NETWORKS AND HETEROGENEOUS MEDIA
©American Institute of Mathematical Sciences
Volume **11**, Number **4**, December **2016**
doi:10.3934/nhm.2016013
pp. **X–XX**

DECAY RATES FOR $1-d$ HEAT-WAVE PLANAR NETWORKS

Zhong-Jie Han

Department of Mathematics
Tianjin University
Tianjin 300072, China

Enrique Zuazua

Departamento de Matemáticas
Facultadde Ciencias, Universidad Autónoma de Madrid
Madrid 28049, Spain

(Communicated by Benedetto Piccoli)

Abstract. The large time decay rates of a transmission problem coupling heat and wave equations on a planar network is discussed.

When all edges evolve according to the heat equation, the uniform exponential decay holds. By the contrary, we show the lack of uniform stability, based on a Geometric Optics high frequency asymptotic expansion, whenever the network involves at least one wave equation.

The (slow) decay rate of this system is further discussed for star-shaped networks. When only one wave equation is present in the network, by the frequency domain approach together with multipliers, we derive a sharp polynomial decay rate. When the network involves more than one wave equation, a weakened observability estimate is obtained, based on which, polynomial and logarithmic decay rates are deduced for smooth initial conditions under certain irrationality conditions on the lengths of the strings entering in the network. These decay rates are intrinsically determined by the wave equations entering in the system and are independent on the heat equations.

1. **Introduction.** In recent years, hyperbolic-parabolic coupled models have been studied extensively due to their applications in analyzing fluid-structure interactions, which are crucial in many scientific and engineering areas, such as airflow along the aircraft, deformation of heart valves, the process of mixing and so on (see [22], [18], [10]). The simplest model in this context is constituted by a $1-d$ wave

2010 *Mathematics Subject Classification.* Primary: 35B40, 35B07, 93B07; Secondary: 35M10, 93D20.

Key words and phrases. Hyperbolic-parabolic coupled system, star-shaped network, Geometric Optics, polynomial decay, logarithmic decay, observability, asymptotic behavior, irrationality conditions.

The first author was supported by the Natural Science Foundation of China grant NSFC-61573252 and China Scholarship Council. The second author was supported by the Advanced Grant DYCON of the European Research Council Executive Agency, FA9550-14-1-0214 of the EOARD-AFOSR, FA9550-15-1-0027 of AFOSR, the MTM2014-52347 Grant of the MINECO and a Humboldt Research Award at the University of Erlangen-Nürnberg.

equation coupled with a $1-d$ heat equation at a point interface (see Fig. 1):

$$\begin{cases} \theta_t(x,t) - \theta_{xx}(x,t) = 0, \quad x \in (-1,0),\ t > 0, \\ u_{tt}(x,t) - u_{xx}(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ \theta(-1,t) = u(1,t) = 0,\ t > 0, \\ \theta(0,t) = u(0,t),\ \theta_x(0,t) = u_x(0,t),\ t > 0. \end{cases} \tag{1}$$

In system above, the heat and wave components are coupled at $x = 0$ through trans-

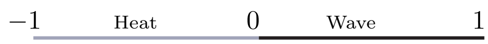

FIGURE 1. A simple hyperbolic-parabolic model

mission conditions ensuring continuity. Two different types of PDEs are coupled only at one point, making the analysis of the qualitative properties of the system like decay rates or controllability delicate to analyze. Zuazua in [32] proved the null controllability of this system with boundary control acting through the wave equation at $x = 1$ by the sidewise method for wave part and Carleman estimate for heat part. Zhang and Zuazua in [29] got the sharp polynomial decay rate for system (1), and further obtained the null controllability with control acting through the heat part at $x = -1$ by the spectral properties. We also refer to [30] for heat-wave system with another transmission condition and [25], [31] for multi-dimensional ones.

The results above on system (1) explain the heat-wave interaction though one single transmission point. The same issues then arise along networks in which various wave and heat equations interact through some joint nodes along a planar network (see Fig. 2 for example). It can be considered as a simplified dynamical model for the interaction of 1-d multi-connected fluids and elastic structures via the interfaces (common nodes).

The main purpose of this paper is to analyze the long time behavior of this kind of networks to explain to which extent the heat components induce decay properties of the energy of the system and how it depends on the topology of the network, the number theoretical conditions of the lengths of the segments, and the location of the heat components. We shall especially focus on the star-shaped heat-wave networks as in Fig. 2. But more general networks will also be discussed.

Firstly, let us describe the transmission problem on star-shaped network in detail. Denote by $e_j,\ j = 1, 2, \cdots, N$ the curves with the interval $(0, \ell_j),\ \ell_j > 0$. Assume that the heat equations arise on the intervals $(0, \ell_k),\ k = 1, 2, \cdots, N_1,\ 1 \le N_1 < N$ in the network with state θ_k, respectively; the wave equations hold on the intervals $(0, \ell_j),\ j = N_1 + 1, N_1 + 2, \cdots, N$ with state $(u_j, u_{j,t})$. Assume that the Dirichlet conditions are fulfilled at the exterior nodes and the geometrical continuity is satisfied at the common node of the network. Then we get the following heat-wave system on star-shaped network:

$$\begin{cases} \theta_{k,t}(x,t) - \theta_{k,xx}(x,t) = 0,\ x \in (0,\ell_k),\ k = 1,2,\cdots,N_1,\ t > 0, \\ u_{j,tt}(x,t) - u_{j,xx}(x,t) = 0,\ x \in (0,\ell_j),\ j = N_1+1, N_1+2, ...N,\ t > 0, \\ \theta_k(\ell_k,t) = u_j(\ell_j,t) = 0,\ k = 1,2,\cdots,N_1, j = N_1+1, N_1+2,\cdots,N,\ t > 0, \\ \theta_k(0,t) = u_j(0,t),\ \forall k = 1,2,\cdots,N_1,\ j = N_1+1, N_1+2,\cdots,N, \\ \sum_{j=N_1+1}^{N} u_{j,x}(0,t) + \sum_{k=1}^{N_1} \theta_{k,x}(0,t) = 0,\ t > 0, \\ \theta^k(t=0) = \theta_k^0,\ k = 1,2,\cdots,N_1, \\ u_j(t=0) = u_j^0,\ u_{j,t}(t=0) = u_j^1, \quad j = N_1+1, N_2+2,\cdots,N, \end{cases} \tag{2}$$

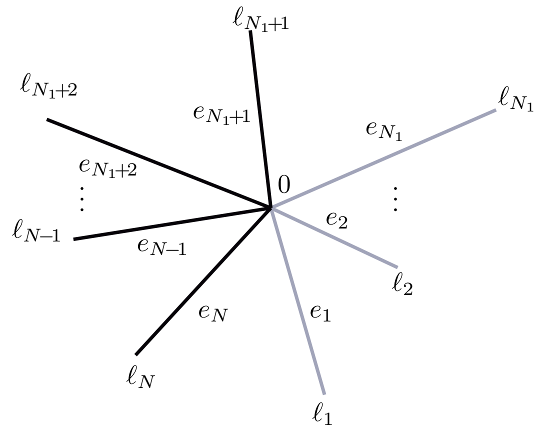

FIGURE 2. Star-shaped network: heat equations (grey), wave equations (black)

where $((\theta_k^0)_{k=1}^{N_1}, (u_j^0)_{k=N_1+1}^{N}, (u_j^1)_{k=N_1+1}^{N})$ is the given initial state.

Remark 1. We can see that system (1) mentioned above can be considered as a special case of this network (2), that is, if $N_1 = 1,\ N = 2$, then the network becomes (1).

System (2) can be rewritten as an abstract Cauchy problem in some appropriate Hilbert space $\mathcal{H}$ as we will see later:

$$\begin{cases} \frac{dU(t)}{dt} = \mathcal{A}U(t), \quad t > 0, \\ U(0) = U_0, \end{cases} \tag{3}$$

where $U(t) = (\theta, u, u_t)^T$ and $U(0) = (\theta^{(0)}, u^{(0)}, u^{(1)})^T \in \mathcal{H}$ is given. It can be proved easily that $\mathcal{A}$ generates a C_0 semigroup by the classic semigroup theory.

The energy of system (2) is defined as follows:

$$E(t) = \frac{1}{2}\sum_{k=1}^{N_1}\int_0^{\ell_k} |\theta_{k,x}|^2 dx + \frac{1}{2}\sum_{j=N_1+1}^{N}\int_0^{\ell_j} (|u_{j,x}|^2 + |u_{j,t}|^2) dx. \tag{4}$$

It satisfies

$$\frac{dE(t)}{dt} = -\sum_{k=1}^{N_1}\int_0^{\ell_k} |\theta_{k,t}|^2 dx \le 0, \tag{5}$$

and therefore the energy is decreasing.

Based on the dissipation law (5), a sufficient and necessary condition for the strong asymptotic stability of system (2) is given in this work. Moreover, from (5), we also find that the dissipative mechanism only acts on the heat components of the network, and affects the wave parts through the common node. This inspires us to further analyze that whether or not the more heat equations are present the networks can lead to the better decay rate of system (2), especially the exponential decay rate, i.e., whether there exist constants $C > 0$ and $\beta > 0$ satisfying

$$E(t) \le CE(0)e^{-\beta t},\ \forall t > 0,$$

for all solution to (2).

