2_{L^2(\Omega)} + \|y_1\|^2_{H^{-1}(\Omega)}\right). \tag{13}$$

This specific control problem for the wave equation can be put in the form **EC** in a suitable functional setting, see [7] and Theorem 1 is applicable. Thus, if γ is sufficiently large, the following problem involving non-smooth penalization is equivalent to **ECD2**:

$$\textbf{EPD2}\begin{cases} \min_{u\in L^2(\gamma\times(0,T))} \quad \frac{1}{2}\int_0^T\int_\Gamma |u(t,x)|^2\,dx\,dt \ + \ \gamma\sqrt{\|y(\cdot,T)\|^2_{L^2(\Omega)} + \|y_t(\cdot,T)\|^2_{H^{-1}(\Omega)}} \\ \qquad\qquad\text{subject to} \\ y_{tt} - \Delta y \ = \ 0 \ \text{ in } \Omega\times(0,T) \\ y(0,\cdot) = y_0, \quad y_t(0,\cdot) = y_1 \ \text{ in } \Omega \\ y(t,\cdot) \ = \ u(t,\cdot) \ \text{ in } \Gamma\times(0,T). \end{cases} \tag{14}$$

2.5. Example 2: The Euler–Bernoulli beam

Let us consider the boundary control problem for a vibrating Euler–Bernoulli beam.

Let $\Omega = (0,1)$ and assume that $y_0 \in H_0^2(\Omega)$, $y_1 \in H_0^1(\Omega)$. Consider the Dirichlet boundary control problem

$$\textbf{ECbeam}\begin{cases} \min_{u\in L^2(0,T)} \ \frac{1}{2}\int_0^T |u(t)|^2\, dt \\ \text{subject to} \\ y_{tt} + y_{xxxx} = 0 \ \text{ in } \Omega \times (0,T) \\ y(0,\cdot) = y_0,\ y_t(0,\cdot) = y_1 \ \text{ in } \Omega \\ y(t,0) = 0,\ y_{xx}(t,0) = u(t) \ \text{ in } (0,T) \\ y(t,1) = 0,\ y_{xx}(t,1) = 0 \ \text{ in } (0,T) \\ y(T,\cdot) = y_t(T,\cdot) = 0 \ \text{ in } \Omega. \end{cases} \tag{15}$$

For all $T > 0$, the system is exactly controllable using L^2-controls ([7], 10.4). Hence, there exists a constant $C_1 > 0$ such that

$$\int_0^T |u(t)|^2\, dt \leqslant C_1 \left(\|y_0\|^2_{H^2(\Omega)} + \|y_1\|^2_{H^1(\Omega)} \right), \tag{16}$$

for the control of minimal norm $u \in L^2(0,T)$.

Theorem 1 is applicable. Thus, if γ is sufficiently large, the following problem with non-smooth penalization is equivalent to **ECbeam**:

$$\textbf{EPbeam}\begin{cases} \min \ \frac{1}{2}\int_0^T |u(t)|^2\, dt + \gamma\sqrt{\|y(\cdot,T)\|^2_{H^2(\Omega)} + \|y_t(\cdot,T)\|^2_{H^1(\Omega)}} \\ \text{subject to} \\ y_{tt} + y_{xxxx} = 0 \ \text{ in } \Omega \times (0,T) \\ y(0,\cdot) = y_0, y_t(0,\cdot) = y_1 \ \text{ in } \Omega \\ y(t,0) = 0,\ y_{xx}(t,0) = u(t) \ \text{ in } (0,T) \\ y(t,1) = y_{xx}(t,1) = 0 \ \text{ in } (0,T). \end{cases} \tag{17}$$

Note that the controllability problem for this 1D beam equation can also be dealt with by the method of moments [10]. This will be discussed in Section 3.

2.6. Example 3: an example with the heat equation

Here, we consider an example with distributed control of the heat equation.

Let a smooth domain Ω in $\mathbb{R}^n$ ($n \in \{1,2,3,...\}$) with boundary Γ and an initial state $y_0 \in L^2(\Omega) = X$ be given. Let $\omega \subset \Omega$ be a given nonempty subdomain of Ω. Consider the distributed optimal control problem

$$\textbf{HEAT2}\begin{cases} \min \frac{1}{2}\|u\|^2_{L^2((0,T)\times\omega)} \quad \text{subject to} \\ y' = \Delta y + 1_\omega u \\ y(t,x) = 0 \ \text{ for all } x \in \Gamma \\ y(0,x) = y_0(x) \\ y(T,x) = 0. \end{cases} \tag{18}$$

The system is exactly controllable using L^2–controls to $x_1 = 0$ for arbitrarily short times $T > 0$, which implies that **HEAT2** has a solution.

Now, we consider the application of Theorem 1 to the problem **HEAT2**. For the penalization, we consider a norm with exponential weights, namely

$$\|f\|_X = \sqrt{\sum_{j=0}^{\infty} \left|\hat{f}(j)\right|^2 \exp(c_0\sqrt{\lambda_j})}, \tag{19}$$

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where as usual $\hat{f}(j)$ denotes the coefficient in the expansion of f in the eigenfunctions of the operator $-\Delta$ with homogeneous Dirichlet boundary conditions on Ω with the corresponding increasing sequence of positive eigenvalues $(\lambda_j)_j$. Then the system governed by the heat equation is exactly controllable to $x_1 = 0$: There exists a constant $C_* > 0$ such that for each initial state $y_0 \in L^2(\Omega)$, there exists a control $u \in L^2((0,T) \times \Omega)$ such that for the generated state, we have $y(T) = 0$ and $\|u\|^2_{L^2((0,T)\times\Omega)} \leqslant C_1 \, \|y_0\|^2_{L^2(\Omega)}$.

Moreover, there exist constants $c_0 > 0$, $C_2 > 0$ such that for each final state $y_T \in X$, there exists a control $v \in L^2((0,T) \times \Omega)$ that steers the system starting from the zero initial state to $y(T) = y_T$ and [11]

$$\|v\|^2_{L^2((0,T)\times\Omega)} \leqslant C_2^2 \sum_{j=0}^{\infty} \left| \hat{y}_T(j) \, \exp(c_0 \sqrt{\lambda_j}) \right|^2 = C_2^2 \, \|y_T\|^2_X. \tag{20}$$

In addition, starting from the zero initial state, controls $u \in L^2((0,T) \times \Omega)$ generate system states $y(T) \in X$. Thus the system satisfies the controllability condition (5), and we can apply Theorem 1, which implies that for γ sufficiently large, the penalization is exact, that is, the problem

$$\textbf{EPHEAT2} \begin{cases} \min \frac{1}{2} \, \|u\|^2_{L^2((0,T)\times\omega)} + \gamma \|y(T,\cdot)\|_X & \text{subject to} \\ y' = & \Delta y + 1_\omega u \\ y(t,x) = 0 & \text{for all } x \in \partial\Omega \\ y(0,x) = & y_0(x) \end{cases} \tag{21}$$

is equivalent to **HEAT2**.

2.7. *An upper bound for the optimal value function*

Theorem 1 implies that for γ sufficiently large, the optimal value of problem **EP** as a function of γ remains constant. In the following Lemma, we give an upper bound for the optimal value that holds also for small values of $\gamma \geqslant 0$. In Section 1.1, we have seen an example where the optimal value function is given as a polynomial in γ of degree two for γ less than the critical value. The following lemma generalizes this result.

Lemma 2

Let the assumptions of Theorem 1 hold. Define

$$\gamma_0 = \inf\{\overline{\gamma} \geqslant 0 : x_\gamma(T) = 0 \ \text{ for all } \ \gamma \geqslant \overline{\gamma}\}.$$

Let

$$J_\gamma(u) = \frac{1}{2} \, \|u\|^2_{L^2(0,T;U)} + \gamma \|x(T)\|_X,$$

denote the objective function of **EP**, u_γ denote the solution of **EP**, u_κ denote the solution of **EC**, and x_2 be the solution of

$$x_2' + Ax_2 = 0, \; x_2(0) = x_0.$$

If $\gamma_0 > 0$, for all $\gamma \in [0, \gamma_0]$ for the optimal value of **EP** as a function of γ, we have the upper bound

$$J_\gamma(u_\gamma) \leqslant \frac{1}{2} \frac{\gamma^2}{\gamma_0^2} \, \|u_\kappa\|^2_{L^2(0,T;U)} + \gamma \left(1 - \frac{\gamma}{\gamma_0}\right) \|x_2(T)\|_X \tag{22}$$

$$\leqslant \gamma_0 \, \|x_2(T)\|_X \,. \tag{23}$$

Note that for $\gamma \geqslant 0$, we also have

$$J_\gamma(u_\gamma) \leqslant \frac{1}{2} \, \|u_\kappa\|^2_{L^2(0,T;U)}.$$

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If $\gamma_0 = 0$, for all $\gamma \geqslant 0$, we have

$$J_\gamma(u_\gamma) = \frac{1}{2}\,\|u_\kappa\|^2_{L^2(0,T;U)} = 0$$

and x_2 is the optimal state.

Proof

Let u_κ denote the solution of **EC**. For $\gamma \in [0,\gamma_0]$, consider the control function $u_0 = \frac{\gamma}{\gamma_0}u_\kappa$. We decompose the initial state in the form $x_0 = \frac{\gamma}{\gamma_0}x_0 + (1-\frac{\gamma}{\gamma_0})x_0$. Because of the linearity of the system, the control u_0 steers the initial state $\frac{\gamma}{\gamma_0}x_0$ to zero at time T. By superposition, we obtain for the state x generated by u_0 the final state $x(T) = (1-\frac{\gamma}{\gamma_0})x_2(T)$. This yields

$$J_\gamma(u_\gamma) \leqslant J_\gamma(\frac{\gamma}{\gamma_0}u_\kappa) = \frac{1}{2}\frac{\gamma^2}{\gamma_0^2}\,\|u_\kappa\|^2_{L^2(0,T;U)} + \gamma\left|1-\frac{\gamma}{\gamma_0}\right|\;\|x_2(T)\|_X$$

and (22) follows.

The definition of γ_0 implies that for all $\gamma > \gamma_0$, the optimal state satisfies the terminal constraint $x_\gamma(T) = 0$. Thus for all $\gamma > \gamma_0$, u_γ is also feasible for **EC**, which yields the inequality $\frac{1}{2}\|u_\kappa\|^2_{L^2(0,T;U)} \leqslant \frac{1}{2}\|u_\gamma\|^2_{L^2(0,T;U)}$. Hence, inserting the control $u = 0$ in the objective function of **EP** with $\gamma > \gamma_0$ yields the inequality

$$\frac{1}{2}\|u_\kappa\|^2 + 0 \leqslant \frac{1}{2}\|u_\gamma\|^2_{L^2(0,T;U)} + 0 \leqslant 0 + \gamma\|x_2(T)\|_X.$$

Because we can choose $\gamma > \gamma_0$ arbitrarily, this implies $\frac{1}{2}\|u_\kappa\|^2 \leqslant \gamma_0\|x_2(T)\|_X$. Together with (22), this yields (23).

If $\gamma_0 = 0$, for all $\gamma > 0$, we have $x_\gamma(T) = 0$. Because u_γ solves **EP**, this implies

$$\frac{1}{2}\,\|u_\gamma\|^2_{L^2(0,T;U)} \leqslant \frac{1}{2}\,\|u_\kappa\|^2_{L^2(0,T;U)}.$$

Hence u_γ is a solution of **EC**. Because the solution of **EC** is unique, this yields $u_\gamma = u_\kappa$ for all $\gamma > 0$. On the other hand, as mentioned earlier, we obtain $\frac{1}{2}\|u_\kappa\|^2 \leqslant \gamma_0\|x_2(T)\|_X = 0$, which implies the assertion for $\gamma_0 = 0$. □

Remark 1

In this section, we have worked in a Hilbert space setting for the state space. The proof of Theorem 1 is based upon the fact that except at zero, Hilbert space norms are differentiable. However, also in a Banach space setting, the corresponding result holds. Note, however, that the application of the Direct Method of the Calculus of Variations for proving the existence of optimal controls of the penalized problems requires the Banach space to be reflexive. With this assumption, it is possible to show a Banach space version of Theorem 1.

3. PENALIZATION OF OPTIMAL CONTROL PROBLEMS WITH MOMENT EQUATIONS

In the applications, the end conditions for control systems are often equivalent to a sequence of moment equations for the control u. For problems of optimal damping of vibrations, in particular the vibrating string and the Euler-Bernoulli beam, this relation is discussed in detail in [10], both for the cases of distributed control and boundary control. Also, problems of optimal control of a Timoshenko beam lead to problems of this structure [12, 13]. The difference between the two models is that for the Timoshenko beam, the time T has to be sufficiently large to allow the exact controllability of the system, whereas for the Euler–Bernoulli beam, exact controllability for arbitrarily short time intervals holds. These problems lead to trigonometric moment problems. In contrast to this,

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problems of optimal control of heating processes lead to exponential moment problems that are also presented in [10].

Now, we present 1D examples with the wave equation to illustrate that the end conditions in the optimal control problems are equivalent to sequences of moment equations. So the end conditions that appear as constraints in the optimal control problems can be replaced by the corresponding sequence of moment equations.

3.1. An example with Dirichlet control

Example 1

Consider the following optimal Dirichlet boundary control system with the wave equation in dimension one:

$$\mathbf{ECD2}\begin{cases} \min \quad \frac{1}{2}\, \|u\|^2_{L^2(0,T)} & \text{subject to} \\ y'' = & y_{xx} \\ y(t,0) = & 0 \\ y(t,1) = & u(t) \\ y(0,x) = & y_0(x) \\ y'(0,x) = & y_1(x) \\ y(T,x) = & 0 \\ y'(T,x) = & 0. \end{cases} \tag{24}$$

with $x \in [0, 1]$, $t \geqslant 0$, and $(y_0, y_1) \in L^2(0,1) \times H^{-1}(0,1)$. The control function u is in $L^2(0,T)$. This is a well-known system that has been considered, for example, in [14] and in [1]. It is exactly controllable using L^2-controls in time $T \geqslant 2$ to $x_1 = 0$. The moment problem corresponding to the end conditions

$$y(T,x) = 0, \;\; y'(T,x) = 0,$$

is stated in [15] (see also [16]), where the system is started at the null state and controlled to a nonzero state, but because of the time-reversability for the wave equation, this is an equivalent problem. The corresponding moment problem is

$$\begin{cases} \min \quad \frac{1}{2}\, \|u\|^2_{L^2(0,T)} & & \text{subject to} \\ \int_0^T u(s)\sin(\pi j s)\, ds &=& -(-1)^j \int_0^1 y_0 \; \sin(j\pi x)\, dx, \\ \int_0^T u(s)\cos(\pi j s)\, ds &=& -(-1)^j \frac{1}{\pi j} \int_0^1 y_1 \; \sin(j\pi x)\, dx, \\ & & \text{for all} \quad j \in \{1,2,3,...\}. \end{cases} \tag{25}$$

3.2. An example with Neumann control

Example 2

Consider the following optimal Neumann boundary control problem with the wave equation.