Note however that, due to the fact that the exponential decay rate fails to hold even in the simplest case of system (1), it is not expected to occur for more general networks either. In fact, the main results of the paper show that this exponential decay rate never occurs. Accordingly, our analysis will be devoted to prove slow decay properties for smooth solutions.

In recent years, there has been an extensive literature on the controllability and decay rate for PDEs' networks with boundary controls, such as wave networks, parabolic networks and so on. We refer, for instance, [9] for the boundary controllability of many kinds of general wave networks by the HUM method; [1], [2], [15] for the explicit decay rates with star and tree shape structures based on the observability estimates; [6], [11], [12], [20] and [28] for the spectral properties of wave and beam networks; [7] for solvability of parabolic networks; and [23], [13] where the stability is discussed for the networks with boundary time delay inputs. Nevertheless, heat-wave networks are different from the pure hyperbolic or parabolic ones because of the heat-wave coupling at joint nodes. Especially, the techniques developed to analyze the observability of pure wave or heat networks, usually can not deal with both of them in a unified way. As far as the authors know, at present, there is no result considering the long time behavior of the heat-wave system on networks.

The first topic we address in this work is to show the non-uniform decay of heat-wave networks, no matter what shapes the networks are, as long as at least one wave equations is involved in the networks. We mainly prove it by building a local approximate ray-like solutions, by means of a careful analysis on the interaction of the wave and heat-like solutions at the joint nodes. Our method is based on the high frequency asymptotic expansion in Geometric Optics (see [25]). We get that the energy of such ray-like solutions are mainly concentrated in the wave parts of the networks and almost completely reflected back to wave parts at the joint nodes, which implies that the norm of the semigroup $S(t)$ corresponding to the heat-wave networks always equal to 1 for any given $t > 0$; and hence uniform decay rate fails. Moreover, this kind of ray-like solutions can be built independent of the topology of the graph, which is in agreement with what is known for the coupling of one single wave equation with one single heat equation.

In view of the negative result on exponential decay, it is natural to further address whether or not the dissipative mechanism in heat-wave networks can produce some slow decay rates, such as polynomial and logarithmical decay under smooth initial conditions. The spectrum of PDEs in networks is hard to be calculated, especially when the mutual ratios of the lengths of each edge in the network are irrational numbers, since the asymptotic spectra contain different branches, very close to each other. Hence, the approach based on spectral analysis, which works well to achieve the polynomial decay rate for heat-wave system with the simple structure in Fig. 1 (see [5], [6]), cannot be applied to analyze the decay rate of general heat-wave networks. Thus, other methods have to be developed.

We mainly consider the decay rate of star-shaped network (2) and divide the problem into two cases:

Case 1) $N - N_1 = 1$, Case 2) $N - N_1 \geq 2$.

For Case 1), we analyze its decay rate by estimating carefully the norm of the resolvent operator along the imaginary axis. Some multipliers are constructed to help us derive a sharp polynomial decay rate. Especially, the sharpness of the obtained decay rate does not change no matter how many heat equations involved, as long as only one wave equation is present in the network.

For Case 2), the polynomial and logarithmic decay rates are derived under smooth initial conditions, based on different properties of the wave equations entering in the network, respectively. To do this, we deduce a weakened observability inequality for system (2), by means of the energy estimate and some known observability results for pure wave networks (see [9]). In this case we do not use resolvent estimates as in Case 1. Contrarily to Case 1, when more than one wave equation is involved in the network, it is difficult to transfer the dissipative effect from the heat components to the wave ones by the transmission conditions at the common node, so as to obtain estimates on the resolvent operator along the imaginary axis. In fact, the relationship between the decay rate and the lengths of the wave edges in the network is delicate and difficult to identify by means of a frequency domain analysis. This can be done using the observability inequality. As we will see later, the decay rate of the network depends on the Diophantine properties of the mutual ratios of the lengths of the wave edges.

It should be noted that although we focus on discussing the decay rate of heat-wave network with star-shaped structure, the methods also can be adapted to the transmission problem between heat and wave equations on tree-shaped or more complex networks. Hence, we also present some results on decay rate for more general planar networks, which can be derived easily out of the techniques proposed in this paper.

The results on this paper yield further light on the decay properties of damped systems of vibrating networks. Further analysis is still required to handle other models such as those in which the heat equations entering in the network are replaced by the system of thermoelasticity.

The rest of the paper is organized as follows. In Section 2, we consider system (2) in an appropriate functional setting. The well-posedness and strong stability are derived. Section 3 is devoted to show that the energy of heat-wave networks can not achieve exponential decay rate. In Section 4, we show the sharp polynomial decay rate of star-shaped network (2) which contains only one wave equation ($N-N_1=1$). Section 5 is devoted to analyze the decay rate of system (2) for the case involved more than one wave equation ($N-N_1\geq 2$). In Section 6, more general planar networks are considered.

2. **Well-posedness and strong stability.** In this section, we shall consider system (2) in an appropriate well-posedness space and discuss the strong asymptotic stability of it. Define

$$\begin{aligned}
L^2(\Omega_h) &= \{f=(f_j)_{j=1}^{N_1} | f_j\in L^2(0,\ell_j),\ \forall j=1,2,\cdots,N_1\},\\
L^2(\Omega_w) &= \{f=(f_j)_{j=N_1+1}^{N} | f_j\in L^2(0,\ell_j),\ \forall j=N_1+1,N_1+2,\cdots,N\},\\
V^m(\Omega_h) &= \{f=(f_j)_{j=1}^{N_1} | f_j\in H^m(0,\ell_j),\ f_j(\ell_j)=0,\forall j=1,2,\cdots,N_1\},\\
V^m(\Omega_w) &= \{f=(f_j)_{j=N_1+1}^{N} | f_j\in H^m(0,\ell_j),\ f_j(\ell_j)=0,\forall j=N_1+1,N_1+2,\cdots,N\}.
\end{aligned}$$

Set the state space $\mathcal{H}$ as follows:

$$\mathcal{H}=\left\{(\theta,u)\in V^1(\Omega_h)\times V^1(\Omega_w)\ \middle|\ \begin{array}{l}\theta_k(0)=u_j(0),\\ k=1,2,\cdots,N_1,\\ j=N_1+1,N_1+2,\cdots,N\end{array}\right\}\times L^2(\Omega_w),$$

equipped with inner product: for $W=(\theta,u,z),\widetilde{W}=(\widetilde{\theta},\widetilde{u},\widetilde{z})\in\mathcal{H}$,

$$(W,\widetilde{W})_{\mathcal{H}}=\sum_{k=1}^{N_1}\int_0^{\ell_k}\theta_{k,x}\overline{\widetilde{\theta}_{k,x}}dx+\sum_{j=N_1+1}^{N}\int_0^{\ell_j}u_{j,x}\overline{\widetilde{u}_{j,x}}dx+\sum_{j=N_1+1}^{N}\int_0^{\ell_j}z_j\overline{\widetilde{z}_j}dx.$$

It is easy to check that $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a Hilbert space. Define the system operator $\mathcal{A}$ in $\mathcal{H}$ as follows:

$$\mathcal{A}\begin{bmatrix}\theta\\ u\\ z\end{bmatrix} = \begin{pmatrix}(\theta_{k,xx})_{k=1}^{N_1}\\ z\\ (u_{j,xx})_{j=N_1+1}^{N}\end{pmatrix},$$

the domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (\theta,u,z)\in\mathcal{H}\cap[H^2(\Omega_h)\times H^2(\Omega_w)\times V^1(\Omega_w)] \;\middle|\; \begin{array}{l}\theta_{k,xx}\in H^1(0,\ell_k),\\ \theta_{k,xx}(\ell_k)=0,\\ \theta_{k,xx}(0)=z_j(0),\\ k=1,2,\cdots,N_1,\\ j=N_1+1,N_1+2,\cdots,N,\\ \sum\limits_{k=1}^{N_1}\theta_{k,x}(0)+\sum\limits_{j=N_1+1}^{N}u_{j,x}(0)=0\end{array}\right\}.$$

We have the following result on the well-posedness and asymptotic stability of system (2).

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then $\mathcal{A}$ is dissipative in $\mathcal{H}$. $\mathcal{A}$ generates a C_0 semigroup of contractions on $\mathcal{H}$. Moreover, the energy of the system decays to zero as $t\to\infty$, if and only if one of the following two conditions is fulfilled,*

1). $N - N_1 = 1$;

2). $N - N_1 \geq 2$ and $\ell_i/\ell_j \notin Q$, $i,j = N_1+1, N_1+2,\cdots,N, i\neq j$.

Proof. Since the well-posedness of system (2) can be proved by the standard semigroup methods (see [11], [12], [24]), we omit it. Hence, we focus on proving the rest part of this theorem. the Proof by contradiction is mainly used here.