$$\mathbf{ECN2}\begin{cases} \min \quad \frac{1}{2}\, \|u\|^2_{L^2(0,T)} & \text{subject to} \\ y'' = & y_{xx} \\ y_x(t,0) = & -u(t) \\ y_x(t,1) = & u(t) \\ y(0,x) = & y_0(x) \\ y'(0,x) = & y_1(x) \\ y(T,x) = & 0 \\ y'(T,x) = & 0. \end{cases} \tag{26}$$

with $x \in [0, 1]$, $t \geqslant 0$, and constant states (y_0, y_1). The control function u is in $L^2(0,T)$. This is a well-known system that has been considered, for example, in [17]. It is exactly controllable using L^2-controls in time $T \geqslant 2$ to $x_1 = 0$. The end conditions

$$y(T,x) = 0, \;\; y'(T,x) = 0,$$

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correspond to the moment problem stated in [17] ((14)–(17)), where the system is started at the null state and controlled to a nonzero state, but because of the time-reversability for the wave equation, this is an equivalent problem. **ECN2** can be written in the following form with a sequence of moment equations as constraints:

$$\begin{cases} \min \ \frac{1}{2}\ \|u\|^2_{L^2(0,T)} & \text{subject to} \\ \int_0^T u(s)\ (s-\frac{T}{2})\ ds \ = & \frac{y_0}{2}-\frac{T}{2}\frac{y_1}{2}, \\ \int_0^T u(s)\ ds \ = & \frac{y_1}{2}, \\ \int_0^T u(s)\sin(2\pi js)\ ds \ = & 0, \\ \int_0^T u(s)\cos(2\pi js)\ ds \ = \ 0, \ \text{for all} & j\in\{1,2,3,...\}. \end{cases} \tag{27}$$

The two examples earlier motivate us to consider problems of optimal exact controllability that are given in the form

$$\textbf{MOM:} \begin{cases} \min \frac{1}{2}\ \|u\|^2_{L^2(0,T;U)} \\ \text{subject to} \qquad \langle u, s_k\rangle_{L^2(0,T;U)} = y_k, \ \text{for all} \quad k\in\{1,2,3,...\}. \end{cases}$$

where $(s_k)_k$ is a given sequence of functions in $L^2(0,T;U)$ and $(y_k)_k \in l^2$ such that there exists $\hat{u} \in L^2(0,T;U)$ that solves the moment problem

$$\langle \hat{u}, s_k\rangle_{L^2(0,T)} = y_k, \ \text{for all} \quad k\in\{1,2,3,...\}.$$

Then **MOM** has a unique solution that we denote by u_κ.

In the sequel, we consider the penalization of **MOM** with l^2-norms and weighted l^∞–norms, where the weights can be chosen according to the regularity of the problem data. Moreover, we consider the corresponding problem with the L^∞-norm of the control as the objective function.

3.3. Penalization by l^2-norms

In this section, we focus on problems with given data of minimal regularity.

For a penalty parameter $\gamma > 0$, we consider the penalized problem

$$\textbf{EPMOM2:} \min \frac{1}{2}\ \|u\|^2_{L^2(0,T;U)} + \gamma\sqrt{\sum_{k=1}^{\infty}\left|\langle u, s_k\rangle_{L^2(0,T;U)} - y_k\right|^2}$$

In the following Theorem, we assume that the sequence $(s_k)_k$ has a biorthogonal Bessel sequence [18].

Theorem 2
Assume that the sequence $(s_k)_k$ has a biorthogonal sequence $(\bar{s}_k)_k$, that is, a Bessel sequence with bound M^2 and that

$$\text{span}\langle s_j,\ j\in\mathbb{N}\rangle \subset \overline{\text{span}\langle \bar{s}_j,\ j\in\mathbb{N}\rangle}. \tag{28}$$

If

$$\gamma > M\|u_\kappa\|_{L^2(0,T;U)}, \tag{29}$$

the solution of **EPMOM2** is independent of γ and equal to the solution of **MOM**.

Proof
Let $(\bar{s}_k)_k$ denote the biorthogonal sequence with

$$\langle \bar{s}_k, s_j\rangle_{L^2(0,T;U)} = \delta_{kj},$$

where δ_{kj} is Kronecker's symbol. Because the sequence $(\bar{s}_k)_k$ is a Bessel sequence, for all sequences $(\alpha_k)_k \in l^2$, we have the inequality

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$$\| \sum_{j=1}^{\infty} \alpha_j \bar{s}_j \|^2_{L^2(0,T;U)} \leqslant M^2 \sum_{j=1}^{\infty} |\alpha_j|^2.$$

We have

$$u_\kappa = \sum_{j=1}^{\infty} y_j \bar{s}_j.$$

Let $d \in L^2(0,T;U)$ be given, $d \neq 0$. Let K denote the closure of the span $\langle \bar{s}_j, j \in N \rangle$. Because the corresponding biorthonormal sequence $(\bar{s}_k)_k$ is complete in K, we can write d in the form

$$d = d^\perp + \sum_{j=1}^{\infty} \delta_j \bar{s}_j,$$

with a sequence $\delta = (\delta_j)_j \in l^2$ and $d^\perp \in K^\perp$. We have

$$\begin{aligned} \left| \langle u_\kappa, d \rangle_{L^2(0,T;U)} \right| = & \left| \langle u_\kappa, \sum_{j=1}^{\infty} \delta_j \bar{s}_j \rangle_{L^2(0,T;U)} \right| \\ \leqslant & \|u_\kappa\|_{L^2(0,T;U)} \left\| \sum_{j=1}^{\infty} \delta_j \bar{s}_j \right\|_{L^2(0,T;U)} \\ \leqslant & \|u_\kappa\|_{L^2(0,T;U)} \; M \sqrt{\sum_{j=1}^{\infty} |\delta_j|^2}. \end{aligned}$$

Let h denote the objective function of **EPMOM2**, that is

$$h(u) = \frac{1}{2} \|u\|^2_{L^2(0,T;U)} + \gamma \sqrt{\sum_{k=1}^{\infty} \left| \langle u, s_k \rangle_{L^2(0,T;U)} - y_k \right|^2}.$$

Then, using (28), we obtain

$$\begin{aligned} h(u_\kappa + d) = & \; h(u_\kappa) + \langle u_\kappa, d \rangle_{L^2(0,T;U)} + \frac{1}{2} \|d\|^2_{L^2(0,T;U)} \\ + & \; \gamma \sqrt{\sum_{k=1}^{\infty} \left| \langle d, s_k \rangle_{L^2(0,T;U)} \right|^2} \\ > & \; h(u_\kappa) - \|u_\kappa\|_{L^2(0,T;U)} \; M \sqrt{\sum_{k=1}^{\infty} |\delta_k|^2} \\ + & \; \gamma \sqrt{\sum_{j=1}^{\infty} \left| \delta_j + \langle d^\perp, s_j \rangle_{L^2(0,T;U)} \right|^2} \\ = & \; h(u_\kappa) + (\gamma - \|u_\kappa\|_{L^2(0,T;U)} \; M) \sqrt{\sum_{j=1}^{\infty} |\delta_j|^2} \\ \geqslant & \; h(u_\kappa), \end{aligned}$$

hence u_κ solves **EPMOM2** and the assertion follows. □

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Remark 2
Proposition 2.3 (ii) in [19] states that the Riesz–Fischer sequences in a separable Hilbert space are precisely the families for which a biorthogonal Bessel sequence exists. Young states in [20] that it is not known whether every incomplete sequence of complex exponentials admits a complete biorthogonal sequence.

Example 3
Consider problem **ECD2** from Example 1. If $T = 2k$, that is, T is an even natural number, the functions

$$s_{2j}(s) = \sqrt{2}\,\sin(\pi j s),\ s_{2j-1}(s) = \sqrt{2}\,\cos(\pi j s),$$

form an orthonormal system in $L^2(0,1)$. Hence, we can choose $\bar{s}_j = s_j$ as biorthogonal system and Theorem 2 is applicable.