" $\Leftarrow$ " If the energy of the system does not decay, then due to the Lyubich-Phóng strong stability theorem (see [19]), we have that there exists at least one $\lambda_0 = i\sigma \in \sigma(\mathcal{A})$, $\sigma\in\mathbb{R}, \sigma\neq 0$ on the imaginary axis. Assume that $W\in\mathcal{D}(\mathcal{A})$ is an eigenvector of $\mathcal{A}$ corresponding to λ_0, where

$$W = \left((\theta_k)_{k=1}^{N_1}, (u_j)_{j=N_1+1}^{N}, \lambda_0(u_j)_{j=N_1+1}^{N}\right)^T.$$

We get

$$0 = \Re\lambda_0\|W\|_{\mathcal{H}}^2 = \Re(\mathcal{A}W,W)_{\mathcal{H}} = -\lambda_0^2\sum_{k=1}^{N_1}\int_0^{\ell_k}\theta_k^2dx = -\sum_{k=1}^{N_1}\int_0^{\ell_k}(\theta_{k,xx})^2dx,$$

which yields $\theta_k = \theta_{k,xx} = 0,\ k=1,2,\cdots,N_1$. Then by the boundary conditions and transmission conditions in (2), $\theta_k(x)$ and $u_j(x)$ satisfy the following equations:

$$\begin{cases}\theta_k(x)=0, \quad k=1,2,\cdots,N_1,\\ \lambda_0^2u_j-u_{j,xx}=0,\ j=N_1+1,N_1+2,\cdots,N,\\ u_j(\ell_j)=u_j(0)=0,\ j=N_1+1,N_1+2,\cdots,N,\\ \sum\limits_{j=N_1+1}^{N}u_{j,x}(0)=0.\end{cases} \tag{6}$$

If $N-N_1=1$, then by the last equation in (6), we get $u_{N,x}(0)=0$, which together with $u_N(0)=0$ implies $u_N(x)=0$. Thus $(\theta,u,\lambda_0u)=0$. It is a contradiction, since $(\theta,u,\lambda_0u)=0$ is an eigenvector corresponding to λ_0.

If $N-N_1\geq 2$, then by a direct calculation, we get $u_j=c_j\sinh\lambda_0 x,\ j=N_1+1,N_1+2,\cdots,N$, which satisfy

$$c_j\sinh\lambda_0\ell_j=0,\ j=N_1+1,N_1+2,\cdots,N,\quad\text{and}\quad\sum_{j=N_1+1}^{N}c_j=0.$$

Note that there are at least $c_{j_1},\ c_{j_2}\neq 0$ for some $j_1,\ j_2\geq N_1+1$, since $(\theta,u,\lambda_0 u)$ is the eigenvector corresponding to λ_0. Hence, $\sinh\lambda_0\ell_{j_1}=\sinh\lambda_0\ell_{j_2}=0$, which deduces that $\frac{\ell_{j_1}}{\ell_{j_2}}\in Q$. It contradicts to the condition 2) in Theorem 2.1.

" $\Rightarrow$ " If there exist $i_0,\ j_0$ satisfying $N_1+1\leq i_0,j_0\leq N,\ \frac{\ell_{i_0}}{\ell_{j_0}}=\frac{p}{q}$, p,q are nonzero integers, then it is easy to check that $\Big((0)_{k=1}^{N_1},(\widehat{u}_j(x))_{j=N_1+1}^{N},\lambda_0(\widehat{u}_j(x))_{j=N_1+1}^{N}\Big)$ is an eigenvector corresponding to eigenvalue $\lambda_0=i\frac{q\pi}{\ell_{j_0}}=i\frac{p\pi}{\ell_{i_0}}$, in which

$$\widehat{u}_{i_0}(x)=\sin(\frac{p\pi x}{\ell_{i_0}}),\ \widehat{u}_{j_0}(x)=-\sin(\frac{q_0\pi x}{\ell_{j_0}})\text{ and }u_j(x)=0,\forall N_1{+}1\leq j\leq N,j\neq i_0,j_0.$$

Note that the above eigenvector is sinusoidal wave concentrated in the wave equations, without any support in the heat ones. Thus we can build the solution $((\theta_k(x,t))_{k=1}^{N_1},(u_j(x,t))_{j=N_1+1}^{N})$ to system (2), such that

$$\theta_k(x,t){=}0,k=1,2,\cdots,N_1,\ u_{i_0}(x,t)=\widehat{u}_{i_0}(x)\cos(\frac{p\pi t}{\ell_{i_0}}),\ u_{j_0}(x,t)=\widehat{u}_{j_0}(x)\cos(\frac{q\pi t}{\ell_{j_0}})$$

and $u_j(x,t)=0,\ N_1+1\leq j\leq N,j\neq i_0,j_0$. Based on the construction of the above solution, we get $E(t)=E(0),\ t\geq 0$, which is a contradiction to that the energy of system decays to zero as $t\to\infty$. Hence, $\ell_i/\ell_j\notin Q,\ i,j=N_1+1,N_1+2,\cdots,N,\ i\neq j$. □

Remark 2. Note that Theorem 2.1 give a sufficient and necessary condition for the strong stability of this system. Thus, we always assume that the conditions in Theorem 2.1 are fulfilled, when discussing the decay rate of system (2).

3. **Lack of exponential decay rate.** This section is devoted to show the lack of exponential decay of heat-wave networks. We have the following result.

Theorem 3.1. *The energy of system (2) does not decay exponentially as $t\to\infty$, as soon as the network involves at least one wave equation.*

Proof. By means of WKB asymptotic analysis (see [25], [5]), we construct a kind of local ray-like approximate solutions for heat-wave networks, based on which, we will show that this kind of solutions does not decay exponentially. Thus, the heat-wave networks have no exponential decay.

Since there exists at least one wave equation in the networks, let us consider the following transmission problem on one joint node in the networks:

$$\begin{cases}\theta_{k,t}(x,t)-\theta_{k,xx}(x,t)=0,\ x\in(0,\ell_k),\ k=1,2,\cdots,N_1,\ t>0,\\ u_{j,tt}(x,t)-u_{j,xx}(x,t)=0,\ x\in(0,\ell_j),\ j=N_1+1,N_1+2,...N,\ t>0,\\ \theta_k(0,t)=u_j(0,t),\quad\forall k=1,2,\cdots,N_1,\ j=N_1+1,N_1+2,\cdots,N,\\ \sum\limits_{j=N_1+1}^{N}u_{j,x}(0,t)+\sum\limits_{k=1}^{N_1}\theta_{k,x}(0,t)=0,\ t>0.\end{cases}\tag{7}$$

For simplification, here we still use the same subscripts as in (2) to describe the transmission conditions at the joint node. We build the ray-like solutions of (7) by the following three steps. Here only the sketch of the construction is given. See Appendix 7.1 for more details.

Step 1). Assume that $\overline{u}^{\epsilon}_{\widehat{j}}(x,t) := e^{i(\tau t+\xi x)/\epsilon}\sum\limits_{n=0}^{\infty}\epsilon^n a^n_{\widehat{j}}(x,t),\ N_1+1\leq \widehat{j}\leq N$ is the incoming wave. We seek an approximate solutions for wave equations in (7) of the following WKB type with linear phase

$$\begin{cases} u^{\epsilon}_{\widehat{j}}(x,t)\sim \overline{u}^{\epsilon}_{\widehat{j}}(x,t)+\widetilde{u}^{\epsilon}_{\widehat{j}}(x,t),\\ u^{\epsilon}_{j}(x,t)\sim \widetilde{u}^{\epsilon}_{j}(x,t),\ j=\widehat{N}+1,N_1+2,\cdots,N, j\neq\widehat{j}, \end{cases} \tag{8}$$

in which

$$\widetilde{u}^{\epsilon}_{j}(x,t) := e^{i(\tau t-\xi x)/\epsilon}\sum_{n=0}^{\infty}\epsilon^{\frac{n}{2}}b^n_j(x,t),\ j=N_1+1,N_1+2,\cdots,N$$

where $\tau\neq 0,\ \xi\neq 0$ are real numbers satisfying $\tau^2=\xi^2$ and $\epsilon\in(0,1)$; the functions $a^n_{\widehat{j}}(x,t), b^n_j(x,t),\ n=0,1,2\cdots$, can be uniquely gotten from the initial conditions imposed at $x=0$: $a^n_{\widehat{j}}(0,t)=a^{n,0}_{\widehat{j}}(t)$ and $b^n_j(0,t)=b^{n,0}_j(t)$, $j=N_1+1,N_1+2,\cdots,N$.

Step 2). Build the approximate solutions for the heat equations in system (7). Since $\theta_i(0,t)=\theta_j(0,t),\ i,j=1,2,\cdots,N_1$ holds, set

$$\theta^{\epsilon}_k(0,t)\sim e^{i\tau t/\epsilon}\sum_{n=0}^{\infty}\epsilon^{n/2}f^n(t),\ k=1,2,\cdots,N_1$$

and

$$\theta^{\epsilon}_k(x,t)\sim e^{i(\tau t/\epsilon+x\widehat{\xi}/\sqrt{\epsilon})}\sum_{n=0}^{\infty}\epsilon^{n/2}B^n_k(x,t),\ k=1,2,\cdots,N_1, \tag{9}$$

where τ is the same as in (8) and

$$i\tau+\widehat{\xi}^2=0,\ \Im\widehat{\xi}>0. \tag{10}$$

$B^n_k(x,t),\ k=1,2,\cdots,N_1,\ n=0,1,2,\cdots$ can be identified uniquely from $f^n,\ n=0,1,2,\cdots$.