Example 4
Consider Example 2. Define the functions $\tilde{s}_1(s) = \frac{s-\frac{T}{2}}{\sqrt{T^3/12}}$, $s_2(s) = \frac{1}{\sqrt{T}}$, and for natural numbers $j \in \{1, 2, 3, ...\}$

$$s_{2j+1}(s) = \sqrt{\frac{2}{T}}\,\sin(2\pi j s),\ s_{2j+2}(s) = \sqrt{\frac{2}{T}}\,\cos(2\pi j s).$$

If T is a natural number, the sequence $(s_j)_{j=2}^{\infty}$ is an orthonormal system. For the corresponding problem of L^2-norm minimal optimal control where the constraint with s_1 is omitted, the assumptions of Theorem 2 hold. To obtain a problem where our assumptions can easily be verified for the complete sequence of moment equations, we orthogonalize $\tilde{s}_1$: We define

$$\hat{s}_1 = \tilde{s}_1 - \sum_{j=2}^{\infty} \langle \tilde{s}_1, s_j \rangle s_j.$$

With this function $\hat{s}_1$, we can replace the first moment equation by

$$\langle \hat{s}_1, u \rangle = \langle \tilde{s}_1, u \rangle - \sum_{j=2}^{\infty} \langle \tilde{s}_1, s_j \rangle\ \langle u, s_j \rangle.$$

In our example, we have

$$\hat{s}_1 = \tilde{s}_1 + \sum_{j=1}^{\infty} \frac{\sqrt{6}}{T} \frac{1}{\pi j} s_{2j+1},$$

and

$$\langle \hat{s}_1, u \rangle = \langle \tilde{s}_1, u \rangle = \frac{y_0}{2} - \frac{T}{2}\frac{y_1}{2}.$$

Then the assumptions of Theorem 2 hold with $s_1 = \hat{s}_1 / \|\hat{s}_1\|$ for $(s_j)_{j=1}^{\infty}$ if $T > 1$ is a natural number.

For general $T > 1$, it is more complicated to verify that the assumptions of Theorem 2 hold, but it is still possible [10]. The verification that the sequence is a Riesz–Fischer sequence is based upon a trigonometric inequality by Ingham [21].

3.4. *Penalization by l^∞-norms*

In this section, we focus on problems with given data that have more than the minimal regularity. Our aim is to show that for a given data with higher regularity, a weaker penalization is still exact.

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For the problem of a vibrating string with Dirichlet boundary control and an initial state with L^∞ regularity, a result of this type (in this case with L^1-penalization) is stated in [1], Theorem 5.1.

Here, we consider a penalization with a weighted norm, and the weights can be chosen according to the regularity of the initial data (terminal data respectively). We introduce weighted l^∞-norms that can be adapted to the regularity of the data. Let $(w_k)_k$ be a sequence of positive weighting parameters, $w_k > 0$. For a sequence $(\delta_k)_k$, we define the weighted l^∞-norm

$$\|\delta\|_{\infty,w} = \sup\{w_k\,|\delta_k|,\ k \in \mathbb{N}\}.$$

For a penalty parameter $\gamma > 0$, we consider the penalized problem

$$\textbf{EPMOM}\infty : \min \frac{1}{2}\ \|u\|^2_{L^2(0,T;U)} + \gamma \sup\left\{w_k \left|\langle u, s_k\rangle_{L^2(0,T;U)} - y_k\right|,\ k \in \mathbb{N}\right\}.$$

In the following Theorem, we assume that the sequence $(s_k)_k$ is a Riesz–Fischer sequence [18], which implies that it is minimal. A definition of minimality is given in [10].

Theorem 3
Assume that the sequence $(s_k)_k$ is a Riesz–Fischer sequence with a biorthogonal sequence $(\bar{s}_k)_k$, that

$$\text{span}\langle s_j,\ j \in \mathbb{N}\rangle \subset \overline{\text{span}\langle \bar{s}_j,\ j \in \mathbb{N}\rangle},$$

and that

$$\sum_{j=1}^{\infty} \frac{1}{w_j} \left| \left\langle \sum_{k=1}^{\infty} y_k \bar{s}_k,\ \bar{s}_j \right\rangle_{L^2(0,T;U)} \right| = b_a < \infty. \tag{30}$$

If

$$\gamma > b_a, \tag{31}$$

the solution of **EPMOM**∞ is independent of γ and equal to the solution of **MOM**.

Proof
Because the sequence $(s_k)_k$ is a Riesz–Fischer sequence, there exists a biorthonormal sequence $(\bar{s}_k)_k$ such that

$$\langle \bar{s}_k, s_j\rangle_{L^2(0,T;U)} = \delta_{kj},$$

where δ_{kj} is Kronecker's symbol. Moreover, we have

$$u_\kappa = \sum_{j=1}^{\infty} y_j \bar{s}_j. \tag{32}$$

The regularity of the optimal control (30) implies

$$\sum_{j=1}^{\infty} \frac{1}{w_j}\ \left|\langle u_\kappa,\ \bar{s}_j\rangle_{L^2(0,T)}\right| = b_a < \infty.$$

Let $d \in L^2(0,T;U)$ be given, $d \neq 0$. Let K denote the closure of the span $\langle \bar{s}_j,\ j \in N\rangle$. Because the corresponding biorthonormal sequence $(\bar{s}_k)_k$ is complete in K, we can write d in the form

$$d = d^\perp + \sum_{j=1}^{\infty} \delta_j \bar{s}_j,$$

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with a sequence $\delta = (\delta_j)_j \in l^2$ and $d^\perp \in K^\perp$. We have

$$\begin{aligned}
\left|\langle u_\kappa, d\rangle_{L^2(0,T;U)}\right| = & \left|\left\langle u_\kappa, \sum_{j=1}^{\infty} \delta_j \bar{s}_j \right\rangle_{L^2(0,T;U)}\right| \\
= & \left|\left\langle u_\kappa, \sum_{j=1}^{\infty} \frac{1}{w_j}\, \delta_j w_j \bar{s}_j \right\rangle_{L^2(0,T;U)}\right| \\
\leqslant & \|\delta\|_{\infty,w} \sum_{j=1}^{\infty} \frac{1}{w_j} \left|\langle u_\kappa, \bar{s}_j\rangle_{L^2(0,T;U)}\right| \\
= & b_a \|\delta\|_{\infty,w},
\end{aligned}$$

where the last equation follows from (30). Let h denote the objective function of **EPMOM**∞, that is

$$h(u) = \frac{1}{2} \|u\|^2_{L^2(0,T;U)} + \gamma \sup\left\{w_k \left|\langle u, s_k\rangle_{L^2(0,T;U)} - y_k\right|,\ k \in \mathbb{N}\right\}.$$

Then we have

$$\begin{aligned}
h(u_\kappa + d) = & \ h(u_\kappa) + \langle u_\kappa, d\rangle_{L^2(0,T;U)} + \frac{1}{2}\|d\|^2_{L^2(0,T;U)} \\
+ & \ \gamma \sup\left\{w_k \left|\langle d, s_k\rangle_{L^2(0,T;U)}\right|,\ k \in \mathbb{N}\right\} \\
> & \ h(u_\kappa) - b_a \|\delta\|_{\infty,w} \\
+ & \ \gamma \sup\left\{w_k \left|\left\langle \sum_{j=1}^{\infty} \delta_j \bar{s}_j, s_k \right\rangle_{L^2(0,T;U)}\right|,\ k \in \mathbb{N}\right\} \\
\geqslant & \ h(u_\kappa) + (\gamma - b_a) \|\delta\|_{\infty,w} \\
\geqslant & \ h(u_\kappa),
\end{aligned}$$

hence u_κ solves **EPMOM**∞ and the assertion follows. □

Remark 3

Assumption (30) is valid if $s_k = \bar{s}_k$ for all $k \in \mathbb{N}$, $(y_k)_k \in l^1$ and $w_j = 1$. This is the case in Example 3 if y_0 and y_1 are sufficiently regular.