Step 3). Glue $u^{\epsilon}_j(x,t)$ and $\theta^{\epsilon}_k(x,t)$ by means of the transmission conditions in (7). Based on the transmission conditions at the joint node, all the functions $b^n_j(x,t),\ N_1+1\leq j\leq N$ and $B^n_k(x,t),\ 1\leq k\leq N_1,\ n=0,1,2,3,\cdots$ in (8) and (9) can be identified uniquely from $a^{n,0}_{\widehat{j}}(t),\ n=0,1,2,\cdots$, that is, the reflected waves in (8) and the solutions (9) to heat equations are determined uniquely by the incoming wave $\overline{u}^{\epsilon}_{\widehat{j}}(x,t)$.

Now, based on the above construction, let us consider the energy absorbed upon reflection. From (9), we deduce that for $j=1,2,\cdots,N_1$, the solutions $\theta^{\epsilon}_k(x,t)$ are all localized in $(0,O(\sqrt{\epsilon}))$. Indeed, out of $(0,O(\sqrt{\epsilon}))$, we get from (10) that $e^{i(\tau t/\epsilon+x\widehat{\xi}/\sqrt{\epsilon})}\to 0$ exponentially as $\epsilon\to 0$, which means that the solutions $\theta^{\epsilon}_k(x,t)$ vanish out of the domain $(0,O(\sqrt{\epsilon}))$ when $\epsilon\to 0$. Thus, for the constructed approximate solutions $(\theta^{\epsilon}_k)^{N_1}_{k=1},(u^{\epsilon}_j)^{N}_{j=N_1+1}$, we calculate directly that the energy dissipation between $t=0$ and $t=T$ is

$$\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}|\theta^{\epsilon}_{k,t}|^2dxdt\approx\sum_{k=1}^{N_1}\int_0^T\int_0^{M\sqrt{\epsilon}}|O(\frac{1}{\epsilon})|^2dxdt=O(\frac{1}{\epsilon^{\frac{3}{2}}}).$$

It is easy to get the total energy is $O(\frac{1}{\epsilon^2})$. Hence, compared to the total energy, the dissipated energy is negligible. Moreover, the negligible energy loss can be quantified as $\sqrt{\epsilon}\%$ of the total energy.

Based on the above constructed approximate solutions, let us show the non-uniform decay of heat-wave networks. Let $S(t), (\|S(t)\| \leq 1)$ be the semigroup of contractions corresponding to heat-wave networks. In order to get the non-uniform decay of this system, it is sufficient to show $\|S(T)\| = 1$ for any given $T > 0$. We consider a ray ℓ of length T, which locates in wave domains in (7). It is reflected at the boundary nodes and the joint nodes according to the law of Geometric Optics. Then a family of solutions is builded, which is concentrated along the ray considered above. When the ray hits the boundary nodes, the reflection follows the simple reflection rule at the Dirichlet boundary; while when the ray intersects the joint nodes like in (7), then the reflection occurs according to the above construction. Note that when the ray intersects the joint nodes as in (7), there are $N-N_1$ reflected waves, each of which is in the wave domain $(0, \ell_j)$, $j = N_1+1, N_1+2, \cdots, N$, respectively.

Firstly, assume that T is small enough that the ray only intersects the joint node one time. By the constructions of the ray-like solutions above, it is easy to see that $\|S(T)\| = 1$. Indeed, from the discussion above, we can get that for the constructed ray-like solutions,

$$\frac{\|(\theta^\epsilon(T), u^\epsilon(T), u_t^\epsilon(T))\|_{\mathcal{H}}}{\|(\theta^\epsilon(0), u^\epsilon(0), u_t^\epsilon(0))\|_{\mathcal{H}}} \approx \frac{\frac{1}{\epsilon^2} - \frac{1}{\epsilon^{\frac{3}{2}}}}{\frac{1}{\epsilon^2}} \to 1, \quad \epsilon \to 0.$$

Secondly, assume T is large. We have known that when the ray intersects the joint node like in (7), there are $N - N_1$ reflected waves (or outgoing waves) which is in each of the $N - N_1$ wave domains, respectively. Then these reflected waves continue to hit other joint nodes or Dirichlet boundaries. Let us consider one of these $N - N_1$ reflected waves:

If this reflected wave hits a joint node in which all wave equations involved, new reflected waves are generated for each wave domain around this node. It is well-known that there is no dissipation and the energy is conservative.

If this reflected wave hits the Dirichlet boundary, then all of the wave will reflect back according to the simple reflection rule at the Dirichlet boundary, which also implies there is no dissipation during the process.

If this reflected wave hits the joint node as in (7), then new reflected waves are generated again in each wave domain, respectively. The dissipated energy is negligible, which has been derived above.

As time goes, more and more reflected waves exist in the wave domains in the networks. However, as T is fixed, there are only finite reflections occurring at the joint nodes or boundaries during the time interval $[0, T]$. Note that by the above analysis, we have known that the dissipated energy is negligible during each reflection at the joint nodes as in (7). Hence, the total loss is also negligible and can be quantified as $M_T\sqrt{\epsilon}\%$ of the total energy, where M_T is a positive constant related to T. Hence, we get

$$\frac{\|(\theta^\epsilon(T), u^\epsilon(T), u_t^\epsilon(T))\|_{\mathcal{H}}}{\|(\theta^\epsilon(0), u^\epsilon(0), u_t^\epsilon(0))\|_{\mathcal{H}}} \approx \frac{\frac{1}{\epsilon^2} - M_T\frac{1}{\epsilon^{\frac{3}{2}}}}{\frac{1}{\epsilon^2}} \to 1, \quad \epsilon \to 0.$$

Thus, $\|S(T)\| = 1$ holds for all $T > 0$. Therefore, the heat-wave networks can not achieve exponential decay rate. The proof is complete. □

4. **Decay rate estimate. Case: $N - N_1 = 1$.** In this section, let us consider the case: a star-shaped network with $N-1$ heat equations and only one wave equation (see Fig. 3). From the last section, we know that the energy of this kind of system always does not decay exponentially. So we shall further discuss the polynomial decay rate of this case. In fact, a sharp polynomial decay rate is derived.

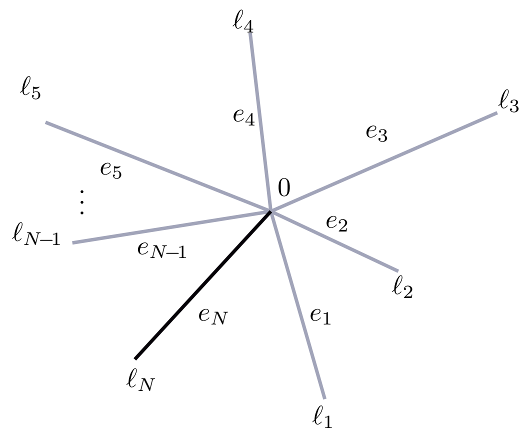

FIGURE 3. network with $N-1$ heat equations (grey) and 1 wave equation (black)

4.1. **Statement of the main result.** Similar to the denotation in system (2), we get the following equations to describe this case:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, \ x \in (0, \ell_N), \ t > 0, \\ \theta_{k,t}(x,t) - \theta_{k,xx}(x,t) = 0, \ x \in (0, \ell_k), \ k = 1, 2, \cdots, N-1, \ t > 0, \\ u(\ell_N, t) = \theta_k(\ell_k, t) = 0, \ t > 0, \ k = 1, 2 \cdots, N-1, \\ \theta_k(0,t) = u(0,t), \ \forall k = 1, 2, 3, \cdots, N-1, \\ \sum_{k=1}^{N-1} \theta_{k,x}(0,t) + u_x(0,t) = 0, \ t > 0, \\ u(t=0) = u^0, \quad u_t(t=0) = u^1, \\ \theta_k(t=0) = \theta_k^0, \ k = 1, 2, \cdots, N-1. \end{cases} \tag{11}$$

Here and in this section below, for convenience, the displacement of the wave equation is always denoted by $u(x,t)$ without any subscript, since there is only one wave equation in this case.

Based on the frequency domain method, we obtain the following result on the stability of system (11).

Theorem 4.1. *The $(S(t))_{t\geq 0}$ associated with system (11) decays polynomially as*

$$\|S(t)W_0\| \leq \frac{C}{t^2} \|W_0\|_{\mathcal{D}(\mathcal{A})}, \tag{12}$$

Moreover, it is the sharp polynomial decay rate for this system.

Remark 3. From Theorem 4.1, we find that the polynomial decay rate of the energy of system (11) is t^{-2}, no matter how many edges described by heat equations in this network. It means that the polynomial decay rate is not changed by adding or reducing the heat equations in the star-shaped networks, as long as the networks contain only one wave equation.

4.2. **Polynomial decay rate (Proof of Theorem 4.1).** In this subsection, we shall deduce the polynomial decay rate of system (11) based on the frequency description. Let us introduce the following lemma from [8] (see also [16]).

Lemma 4.2. *A C_0 semigroup $e^{t\mathcal{A}}$ of contractions on a Hilbert space satisfies*

$$\|e^{t\mathcal{A}}U_0\| \le Ct^{-\frac{1}{\ell}}\|U_0\|_{\mathcal{D}(\mathcal{A})}, \quad \forall U_0 \in \mathcal{D}(\mathcal{A}),\ t \to \infty$$

for some constant $C>0$, if and only if the following conditions hold:
1). $\{i\beta | \beta \in \mathbb{R}\} \subset \rho(\mathcal{A})$;
2). $\limsup\limits_{|\beta|\to\infty} \frac{1}{\beta^\ell}\|(i\beta - \mathcal{A})^{-1}\| < \infty$.