Now, we discuss the choice of the weights $(w_j)_j$ depending on the regularity of the problem data. Consider problem **ECD2** from Example 1. If $T = 2k$, that is, T is an even natural number, the functions

$$s_{2j}(s) = \sqrt{2}\,\sin(\pi j s),\ s_{2j-1}(s) = \sqrt{2}\,\cos(\pi j s),$$

form an orthonormal system in $L^2(0,1)$. Hence, we can choose $\bar{s}_j = s_j$ as biorthogonal system. In this case (30) is equivalent to

$$\sum_{j=1}^{\infty} \frac{1}{w_j} |y_j| < \infty. \tag{33}$$

Let $q \geqslant 2$. If y_0 and $Y_1 = \int y_1$ are in $L^q(0,1)$, for the corresponding Fourier coefficients, we have $(\hat{y}_0(j))_j$, $(\hat{Y}_1(j))_j \in l^p$, where $\frac{1}{p} + \frac{1}{q} = 1$ ([22], Chapter 4). In particular, (25) implies that we have $(y_j)_j \in l^p$. Thus (33) implies that for any sequence $(w_j)_j$ such that $(\frac{1}{w_j})_j \in l^q$, condition (30) holds. For example, with $\alpha > 1$, we can choose $w_j = j^{\alpha/q}$.

If the periodic extensions of y_0 and $Y_1 = \int y_1$ are $(r-1)$ times absolutely continuous and $r \geqslant 1$, for the corresponding Fourier coefficients, we have $(\hat{y}_0(j)\, j^r)_j$, $(\hat{Y}_1(j)\, j^r)_j \in l^2$ [22]. In particular, (25) implies that we have $(y_j\, j^r)_j \in l^2$. Thus (33) implies that for any sequence $(w_j)_j$ such that $(\frac{1}{j^r\, w_j})_j \in l^2$, condition (30) holds. For example, with $\alpha > 1$, we can choose $w_j = j^{\alpha - r}$. In particular, this implies that for $r \geqslant 2$, we can choose weights w_j with $\lim_{j\to\infty} w_j = 0$.

Consider now the case that $(\bar{s}_k)_k$ is a sequence of exponentials $(\exp(i\lambda_k t))_k$ with $\lambda_{n+1} - \lambda_n \geqslant \iota > 0$. Let $T = \pi/\iota$. Then Ingham's trigonometric L^1-inequality [23] states that

$$\|(\langle d,\, s_j\rangle_{L^2(0,T;U)})_j\|_\infty \leqslant \frac{1}{T}\int_{-T}^{T}\left|\sum_{j=1}^{\infty}\langle d,\, s_j\rangle_{L^2(0,T;U)}\,\bar{s}_j\right|\, dt.$$

Now the proof of Theorem 3 implies that in this case and if the assumption of Theorem 3 hold with $w_j = 1$ also, the penalization

$$\min \frac{1}{2}\, \|u\|^2_{L^2(0,T;U)} + \gamma\, \frac{1}{T}\int_{-T}^{T}\left|\sum_{j=1}^{\infty}\left(\langle u, s_j\rangle_{L^2(0,T;U)} - y_j\right)\bar{s}_j\right|\, dt,$$

is exact. This study is partly motivated by Theorem 5.1 in [1] where the L^1-penalization of the terminal state of the wave equation is considered, and it is shown that for an initial state $(y_0, y_1) \in L^\infty(0,1) \times W^{-1,\infty}(0,1)$, this penalization is exact.

3.5. *Penalization by l^1-norms*

In this section, we consider problems of L^∞-norm minimal control (Example 5). Such problems occur, for example, as auxiliary problems for the solutions of problems of time-optimal control under L^∞-norm control constraints. Analytic solutions of some L^∞-optimal Neumann boundary control problems for the wave equation are presented in [24] and [25].

Example 5
Let $(y_0, y_1) \in W^{1,\infty}(0,1) \times L^\infty(0,1)$ be given. We assume that y_0 and y_1 are even with respect to the point 1/2, that is, $y_\sigma(1/2 + x) = y_\sigma(1/2 - x)$, $\sigma \in \{0, 1\}$. Consider the Neumann optimal control problem with L^∞ objective function

$$\textbf{ECN}\infty \begin{cases} \min \quad \|u\|_{L^\infty(0,T)} & \text{subject to} \\ y'' = & y_{xx} \\ y_x(t,0) = & -u(t) \\ y_x(t,1) = & u(t) \\ y(0,x) = & y_0(x) \\ y'(0,x) = & y_1(x) \\ y(T,x) = & 0 \\ y'(T,x) = & 0. \end{cases} \tag{34}$$

In [24], it is shown that this problem is equivalent to the problem

$$\begin{cases} \min \quad \|u\|_{L^\infty(0,T)} \quad \text{subject to} \\ \int_0^T u(s)\, ds = \frac{y_1^0}{2}, \\ \int_0^T u(s)\,(s - \frac{T}{2})\, ds = \frac{y_0^0}{2} - \frac{T\, y_1^0}{4}, \\ \int_0^T u(s)\sin(2\pi j s)\, ds = \pi j\, \frac{y_0^{2j}}{\sqrt{2}}, \\ \int_0^T u(s)\cos(2\pi j s)\, ds = \frac{y_1^{2j}}{2\sqrt{2}},\ j \in \{1,2,3,...\}, \end{cases} \tag{35}$$

where $y_\sigma^j = \int_0^1 y_\sigma(x)\varphi_j(x)\, dx$ ($\sigma \in \{0,\, 1\}$) and $\varphi_0(x) = 1$, $\varphi_j(x) = \sqrt{2}\cos(j\pi x)$.

The form (35) of the Neumann optimal boundary control problems allows to show an interesting result on the structure of the optimal controls: For initial states with velocity zero, that is, $y_1 = 0$, there exists an optimal control that is odd with respect to the midpoint of the time interval. More precisely, we have the following:

Lemma 3
Assume that $T \geqslant 2$ is a natural number. There exists an optimal control u that solves (35) and is odd with respect to the point $T/2$ on $[0, T]$, that is

$$u(\tfrac{T}{2} + t) = -u(\tfrac{T}{2} - t), \tag{36}$$

for all $t \in [0, T/2]$, if and only if $y_1 = 0$.

Lemma 3 is related with a similar result by Bennighof and Boucher where, as in Example 2 ,only constant states (y_0, y_1) but arbitrary times T are considered [26].

Proof
Assume that $y_1 = 0$. Let an optimal control u for (35) be given. Define $\bar{u}(t) = [u(t) - u(T - t)]/2$. Then $\bar{u}$ satisfies (36) and $\|\bar{u}\|_{L^\infty(0,T)} \leqslant \|u\|_{L^\infty(0,T)}$. Moreover, we have $\int_0^T \bar{u}(s)\, ds = 0$ and $\int_0^T \bar{u}(s)\, (s - \frac{T}{2})\, ds = \int_0^T u(s)\, (s - \frac{T}{2})\, ds$. In addition, $\int_0^T \bar{u}(s) \sin(2\pi j s)\, ds = \int_0^T u(s) \sin(2\pi j s)\, ds$ and $\int_0^T \bar{u}(s) \cos(2\pi j s)\, ds = 0$. Hence $\bar{u}$ is feasible for (35), and because $\|\bar{u}\|_{L^\infty(0,T)} \leqslant \|u\|_{L^\infty(0,T)}$, this implies that $\bar{u}$ is also a solution of (35). On the other hand, if an odd function solves (35), the corresponding moment equations imply $y_1^0 = 0$ and $y_1^{2j} = 0$ for all $j \in \mathbb{N}$, hence $y_1 = 0$. □

If $T \geqslant 2$ is a natural number, $y_1 = 0$ and y_0 is constant, an optimal control (which is in this special case unique) is given by $u(t) = -2\, y_0/T^2$ for $t \in [0, T/2]$ and $u(t) = 2\, y_0/T^2$ for $t \in (T/2, T]$. This illustrates that in general, **EC**∞ does not have periodic solutions. It is important that in general, **EC**∞ does not have solutions with bang–bang structure. Examples are given in [24]. The values of the control that are not extremal are particularly difficult to compute.

Our problems have the form

$$\mathbf{MOM}\infty : \begin{cases} \min & \|u\|_{L^\infty(0,T;U)} \\ \text{subject to} & \langle u, s_k \rangle_{L^2(0,T;U)} = y_k,\ k \in \mathbb{N}, \end{cases}$$

where $(s_k)_k$ is a given sequence of functions in $L^2(0, T; U)$ and $(y_k)_k \in l^2$ such that there exists $\hat{u} \in L^\infty(0, T; U)$ such that

$$\langle \hat{u}, s_k \rangle_{L^2(0,T)} = y_k,\ k \in \mathbb{N}.$$

Assume that the sequence $(s_k)_k$ is minimal with a biorthogonal sequence $(\bar{s}_k)_k$, where $\bar{s}_k \in L^\infty(0, T; U)$. For a penalty parameter $\gamma > 0$, we consider the l^1-norm penalized problem

$$\mathbf{EPMOM1} : \min\ \|u\|_{L^\infty(0,T;U)} + \gamma \sum_{k=1}^{\infty} \|\bar{s}_k\|_{L^\infty(0,T;U)}\ \left|\langle u, s_k \rangle_{L^2(0,T;U)} - y_k\right|.$$

The following Theorem is applicable, for example, for problems with the heat equation (Example 2.6), where the corresponding biorthogonal sequence is typically not uniformly bounded in L^∞ but grows exponentially.

Theorem 4
Assume that the sequence $(s_k)_k$ is minimal with a biorthogonal sequence $(\bar{s}_k)_k$ in $L^\infty(0, T; U)$ and that

$$\operatorname{span}\langle s_j,\ j \in \mathbb{N}\rangle \subset \overline{\operatorname{span}\langle \bar{s}_j,\ j \in \mathbb{N}\rangle}.$$

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For a penalty parameter $\gamma > 0$, we consider the weighted l^1-norm penalized problem **EPMOM1**. If γ is sufficiently large, more precisely, if

$$\gamma > 1, \tag{37}$$

the solution of **MOM**∞ is also a solution of **EPMOM1**. In particular, for $\gamma > 1$, the optimal value of **EPMOM1** is independent of γ.

Remark 4
In contrast to the previous lower bounds for γ that imply the exactness of the penalization of the terminal state in Theorem 4, neither the optimal control nor the initial state nor the right-hand sides y_k of the moment equations that depend in turn on the inital state appear explicitly in the bound. This is because in the proof of Theorem 4, we use an inequality of a different type because in the control cost in contrast to the other cases, no square appears. Note that in Theorem 5, the situation is similar: Also for the L^1-control cost, the lower bound on γ is independent of the initial state and the optimal control.

Proof
Because sequence $(s_k)_k$ is minimal, there exists a biorthonormal sequence $(\bar{s}_k)_k$ such that

$$\langle \bar{s}_k, s_j \rangle_{L^2(0,T;U)} = \delta_{kj}.$$

Let $d \in L^\infty(0,T;U)$ be given, $d \neq 0$. Let K denote the closure of the span $\langle \bar{s}_j, j \in \mathbb{N} \rangle$. Because the corresponding biorthonormal sequence $(\bar{s}_k)_k$ is complete in K, we can write d in the form

$$d = d^\perp + \sum_{j=1}^{\infty} \delta_j \bar{s}_j,$$

with a sequence $\delta = (\delta_j)_j \in l^2$ and $d^\perp \in K^\perp$. Let $u_\kappa \in L^\infty(0,T;U)$ denote a solution of **MOM**∞. For all $d^\perp \in K^\perp$, $u_\kappa + d^\perp$ satisfies the moment equations. Because of the optimality of u_κ, this implies

$$\|u_\kappa + d^\perp\|_{L^\infty(0,T;U)} \geqslant \|u_\kappa\|_{L^\infty(0,T;U)} = h(u_\kappa).$$

Because of the biorthogonality, we have

$$\sum_{k=1}^{\infty} \|\bar{s}_k\|_{L^\infty(0,T;U)} \left| \langle d, s_k \rangle_{L^2(0,T;U)} \right| \quad = \quad \sum_{k=1}^{\infty} \|\bar{s}_k\|_{L^\infty(0,T;U)} \, |\delta_k| \, .$$

Moreover, we have

$$\| \sum_{j=1}^{\infty} \delta_j \bar{s}_j \|_{L^\infty(0,T;U)} \quad \leqslant \quad \sum_{j=1}^{\infty} |\delta_j| \, \|\bar{s}_j\|_{L^\infty(0,T;U)}.$$

Let h denote the objective function of **EPMOM1**, that is

$$h(u) = \|u\|_{L^\infty(0,T;U)} + \gamma \sum_{k=1}^{\infty} \|\bar{s}_k\|_{L^\infty(0,T;U)} \left| \langle u, s_k \rangle_{L^2(0,T;U)} - y_k \right| .$$

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Then, we have

$$\begin{aligned} h(u_\kappa + d) \geqslant\ & \|u_\kappa + d^\perp\|_{L^\infty(0,T;U)} - \|\sum_{j=1}^\infty \delta_j \bar{s}_j\|_{L^\infty(0,T;U)} \\ +\ & \gamma \sum_{k=1}^\infty \|\bar{s}_k\|_{L^\infty(0,T;U)}\ \left|\langle d, s_k\rangle_{L^2(0,T;U)}\right| \\ \geqslant\ & h(u_\kappa) - \sum_{k=1}^\infty \|\bar{s}_k\|_{L^\infty(0,T;U)}\ |\delta_k| + \gamma \sum_{k=1}^\infty \|\bar{s}_k\|_{L^\infty(0,T;U)}\ |\delta_k| \\ =\ & h(u_\kappa) + (\gamma - 1) \sum_{k=1}^\infty \|\bar{s}_k\|_{L^\infty(0,T;U)}\ |\delta_k| \\ \geqslant\ & h(u_\kappa), \end{aligned}$$

hence u_κ solves **EPMOM1** and the assertion follows. □

Example 6

Consider the optimal Dirichlet boundary problem with the wave equation from Example 1 with the objective function replaced by the L^∞-norm.

To make sure that **MOM**∞ has a solution, we have to assume higher regularity of the initial state, namely, $(y_0, y_1) \in L^\infty(0,1) \times W^{-1,\infty}(0,1)$. The weakness of the bang–bang principle for L^∞ Dirichlet boundary control problems is studied in [27].

For natural numbers $j \in \{1, 2, 3, ...\}$ define the functions

$$s_{2j}(s) = \sin(\pi j s),\ s_{2j-1}(s) = \cos(\pi j s).$$

If $T = 2$, which is the minimal time where exact controllability holds, the family $(s_j)_j$ together with a constant function $s_0 = \frac{1}{\sqrt{2}}$ form a complete orthonormal system that is uniformly bounded in $L^\infty(0,T)$.

If $T \geqslant 4$ is an even integer, the family $(s_j)_j$ forms an orthogonal system that can be normalized to an orthonormal system that is uniformly bounded in $L^\infty(0,T)$.

4. PENALIZATION FOR L^1-NORM OPTIMAL CONTROL PROBLEMS

In order to complete our study, we also consider optimal control problems where the control cost is given by an L^1-norm. A motivating example is the minimization of the total fuel consumption of a vehicle, the control variable being the motor thrust [28]. The L^1-frame is also a natural setting for the corresponding time-optimal control problem with fuel constraints.

These topics have been widely discussed in the PDE setting. For instance, time-optimal control problems for the heat equation are discussed in [29]. Problems of L^1-norm optimal Dirichlet boundary control problems for the wave equation have, for example, been considered, for example, in [30]. In order to allow for states in spaces of the type $C([0,T]; L^1(\Omega))$, Ω being an open subset of a finite-dimensional Euclidean space, we assume that X and U are Banach spaces. As before, let A: $\mathcal{D}(A) \subset X \to X$ be the generator of a strongly continuous semigroup and let B denote an admissible control operator. Again, for $x_0 \in X$, we consider a control system of the form

$$\begin{cases} x'(t) + Ax = Bu, \\ \qquad x(0) = x_0. \end{cases} \tag{38}$$

Let a time $T > 0$ be given. In this section, we assume that for all $u \in L^1(0,T;U)$, the Cauchy problem (38) has a unique solution $x \in C([0,T]; X)$.