Proof of Theorem 4.1. We mainly prove this theorem by checking the two conditions in Lemma 4.2. Note that the condition 1) has been proved in Theorem 2.1. Thus it is sufficient to show that

$$\lim_{\sigma\to\infty} \sup \frac{1}{\sigma^{\frac{1}{2}}}\|(i\sigma I - \mathcal{A})^{-1}\| < \infty. \tag{13}$$

The trick proposed by Liu *et al.* [14] and [16] is employed to show the above estimation. If (13) is false, then there exists $T_n = \frac{1}{\sigma_n^{\frac{1}{2}}}(i\sigma_n I - \mathcal{A})^{-1}$, such that $\|T_n\|_{\mathcal{H}} \to \infty, \quad n \to \infty$. By Banach-Steinhaus theorem, there exists $F \in \mathcal{H}$ such that

$$T_n F = \frac{1}{\sigma_n^{\frac{1}{2}}}(i\sigma_n I - \mathcal{A})^{-1}F = \widetilde{W}_n \to \infty, \text{ in the sense of norm, } n \to \infty.$$

Thus,

$$\sigma_n^{\frac{1}{2}}(i\sigma_n I - \mathcal{A})\frac{\widetilde{W}_n}{\|\widetilde{W}_n\|} = \frac{F}{\|\widetilde{W}_n\|} \to 0, \text{ in the sense of norm, } n \to \infty.$$

Hence, there exists a sequence $W^n = (((\theta_j^n)_{j=1}^{N-1}, u^n, z^n) \in \mathcal{D}(\mathcal{A})$ with $\|W^n\|_{\mathcal{H}} = 1$, and a sequence $\sigma_n \in \mathbb{R}$ with $\sigma_n \to \infty$ such that $\lim\limits_{n\to\infty} \sigma_n^{\frac{1}{2}}\|(i\sigma_n I - \mathcal{A})W^n\|_{\mathcal{H}} = 0$, i.e.,

$$\sigma_n^{\frac{1}{2}}[i\sigma_n u^n - z^n] \quad \to \quad 0, \quad in\ H^1(0,\ell_N), \tag{14}$$

$$\sigma_n^{\frac{1}{2}}[i\sigma_n z^n - u^n_{xx}] \quad \to \quad 0, \quad in\ L^2(0,\ell_N), \tag{15}$$

$$\sigma_n^{\frac{1}{2}}[i\sigma_n \theta_k^n - \theta^n_{k,xx}] \quad \to \quad 0, \quad in\ H^1(0,\ell_k),\ k = 1,2,\cdots,N-1. \tag{16}$$

Note that $\sigma_n^{\frac{1}{2}}\sum\limits_{k=1}^{N-1}\int_0^{\ell_k}|\theta^n_{k,xx}|^2dx = \Re(\sigma_n^{\frac{1}{2}}(i\sigma_n - \mathcal{A})W^n, W^n)_{\mathcal{H}} \to 0$. Hence,

$$\sigma_n^{\frac{1}{4}}\theta^n_{k,xx} \to 0, \quad in\ L^2(0,\ell_k),\ k = 1,2,\cdots,N-1. \tag{17}$$

Then from (16), it is easy to get that

$$\sigma_n^{\frac{5}{4}}\theta_k^n \to 0, \quad in\ L^2(0,\ell_k),\ k = 1,2,\cdots,N-1. \tag{18}$$

Using Gagliardo-Nirenberg inequality (see [17]),

$$\begin{aligned}
\|\sigma_n\theta_k^n\|_{L^\infty} &\le C_1\|\sigma_n\theta^n_{k,xx}\|_{L^2}^{\frac{1}{4}}\|\sigma_n\theta_k^n\|_{L^2}^{\frac{3}{4}} + C_2\|\sigma_n\theta_k^n\|_{L^2} \\
&= C_1\left\|\sigma_n^{\frac{1}{4}}\theta^n_{k,xx}\right\|_{L^2}^{\frac{1}{4}}\left\|\sigma_n^{\frac{5}{4}}\theta_k^n\right\|_{L^2}^{\frac{3}{4}} + C_2\|\sigma_n\theta_k^n\|_{L^2} \\
&\to 0, \quad n \to \infty,
\end{aligned}$$

where $C_j,\ j=1,2$ are positive constants. Thus,

$$\sigma_n\theta_k^n(0)\to 0,\quad n\to\infty,\ k=1,2,\cdots,N-1. \tag{19}$$

Similarly, using Gagliardo-Nirenberg inequality again on $\theta_{j,x}^n$, we get

$$\|\theta_{k,x}^n\|_{L^\infty}\le C_1\|\theta_{k,xx}^n\|_{L^2}^{\frac12}\|\theta_{k,x}^n\|_{L^2}^{\frac12}+C_2\|\theta_{k,x}^n\|_{L^2},\quad k=1,2,\cdots,N-1, \tag{20}$$

which implies that $\|\theta_{k,x}^n\|_{L^\infty},\ k=1,2,\cdots,N-1$ are bounded and hence

$$|\theta_{k,x}^n(0)|,\ |\theta_{k,x}^n(\ell_k)|,\ k=1,2\cdots,N-1$$

are bounded. Then, taking the inner product of (16) with $\sigma_n\theta_k(x)$, we have

$$(i\sigma_n\theta_k^n,\sigma_n\theta_k^n)-(\theta_{k,xx}^n,\sigma_n\theta_k^n)\to 0, k=1,2,\cdots,N-1.$$

Thus, by integration by parts,

$$(i\sigma_n\theta_k^n,\sigma_n\theta_k^n)-\theta_{k,x}^n\overline{\sigma_n\theta_k^n}|_0^{\ell_k}+(\theta_{k,x}^n,\sigma_n\theta_{k,x}^n)\to 0,\quad k=1,2,\cdots,N-1. \tag{21}$$

Note that $(i\sigma_n\theta_k^n,\sigma_n\theta_k^n)$ is convergent to 0 due to (18). Then, by the boundedness of $|\theta_{k,x}^n(0)|$ and (19),

$$\sigma_n^{\frac12}\theta_{k,x}^n\to 0,\quad n\to\infty,\quad in\ L^2(0,\ell_k),\ k=1,2,\cdots,N-1. \tag{22}$$

Then using Gagliardo-Nirenberg inequality again, we have

$$\|\sigma_n^{\frac14}\theta_{k,x}^n\|_{L^\infty}\le C_1\|\theta_{k,xx}^n\|_{L^2}^{\frac12}\|\sigma_n^{\frac12}\theta_{k,x}^n\|_{L^2}^{\frac12}+C_2\|\sigma_n^{\frac14}\theta_{k,x}^n\|_{L^2}\to 0,\ k=1,2,\cdots,N-1, \tag{23}$$

which implies that

$$|\sigma_n^{\frac14}\theta_{k,x}^n(0)|\to 0,\quad n\to\infty,\ k=1,2,\cdots,N-1. \tag{24}$$

Thus, using the transmission conditions in system (11), that is, $\theta_k^n(0)=u^n(0)$ and $\sum_{k=1}^{N-1}\theta_{k,x}^n(0)+u_x^n(0)=0$, together with (19) and (24), we deduce that

$$\sigma_n u^n(0),\ u_x^n(0)\to 0,\quad n\to\infty. \tag{25}$$

Replacing z^n in (14) by $i\sigma_n u^n$ in view of (15) yields

$$-\sigma_n^2u^n-u_{xx}^n\to 0,\quad in\ L^2(0,\ell_N).$$

Taking the inner product of the above with $(\ell_N-x)u_x^n$, we have

$$((\ell_N-x)u_x^n,-\sigma_n^2u^n)-((\ell_N-x)u_x^n,u_{xx}^n)\to 0,\quad n\to\infty. \tag{26}$$

Note that

$$((\ell_N-x)u_x^n,-\sigma_n^2u^n)=-\sigma_n^2(\ell_N-x)u^n\overline{u^n}|_0^{\ell_N}+\int_0^{\ell_N}\sigma_n^2u^n\overline{[-u^n+(\ell_N-x)u_x^n]}dx.$$

Thus, by (25),

$$\Re((\ell_N-x)u_x^n,-\sigma_n^2u^n)\to-\frac12\int_0^{\ell_N}\sigma_n^2|u^n|^2dx. \tag{27}$$

On the other hand, integrating the second term in (26) by parts yields

$$((\ell_N-x)u_x^n,u_{xx}^n)=(\ell_N-x)u_x^n\overline{u_x^n}|_0^{\ell_N}-\int_0^{\ell_N}[-u_x^n+(\ell_N-x)u_{xx}^n]\overline{u_x^n}dx. \tag{28}$$

By (28),

$$\Re((\ell_n-x)u_x^n,u_{xx}^n)=\frac12\int_0^{\ell_N}|u_x^n|^2dx. \tag{29}$$

Hence, by (26)–(29),

$$2\Re\{((\ell_N-x)u_x^n,-\sigma_n^2u^n)-((\ell_N-x)u_x^n,u_{xx}^n)\}=-\int_0^{\ell_N}\sigma_n^2|u^n|^2dx-\int_0^{\ell_N}|u_x^n|^2dx\to 0. \tag{30}$$

Therefore,

$$\sigma_n u^n,\ u_x^n\to 0,\quad in\ \ L^2(0,\ell_N),\ n\to\infty. \tag{31}$$

So by (14), we get

$$z^n\to 0,\quad in\ \ L^2(0,\ell_N),\ n\to\infty. \tag{32}$$

Hence, by (24), (31), (32), we have

$$W^n=((\theta_k^n)_{k=1}^{N-1},u^n,z^n)\to 0,\ in\ \ \mathcal{H},\quad n\to\infty,$$

which contradicts to $\|W^n\|_{\mathcal{H}}=1$. Hence, (13) holds. Thus By Lemma 4.2, we get (12).