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Assume that (38) is exactly controllable using L^1-controls in time T, that is, there exists a constant $C_1 > 0$ such that for all initial states $x_0 \in X$ and all final states $x_1 \in X$, there is a control $u \in L^1(0, T; U)$ such that the solution $x \in C([0, T]; X)$ of (38) satisfies

$$\begin{cases} x(T) = x_1, \\ \|u\|_{L^1(0,T;U)} \leqslant C_1 \left(\|x_0\|_X + \|x_1\|_X\right). \end{cases} \tag{39}$$

Let us return to the motivating example from Section 1.1. If we replace the objective function by the L^1-norm of the control, we obtain the optimal control problem

$$\begin{cases} \min_{u \in L^1(0,1)} & \|u\|^2_{L^1(0,1)} \\ \text{subject to} & \\ y'(t) - y(t) = & \exp(t)\, u(t) \\ y(0) = & -1 \\ y(1) = & 0. \end{cases} \tag{40}$$

As in Section 1.1, the constant function $u_\kappa(t) = 1$ is an optimal control. However, it is not uniquely determined, because every control with L^1-norm equal to 1 that satisfies the moment equation $\int_0^1 u(s)\, ds = 1$ is also an optimal control. So, we see that we have a huge set of optimal controls. In particular, we can find sequences of optimal controls that approximate Dirac measures, for example, by considering for $k \in \{1, 2, 3, ...\}$ the optimal controls

$$u_k(t) = \begin{cases} 2^k & \text{for} \quad t \in [0, \frac{1}{2^k}], \\ 0 & \text{for} \quad t \in (\frac{1}{2^k}, 1]. \end{cases}$$

Example (40) illustrates that nonuniqueness is an important issue for L^1-norm optimal controls. Another important issue is nonexistence. Consider the problem

$$\begin{cases} \inf_{u \in L^1(0,1)} & \|u\|^2_{L^1(0,1)} \\ \text{subject to} & \\ y'(t) - y(t) = & u(t) \\ y(0) = & -1 \\ y(1) = & 0. \end{cases} \tag{41}$$

Suppose that u^* is a solution of (41). Then the control

$$\tilde{u}(t) = \begin{cases} u^*(t) + \exp\left(-\frac{1}{2}\right) u^*\left(t + \frac{1}{2}\right) & \text{for} \quad t \in \left[0, \frac{1}{2}\right], \\ 0 & \text{for} \quad t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

also generates a state that satisfies the end condition $y(1) = 0$ because it satisfies the moment equation $\int_0^1 \tilde{u}(t)\, \mathrm{e}^{-t}\, dt = \int_0^1 u^*(t)\, \mathrm{e}^{-t}\, dt = 1$. We have $\int_0^1 |\tilde{u}(t)|\, dt \leqslant \int_0^{1/2} |u^*(t)|\, dt + \int_{1/2}^1 |u^*(t)| \exp(-1/2)\, dt$. Because we have supposed u^* to be a solution of (41), this implies that $u^*(t) = 0$ for $t \in [1/2, 1]$ almost everywhere. By induction, we show that for all $k \in \{1, 2, 3, ..\}$, we have $u^*(t) = 0$ for $t \in [1/2^k, 1]$ almost everywhere. For this purpose, define

$$\tilde{u}_k(t) = \begin{cases} u^*(t) + \exp\left(-\frac{1}{2^k}\right) u^*\left(t + \frac{1}{2^k}\right) & \text{for} \quad t \in \left[0, \frac{1}{2^k}\right], \\ 0 & \text{for} \quad t \in \left[\frac{1}{2^k}, 1\right]. \end{cases}$$

Then assuming that $u^* = 0$ for $t \in [\frac{1}{2^{k-1}}, 1]$ almost everywhere, we obtain $\int_0^1 \tilde{u}_k(t)\, \mathrm{e}^{-t}\, dt = \int_0^{1/2^{k-1}} u^*(t)\, \mathrm{e}^{-t}\, dt = \int_0^1 u^*(t)\, \mathrm{e}^{-t}\, dt = 1$. This implies that $\tilde{u}_k$ generates a state that satisfies the end condition $y(1) = 0$.

We have $\int_0^1 |\tilde{u}_k(t)|\, dt \leqslant \int_0^{1/2^k} |u^*(t)|\, dt + \int_{1/2^k}^{1/2^{k-1}} |u^*(t)| \exp(-1/2^k)\, dt$. Because we have supposed u^* to be a solution of (41), this implies that $u^*(t) = 0$ for $t \in [1/2^k, 1/2^{k-1}]$ almost everywhere.

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Because $u^*(t) = 0$ for $t \in [1/2^k, 1]$ almost everywhere for all $k \in \{1, 2, 3, ...\}$, $u^*(t)$ cannot exist as an L^1-function, which implies that (41) does not have a solution.

This is because of the lack of closeness of L^1 with respect to the weak convergence in the sense of measures. The closure of the optimization problem leads to consider controls that are in $\mathcal{M}(0, T; U)$, where $\mathcal{M}$ denotes the space of measures. The same optimal control problems that we consider with controls in $L^1(0, T; U)$ can be considered in $\mathcal{M}(0, T; U)$. In that functional setting, the existence of optimal controls becomes automatic by the Direct Method of the Calculus of Variations. In some cases, optimal controls achieved that way turn out to become more regular and belong to $L^1(0, T; U)$. But for this to be proved, one has to analyze in detail the optimality system and use the regularizing effects of the state and adjoint equations.

The fact that controls might be more smooth (in particular with respect to the time variable) than what they are originally imposed to be has been observed in different contexts. For instance, in [31], it is observed that for the interior control of the wave equations, controls that are optimal in $L^2(0, T; L^2(\omega))$ belong actually to $C(0, T; L^2(\omega))$. Note, however, that this does not occur in the context of boundary control. In [2], it is stated that both for boundary and internal control, even if controls are computed to be optimal in L^2, if the data to be controlled are more smooth, the resulting controls can be actually H^s. This is actually a property of ellipticity or regularizing effect of the Gramian operator.

In order to deal with the situation of possibly empty solution sets and solution sets with more than one element, let us look at the notion of the equivalence of optimization problems.

Definition 1
Two optimization problems are called *equivalent*, if they have the same (possibly empty) set of solutions.

Now, we consider the penalization with L^1-controls:

$$\textbf{EP1}: \begin{cases} \inf \ \|u\|_{L^1(0,T;U)} + \gamma \|x(T)\|_X \\ \text{subject to} \quad x(0) = x_0,\ x'(t) + Ax = Bu. \end{cases}$$

In problem **EP1**, the end condition $x(T) = 0$ does not appear. In the statement of **EP1**, we have written inf instead of min to emphasize that there are cases where the solution set is empty.

Theorem 5 states that if γ is sufficiently large, the optimization problems **EP1** are equivalent to optimization problems with L^1-control cost and terminal constraints. Thus, also in the L^1-case, the penalization provides a nice way to avoid the terminal constraint. However, it does not avoid the issues of non-uniqueness of L^1-norm minimal controls and possible non-existence of such controls. Note that as we have stated in Remark 4, because the L^1-norm appears in the objective function of the penalized problem without a square, in Theorem 5, the lower bound for γ is independent of the initial state.

Theorem 5
Assume that the system is exactly controllable using L^1-controls in time T. If

$$\gamma > C_1, \tag{42}$$

the set of solutions of problem **EP1** is independent of γ and equal to the set of solutions of the optimal control problem

$$\textbf{EC1}: \begin{cases} \inf \ \|u\|_{L^1(0,T;U)} \\ \text{subject to} \quad x(0) = x_0,\ x'(t) + Ax = Bu \\ \qquad\qquad\quad x(T) = 0. \end{cases}$$

Hence, if (42) holds, **EP1** and **EC1** are equivalent.