Now, let us further show the sharpness of the polynomial decay rate t^{-2}, i.e., the decay rate can not be faster than t^{-2} for system (11). For this aim, we first get the following result.

Lemma 4.3. *There exists at least one sequence (σ_n,F_n) satisfying $\sigma_n\to+\infty,\ n\to\infty$ and*

$$\|(i\sigma_nI-\mathcal{A})^{-1}F_n\|^2\geq\widetilde{C}_1\sigma_n+\widetilde{C}_2. \tag{33}$$

where $F^n\in\mathcal{H}$ and $\|F^n\|_{\mathcal{H}}$ is bounded; $\widetilde{C}_j,\ j=1,2$ are some positive constants.

Proof. See Appendix 7.2. ☐

Based on this lemma, let us prove the sharpness by deducing contradiction if the decay rate can be improved. If t^{-2} is not the sharp polynomial decay rate, then there exists a small constant $\epsilon>0$ such that $\|S(t)W_0\|\leq Ct^{-2-\epsilon}\|W_0\|_{\mathcal{D}(\mathcal{A})}$. By Lemma 4.2, we have that

$$\lim_{|\sigma|\to\infty}\sup\frac{1}{\frac{1}{\sigma^{2+\epsilon}}}\|(i\sigma-\mathcal{A})^{-1}\|<\infty,$$

which implies that when σ is sufficiently large,

$$\|(i\sigma-\mathcal{A})^{-1}\|^2\leq C\sigma^{\frac{2}{2+\epsilon}}. \tag{34}$$

On the other hand, by (33), we have that there at least exists one (σ_n,F_n) such that

$$\|(i\sigma_nI-\mathcal{A})^{-1}F_n\|^2\geq\widetilde{C}_1\sigma_n+\widetilde{C}_2,\quad \sigma_n\to+\infty,$$

which contradicts to (34). Therefore, t^{-2} is the sharp decay rate for the solution of system (11). The proof of Theorem 4.1 is complete. ☐

5. **Decay rate estimate. Case:** $N-N_1\geq 2$**.** In the previous section, we have discussed the decay rate of system (2), which contains only one wave equation in the network. This section is devoted to discuss the long time behavior of the system (2) for the rest case, in which more than one wave equation involved.

5.1. **Statement of the main result.** In order to further discuss the decay rate of the energy of system (2), we introduce the definitions of some sets for the irrational number from [9].

Definition 5.1. ([9]) 1. Set B_ϵ: for all $\epsilon>0$ there exists a set $B_\epsilon\subset\mathbb{R}$, such that the Lebesgue measure of $\mathbb{R}\setminus B_\epsilon$ is equal to zero, and a constant $C_\epsilon>0$ for which, if $\xi\in B_\epsilon$, then $|||\xi m|||\geq\frac{C_\epsilon}{m^{1+\epsilon}}$, where $|||\eta|||$ is the distance from η to the set $\mathbb{Z}$: $|||\eta|||:=\min\limits_{\eta-x\in\mathbb{Z}}|x|$.

2. Set $\mathcal{S}$: the set of all real numbers ρ such that $\rho\notin Q$ and so that its expansion as a continued fraction $[0,a_1,a_2,\cdots,a_n,\cdots]$ is such that (a_n) is bounded. In particular $\mathcal{S}$ is contained in the sets B_ϵ for every $\epsilon>0$.

Definition 5.2. ([9]) We call that real numbers $\ell_1,\ell_2,\cdots,\ell_m$ verify the conditions (S), if $\ell_1,\ell_2,\cdots,\ell_m$ are linearly independent over the field Q of rational numbers; and the ratios ℓ_i/ℓ_j are algebraic numbers for $i,j=1,2,\cdots,m$.

Then we have the following weakened observability estimate for system (2).

Theorem 5.3. *There exists positive constants T and C, such that*

$$\begin{aligned}&C\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{\widetilde{s}}\theta_k(x,t)}{\partial t^{\widetilde{s}}})^2+(\frac{\partial^{1+\widetilde{s}}\theta_k(x,t)}{\partial t^{1+\widetilde{s}}})^2+(\frac{\partial^{2+\widetilde{s}}\theta_k(x,t)}{\partial t^{2+\widetilde{s}}})^2]dxdt\\ \geq\;&\|(\theta^0,u^0,u^1)\|^2_{\mathcal{H}},\end{aligned}\tag{35}$$

where $\widetilde{s}$ is given as follows:

- *if $\frac{\ell_i}{\ell_j}\in B_\epsilon,\ i,j=N_1+1,N_1+2,\cdots,N,i\neq j$, then $\widetilde{s}=1+[N-N_1-1+\epsilon]$;*
- *if $\frac{\ell_i}{\ell_j}\in\mathcal{S},\ i,j=N_1+1,N_1+2,\cdots,N,\ i\neq j$, then $\widetilde{s}=N-N_1$;*
- *if $\ell_j\ (j=N_1+1,N_1+2,\cdots,N)$ satisfy the condition (S), then $\widetilde{s}=1+[1+\epsilon]$.*

Here $[\cdot]$ is denoted by the integer part of its inside.

Based on the observability estimate, we show that system (2) can achieve polynomial decay rate.

Theorem 5.4. *For any $(\theta^0,u^0,u^1)\in\mathcal{D}(\mathcal{A})$, there always exists a constant $C>0$ such that the energy of system (2) satisfies*

$$E(t)\leq C\frac{1}{t^{\frac{1}{\widetilde{s}+1}}}\|(\theta^0,u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A})},\ \forall t\geq 0,\tag{36}$$

where $\widetilde{s}$ is the same as in Theorem 5.3.

In the above theorem, we have derived that when the lengths of strings in the network satisfy the conditions in Theorem 5.3, the polynomial decay rate holds for the system. However, if all of these conditions are not satisfied, that is, some of $\frac{\ell_i}{\ell_j},\ i,j=N_1+1,N_1+2,\cdots,N$ do not belong to B_ϵ, $\mathcal{S}$ and $\ell_j(j=N_1+1,N_1+2,\cdots,N)$ do not satisfy the condition (S), the decay rate of system (2) becomes complicated and interesting. Note that under this assumption, the result in Theorem 5.4 no longer holds, since one can not find a suitable constant $\widetilde{s}>0$ for (35). It means that the energy of the system does not decay polynomially for any smooth initial conditions under this assumption. However, we prove that system (2) can achieve logarithmic decay rate for a special case. The result on logarithmic decay rate of system (2) will be given in subsection 5.4.

Remark 4. While the resolvent method yields optimal decay rates in last section, the method based on replacing the heat-wave networks by the pure wave ones and using observability inequalities, does not lead to sharp decay rates. Please see Appendix 7.4.

5.2. **Observability estimate (Proof of Theorem 5.3).** In order to deduce the observability inequality in Theorem 5.3, we divide the system (2) into two systems. Set

$$(\theta, u, z) = (p, w, v) + (\widetilde{p}, \widetilde{w}, \widetilde{v}), \tag{37}$$

where (p, u, v) satisfies

$$\begin{cases} p_{k,tt}(x,t) - p_{k,xx}(x,t) = 0, \ x \in (0, \ell_k), \ k = 1, 2, \cdots, N_1, \ t > 0, \\ w_{j,tt}(x,t) - w_{j,xx}(x,t) = 0, \ x \in (0, \ell_j), \ j = N_1+1, N_1+2, ...N, \ t > 0, \\ p_k(\ell_k, t) = w_j(\ell_j, t) = 0, \ k = 1, 2, \cdots, N_1, \ j = N_1+1, N_1+2, \cdots, N, \ t > 0, \\ p_k(0,t) = w_j(0,t), \ \forall k = 1, 2, \cdots, N_1, \ j = N_1+1, N_1+2, \cdots, N, \ t > 0, \\ \sum\limits_{j=N_1+1}^{N} w_{j,x}(0,t) + \sum\limits_{k=1}^{N_1} p_{k,x}(0,t) = 0, \ t > 0, \\ p_k(t=0) = \theta_k^0, \quad p_{k,t}(t=0) = \theta_{k,xx}^0, \ k = 1, 2, \cdots, N_1, \\ w_j(t=0) = u_j^0, \quad w_{j,t}(t=0) = u_j^1, \ j = N_1+1, N_1+2, \cdots, N, \end{cases} \tag{38}$$

and $(\widetilde{p}, \widetilde{w}, \widetilde{v})$ satisfies

$$\begin{cases} \widetilde{p}_{k,tt}(x,t) - \widetilde{p}_{k,xx}(x,t) = \theta_{k,tt}(x,t) - \theta_{k,t}(x,t), x \in (0, \ell_k), k = 1, 2, \cdots, N_1, t > 0, \\ \widetilde{w}_{j,tt}(x,t) - \widetilde{w}_{j,xx}(x,t) = 0, \ x \in (0, \ell_j), j = N_1+1, N_1+2, ...N, \ t > 0, \\ \widetilde{p}_k(\ell_k, t) = \widetilde{w}_j(\ell_j, t) = 0, \ \forall k = 1, 2, \cdots, N_1, \ j = N_1+1, N_1+2, \cdots, N, \ t > 0, \\ \widetilde{p}_k(0,t) = \widetilde{w}_j(0,t), \ \forall k = 1, 2, \cdots, N_1, \ j = N_1+1, N_1+2, \cdots, N, \ t > 0, \\ \sum\limits_{j=N_1+1}^{N} \widetilde{w}_{j,x}(0,t) + \sum\limits_{k=1}^{N_1} \widetilde{p}_{k,x}(0,t) = 0, \ t > 0, \\ \widetilde{p}_k(t=0) = 0, \quad \widetilde{p}_{k,t}(t=0) = 0, \ k = 1, 2, \cdots, N_1, \\ \widetilde{w}_j(t=0) = 0, \quad \widetilde{w}_{j,t}(t=0) = 0, \ j = N_1+1, N_1+2, \cdots, N. \end{cases} \tag{39}$$

Firstly, we consider system (38). Assume that λ_n are the eigenvalues of the corresponding operator A for system (38) and $\phi_n = (\phi_j^n)_{j=1}^N$ is its corresponding eigenvector. Note that system (38) is a conservative wave system. Thus, the initial state of system (38) can be expanded as follows

$$(\theta^0, u^0)^T = \sum_{n \geq 1} a_n \phi_n(x), \quad (\theta_{xx}^0, u^1) = \sum_{n \geq 1} b_n \phi_n(x). \tag{40}$$

Proposition 1. *There exists a positive constant $T > 0$ such that*

$$\sum_{k=1}^{N_1} \int_0^T \int_0^{\ell_k} [p_{k,t}^2 + p_k^2] dx dt \geq \sum_{n \geq 1} \gamma_n^2 [\lambda_n^2 a_n^2 + b_n^2], \tag{41}$$

where $\gamma_n^2 > 0$ is the weights, which determined by the lengths of the strings involved in the network.

Proof. See Appendix 7.3. □

Remark 5. Note that by the proof of Proposition 1, together with [9], we can see that the weights γ_n in (41) satisfy

$$\gamma_n = \max_{i=N_1+1,N_1+2,\cdots,N} \prod_{j=N_1+1,\ j\neq i}^{N} |\sin(\lambda_n \ell_j)|,\ \forall n \geq 1,$$

and the condition $\inf_{n>0} \gamma_n^2 = c > 0$ does not hold for the star-shaped network (2) and other general networks. In fact, it always holds that $\liminf_{n\to\infty} \gamma_n^2 = 0$. The weights γ_n^2 can be determined by the ratios $\frac{\ell_i}{\ell_j}$, where $\ell_i, \ell_j,\ i,j = N_1+1, N_1+2, \cdots, N,\ i \neq j$.

When the irrational numbers ℓ_i/ℓ_j belong to different sets, we obtain different estimates for γ_k, respectively (see [9], [27]).

Lemma 5.5. *Let λ_n be the eigenvalues of the corresponding operator A for system (38) and γ_n is the same as in (41). Then*

(1) if $\frac{\ell_i}{\ell_j} \in B_\epsilon,\ i,j = N_1+1, N_1+2, \cdots, N,\ i \neq j$, then $\gamma_n \geq \frac{c_\epsilon}{\lambda_n^{N-N_1-1+\epsilon}},\ n \geq 1,\ \epsilon > 0$;

(2) if $\frac{\ell_i}{\ell_j} \in \mathcal{S},\ i,j = N_1+1, N_1+2, \cdots, N,\ i \neq j$, then $\gamma_n \geq \frac{c}{\lambda_n^{N-N_1-1}},\ n \geq 1$;

(3) if ℓ_j $(j = N_1+1, N_1+2, \cdots, N)$ satisfy the the condition (S), then $\gamma_n \geq \frac{c_\epsilon}{\lambda_n^{1+\epsilon}},\ n \geq 1,\ \epsilon > 0$.

From Lemma 5.5, we know that under the above three conditions on the lengths of the strings in the network, there always exists $s > 0$ such that

$$\gamma_n \geq \lambda_n^{-s},\ n \geq 1. \tag{42}$$

Set

$$\widetilde{s} := [s] + 1, \tag{43}$$

where $[s]$ is the integer part of s given as in (42).

Corollary 1. *There exists a positive constant T such that*

$$\sum_{k=1}^{N_1} \int_0^T \int_0^{\ell_k} [(\frac{\partial^{1+\widetilde{s}} p_k(x,t)}{\partial t^{1+\widetilde{s}}})^2 + (\frac{\partial^{\widetilde{s}} p_k(x,t)}{\partial t^{\widetilde{s}}})^2] dxdt \geq \|(\theta^0, u^0, u^1)\|_{\mathcal{H}}^2. \tag{44}$$

Proof. Let $\widetilde{s}$ be defined as (43). By Proposition 1, it is easy to check that

$$\sum_{k=1}^{N_1} \int_0^T \int_0^{\ell_k} [p_{k,t}^2 + p_k^2] dxdt \geq \sum_{n\geq 1} \lambda_n^{-2\widetilde{s}} [\lambda_n^2 a_n^2 + b_n^2]. \tag{45}$$

Set

$$\begin{pmatrix} \widetilde{\theta}^0 \\ \widetilde{u}^0 \end{pmatrix} = A^{\widetilde{s}} \begin{pmatrix} \theta^0 \\ u^0 \end{pmatrix} = A^{\widetilde{s}} \sum_{n\geq 1} a_n \phi_n = \sum_{n\geq 1} \lambda_n^{\widetilde{s}} a_n \phi_n.$$

Similarly, we have

$$\begin{pmatrix} \widetilde{\theta}_{xx}^0 \\ \widetilde{u}^1 \end{pmatrix} = A^{\widetilde{s}} \begin{pmatrix} \theta_{xx}^0 \\ u^1 \end{pmatrix} = A^{\widetilde{s}} \sum_{n\geq 1} b_n \phi_n = \sum_{n\geq 1} \lambda_n^{\widetilde{s}} b_n \phi_n,$$

where (θ^0, u^0) and (θ^0_{xx}, u^1) are the same as (40). It is easy to check that $(\frac{\partial^{\widetilde{s}}p}{\partial t^{\widetilde{s}}}, \frac{\partial^{\widetilde{s}}w}{\partial t^{\widetilde{s}}}, \frac{\partial^{\widetilde{s}}v}{\partial t^{\widetilde{s}}})$ is a solution to system (38) with initial condition $(\begin{pmatrix} \widetilde{\theta}^0 \\ \widetilde{u}^0 \end{pmatrix}, \begin{pmatrix} \widetilde{\theta}^0_{xx} \\ \widetilde{u}^1 \end{pmatrix})$. Therefore, by (45), we get that

$$\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{1+\widetilde{s}}p_k}{\partial t^{1+\widetilde{s}}})^2+(\frac{\partial^{\widetilde{s}}p_k}{\partial t^{\widetilde{s}}})^2]dxdt \geq \sum_{n\geq 1}\lambda_n^{-2\widetilde{s}}\lambda_n^{2\widetilde{s}}[\lambda_n^2a_n^2+b_n^2] = \sum_{n\geq 1}[\lambda_n^2a_n^2+b_n^2].$$

Note that $\sum_{n\geq 1}[\lambda_n^2a_n^2+b_n^2] \geq \|(\theta^0, u^0, u^1)\|_{\mathcal{H}}$. Hence (44) follows. The proof is complete. □

Secondly, let us consider system (39). We get the following result.