Proof
First, we consider the case that the set of solutions of **EC1** is nonempty. In this case, we can choose an element $u_\kappa \in L^1(0,T;U)$ that is a solution **EC1** (the solution of **EC1** is in general not uniquely determined) and let x_κ denote the corresponding state. Then we have

$$x_\kappa(T) = 0. \tag{43}$$

For $u \in L^1(0,T;U)$ define

$$J_\gamma(u) = \|u\|_{L^1(0,T;U)} + \gamma \|x(T)\|_X.$$

Then we have $J_\gamma(u_\kappa) = \|u_\kappa\|_{L^1(0,T;U)}$.
Let $d \in L^1(0,T;U)$ be given, $d \neq 0$. Define x_d as the solution of the initial value problem

$$\begin{cases} x_d'(t) + Ax_d = Bd, \\ x_d(0) = 0. \end{cases} \tag{44}$$

Then the control $u = u_\kappa + d$ generates the state $x_\kappa + x_d$ that solves (38). On account of (39), there exists a control $d_{\min} \in L^1(0,T;U)$ such that for the solution of

$$\begin{cases} x_{\min}'(t) + Ax_{\min} = Bd_{\min}, \\ x_{\min}(0) = 0, \end{cases} \tag{45}$$

we have $x_{\min}(T) = x_d(T)$ and

$$\|d_{\min}\|_{L^1(0,T;U)} \leqslant C_1 \|x_d(T)\|_X.$$

Define $\delta = d - d_{\min}$. Then, for the solution of

$$\begin{cases} x_\delta'(t) + Ax_\delta = B\delta, \\ x_\delta(0) = 0, \end{cases} \tag{46}$$

we have $x_\delta(T) = 0$. Thus $u_\kappa + \delta$ is admissible for **EC1**. Because u_κ is a solution of **EC1**, this implies

$$\|u_\kappa + \delta\|_{L^1(0,T;U)} \geqslant \|u_\kappa\|_{L^1(0,T;U)}.$$

For the objective function, this implies

$$\begin{aligned} J_\gamma(u_\kappa + d) &= \|u_\kappa + d\|_{L^1(0,T;U)} + \gamma\|x_d(T)\|_X \\ &= \|u_\kappa + \delta + d_{\min}\|_{L^1(0,T;U)} + \gamma\|x_d(T)\|_X \\ &\geqslant \|u_\kappa + \delta\|_{L^1(0,T;U)} - \|d_{min}\|_{L^1(0,T;U)} + \gamma\|x_d(T)\|_X \\ &\geqslant \|u_\kappa\|_{L^1(0,T;U)} - \|d_{min}\|_{L^1(0,T;U)} + (\gamma/C_1)\|d_{min}\|_{L^1(0,T;U)} \\ &= \|u_\kappa\|_{L^1(0,T;U)} + \left(\frac{\gamma}{C_1} - 1\right) \|d_{\min}\|_{L^1(0,T;U)} \\ &\geqslant \|u_\kappa\|_{L^1(0,T;U)} = J_\gamma(u_\kappa). \end{aligned}$$

This implies that u_κ is a solution of **EP1**. Hence, the set of solutions of **EC1** is a subset of the set of solutions of **EP1**.

Moreover, if $u_\kappa + d$ is not a solution of **EC1**, we have $x_d(T) \neq 0$ (and thus $d_{\min} \neq 0$) or $\|u_\kappa + d\|_{L^1(0,T;U)} > \|u_\kappa\|_{L^1(0,T;U)}$. Similarly, as provided earlier, this yields the inequality $J_\gamma(u_\kappa + d) > \|u_\kappa\|_{L^1(0,T;U)}$; hence $u_\kappa + d$ is also not a solution of **EP1**.

This implies that if the set of solutions of **EC1** is nonempty, it is equal to the set of solutions of **EP1**.

Now, we consider the case that the set of solutions of **EC1** is empty. Let v denote the optimal value of **EC1**. For all controls $u_a \in L^1(0,T;U)$ that generate a state x_a with $x_a(T) = 0$, we have

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$\|u_a\|_{L^1(0,T;U)} > \upsilon$. Moreover, for all $\varepsilon > 0$, there exists a control u^ε such that $x^\varepsilon(T) = 0$ and $\|u^\varepsilon\|_{L^1(0,T;U)} \leqslant \upsilon + \varepsilon$.

Suppose that **EP1** has a solution. If for a solution u_s of **EP1**, we have $x_s(T) = 0$, this implies $J_\gamma(u_s) = \|u_s\|_{L^1(0,T;U)} > \upsilon$, which is a contradiction, because with the choice $\varepsilon = \frac{1}{2}\left(\|u_s\|_{L^1(0,T;U)} - \upsilon\right)$, we can find a control with $J_\gamma(u^\varepsilon) = \|u^\varepsilon\|_{L^1(0,T;U)} < \|u_s\|_{L^1(0,T;U)} = J_\gamma(u_s)$. Thus, for every solution of **EP1**, we have $x_s(T) \neq 0$. On account of (39), there exists a control $d_{\min} \in L^1(0,T;U)$ such that for the solution of

$$\begin{cases} x'_{\min}(t) + Ax_{\min} = Bd_{\min}, \\ x_{\min}(0) = 0, \end{cases} \tag{47}$$

we have $x_{\min}(T) = x_s(T)$ and

$$\|d_{\min}\|_{L^1(0,T;U)} \leqslant C_1 \|x_s(T)\|_X.$$

Define $\delta = u_s - d_{\min}$. Then for the solution of

$$\begin{cases} x'_\delta(t) + Ax_\delta = B\delta, \\ x_\delta(0) = x_0, \end{cases} \tag{48}$$

we have $x_\delta(T) = 0$. This implies

$$\begin{aligned} J_\gamma(\delta) = &\ \|\delta\|_{L^1(0,T;U)} \\ = &\ \|u_s - d_{\min}\|_{L^1(0,T;U)} \\ \leqslant &\ \|u_s\|_{L^1(0,T;U)} + \|d_{\min}\|_{L^1(0,T;U)} \\ \leqslant &\ \|u_s\|_{L^1(0,T;U)} + C_1 \|x_s(T)\|_X \\ < &\ \|u_s\|_{L^1(0,T;U)} + \gamma \|x_s(T)\|_X \\ = &\ J_\gamma(u_s). \end{aligned}$$

This is a contradiction because u_s was chosen from the set of solutions of **EP1**. Thus, also **EP1** does not have a solution, and the assertion follows. □

5. CONCLUSION

In the statement of optimal control problems with finite time horizon for time-dependent systems that are exactly controllable, often, it makes sense to require that at the terminal time, the system state is equal to a desired terminal state by including the appropriate terminal constraint.

We have shown that often these end conditions can be replaced by a non-smooth penalty term in the objective function. If the corresponding penalty parameter is sufficiently large, minimizing the objective function with the penalty term yields optimal controls that generate states that satisfy the end conditions exactly. This is a useful tool for the analysis of numerical algorithms, because most reasonable numerical algorithms proceed similarly, for example, by coupling the end conditions to the objective via a Lagrange multiplier approach. Our approach can be used to analyze the behavior of merit functions with a non-smooth penalty term.

The non-smooth penalty term leads to objective functions that are not differentiable. There exist many efficient methods for solving non-smooth convex optimization problems (for example [32], [33]. Another approach to deal with this problem numerically is to approximate the non-smooth penalty term by family of smooth functions. For problems with inequality constraints, this approach has been studied in [34] and [35].

Discretizations that are based upon moment equations have been considered in [36], where the moment equations are replaced by moment inequalities to achieve a regularization of the solutions of the discretized problem. The numerical treatment of unconstrained problems that are based upon the exact penalization for moment constraints needs to be studied in more detail in future research.

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