Proposition 2. *There exists positive constants T and C such that*

$$\begin{aligned} &C\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{\widetilde{s}}\theta_k(x,t)}{\partial t^{\widetilde{s}}})^2+(\frac{\partial^{1+\widetilde{s}}\theta_k(x,t)}{\partial t^{1+\widetilde{s}}})^2+(\frac{\partial^{2+\widetilde{s}}\theta_k(x,t)}{\partial t^{2+\widetilde{s}}})^2]dxdt \\ \geq\ & \sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{\widetilde{s}}\widetilde{p}_k(x,t)}{\partial t^{\widetilde{s}}})^2+(\frac{\partial^{1+\widetilde{s}}\widetilde{p}_k(x,t)}{\partial t^{1+\widetilde{s}}})^2]dxdt. \end{aligned} \tag{46}$$

Proof. Set

$$\begin{cases} \widetilde{\widetilde{p}}_k = \frac{\partial^{\widetilde{s}}\widetilde{p}_k}{\partial t^{\widetilde{s}}},\ k=1,2,\cdots,N_1, \\ \widetilde{\widetilde{w}}_j = \frac{\partial^{\widetilde{s}}\widetilde{w}_j}{\partial t^{\widetilde{s}}},\ j=N_1+1,N_1+2,\cdots,N. \end{cases} \tag{47}$$

It is easy to see that $\widetilde{\widetilde{p}}_k, \widetilde{\widetilde{w}}_j$ satisfy the following equations:

$$\begin{cases} \widetilde{\widetilde{p}}_{k,tt}(x,t)-\widetilde{\widetilde{p}}_{k,xx}(x,t)=\frac{\partial^{2+\widetilde{s}}\theta_k(x,t)}{\partial t^{2+\widetilde{s}}}-\frac{\partial^{1+\widetilde{s}}\theta_k(x,t)}{\partial t^{1+\widetilde{s}}},\ x\in(0,\ell_k),\ k=1,2,\cdots,N_1, \\ \widetilde{\widetilde{w}}_{j,tt}(x,t)-\widetilde{\widetilde{w}}_{j,xx}(x,t)=0,\ x\in(0,\ell_j),\ j=N_1+1,N_1+2,...N, \\ \widetilde{\widetilde{p}}_k(\ell_k,t)=\widetilde{\widetilde{w}}_j(\ell_j,t)=0,\ k=1,2,\cdots,N_1,\ j=N_1+1,N_1+2,\cdots,N, \\ \widetilde{\widetilde{p}}_k(0,t)=\widetilde{\widetilde{w}}_j(0,t),\ \forall k=1,2,\cdots,N_1,\ j=N_1+1,N_1+2,\cdots,N, \\ \sum_{j=N_1+1}^{N}\widetilde{\widetilde{w}}_{j,x}(0,t)+\sum_{k=1}^{N_1}\widetilde{\widetilde{p}}_{k,x}(0,t)=0, \\ \widetilde{\widetilde{p}}_k(t=0)=0,\quad \widetilde{\widetilde{p}}_{k,t}(t=0)=0,\ k=1,2,\cdots,N_1, \\ \widetilde{\widetilde{w}}_j(t=0)=0,\quad \widetilde{\widetilde{w}}_{j,t}(t=0)=0,\ j=N_1+1,N_1+2,\cdots,N. \end{cases} \tag{48}$$

Let

$$E_{(\widetilde{\widetilde{p}},\widetilde{\widetilde{p}}_t,\widetilde{\widetilde{w}},\widetilde{\widetilde{w}}_t)}(t)=\sum_{k=1}^{N_1}\int_0^{\ell_k}[\widetilde{\widetilde{p}}^2_{k,x}+\widetilde{\widetilde{p}}^2_{k,t}]dx+\sum_{j=N_1+1}^{N}\int_0^{\ell_j}[\widetilde{\widetilde{w}}^2_{j,x}+\widetilde{\widetilde{w}}^2_{j,t}]dx$$

be the energy of system (48). It is easy to check that

$$\frac{dE_{(\widetilde{\widetilde{p}},\widetilde{\widetilde{p}}_t,\widetilde{\widetilde{w}},\widetilde{\widetilde{w}}_t)}(t)}{dt}=\sum_{k=1}^{N_1}\int_0^{\ell_k}\widetilde{\widetilde{p}}_{k,t}(x,t)[\frac{\partial^{2+\widetilde{s}}\theta_k(x,t)}{\partial t^{2+\widetilde{s}}}-\frac{\partial^{1+\widetilde{s}}\theta_k(x,t)}{\partial t^{1+\widetilde{s}}}]dx. \tag{49}$$

Hence,

$$\begin{aligned}
&E_{(\widetilde{\widetilde{p}},\widetilde{\widetilde{p}}_t,\widetilde{\widetilde{w}},\widetilde{\widetilde{w}}_t)}(t)-E_{(\widetilde{\widetilde{p}},\widetilde{\widetilde{p}}_t,\widetilde{\widetilde{w}},\widetilde{\widetilde{w}}_t)}(0)=E_{(\widetilde{\widetilde{p}},\widetilde{\widetilde{p}}_t,\widetilde{\widetilde{w}},\widetilde{\widetilde{w}}_t)}(t)\\
=&\sum_{k=1}^{N_1}\int_0^t\int_0^{\ell_k}\widetilde{\widetilde{p}}_{k,t}(x,t)[\frac{\partial^{2+\widetilde{s}}\theta_k(x,t)}{\partial t^{2+\widetilde{s}}}-\frac{\partial^{1+\widetilde{s}}\theta_k(x,t)}{\partial t^{1+\widetilde{s}}}]dxdt\\
\le&\frac{1}{2}\sum_{k=1}^{N_1}\Bigg[\int_0^t\int_0^{\ell_k}(\widetilde{\widetilde{p}}_{k,t}(x,\tau))^2dxd\tau\\
&+\int_0^t\int_0^{\ell_k}[\frac{\partial^{2+\widetilde{s}}\theta_k(x,\tau)}{\partial\tau^{2+\widetilde{s}}}-\frac{\partial^{1+\widetilde{s}}\theta_k(x,\tau)}{\partial\tau^{1+\widetilde{s}}}]^2dxd\tau\Bigg]\\
\le&\frac{1}{2}\Bigg[\int_0^t E_{(\widetilde{\widetilde{p}},\widetilde{\widetilde{p}}_t,\widetilde{\widetilde{w}},\widetilde{\widetilde{w}}_t)}(\tau)d\tau+\sum_{k=1}^{N_1}\int_0^t\int_0^{\ell_k}[\frac{\partial^{2+\widetilde{s}}\theta_k(x,\tau)}{\partial\tau^{2+\widetilde{s}}}-\frac{\partial^{1+\widetilde{s}}\theta_k(x,\tau)}{\partial\tau^{1+\widetilde{s}}}]^2dxds\Bigg]. \qquad (50)
\end{aligned}$$

By Grönwall's inequality, we get

$$E_{(\widetilde{\widetilde{p}},\widetilde{\widetilde{p}}_t,\widetilde{\widetilde{w}},\widetilde{\widetilde{w}}_t)}(t)\le\widetilde{C}_t\sum_{k=1}^{N_1}\int_0^t\int_0^{\ell_k}[\frac{\partial^{2+\widetilde{s}}\theta_k(x,t)}{\partial t^{2+\widetilde{s}}}-\frac{\partial^{1+\widetilde{s}}\theta_k(x,t)}{\partial t^{1+\widetilde{s}}}]^2dxdt,\quad\forall t\in[0,T].$$

Thus,

$$\begin{aligned}
&\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[\widetilde{\widetilde{p}}_{k,t}(x,t)]^2dxdt\\
\le\;&\int_0^T E_{(\widetilde{\widetilde{p}},\widetilde{\widetilde{p}}_t,\widetilde{\widetilde{w}},\widetilde{\widetilde{w}}_t)}dxdt\\
\le\;&T\widetilde{C}_T\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[\frac{\partial^{2+\widetilde{s}}\theta_k(x,t)}{\partial t^{2+\widetilde{s}}}-\frac{\partial^{1+\widetilde{s}}\theta_k(x,t)}{\partial t^{1+\widetilde{s}}}]^2dxdt\\
\le\;&C\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{2+\widetilde{s}}\theta_k(x,t)}{\partial t^{2+\widetilde{s}}})^2+(\frac{\partial^{1+\widetilde{s}}\theta_k(x,t)}{\partial t^{1+\widetilde{s}}})^2]dxdt. \qquad (51)
\end{aligned}$$

It is easy to see that $\int_0^t\widetilde{\widetilde{p}}_kdt,\ \int_0^t\widetilde{\widetilde{w}}_jdt,\ k=1,2,\cdots,N_1,\ j=N_1+1,N_1+2,\cdots,N$ also satisfy equation (48) with $\frac{\partial^{2+\widetilde{s}}\theta_k(x,t)}{\partial t^{2+\widetilde{s}}}-\frac{\partial^{1+\widetilde{s}}\theta_k(x,t)}{\partial t^{1+\widetilde{s}}}$ replaced by $\frac{\partial^{1+\widetilde{s}}\theta_k(x,t)}{\partial t^{1+\widetilde{s}}}-\frac{\partial^{\widetilde{s}}\theta_k(x,t)}{\partial t^{\widetilde{s}}}$. Using the similar discussion, we get that

$$\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[\widetilde{\widetilde{p}}_k(x,t)]^2dxdt\le C\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{1+\widetilde{s}}\theta_k(x,t)}{\partial t^{1+\widetilde{s}}})^2+(\frac{\partial^{\widetilde{s}}\theta_k(x,t)}{\partial t^{\widetilde{s}}})^2]dxdt. \qquad (52)$$

Therefore, by (47), (51) and (52), there exist positive constants T and C, such that

$$\begin{aligned}
&C\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{\widetilde{s}}\theta_k}{\partial t^{\widetilde{s}}})^2+(\frac{\partial^{1+\widetilde{s}}\theta_k}{\partial t^{1+\widetilde{s}}})^2+(\frac{\partial^{2+\widetilde{s}}\theta_k}{\partial t^{2+\widetilde{s}}})^2]dxdt\\
\ge\;&\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{1+\widetilde{s}}\widetilde{p}_k}{\partial t^{1+\widetilde{s}}})^2+(\frac{\partial^{\widetilde{s}}\widetilde{p}_k}{\partial t^{\widetilde{s}}})^2]dxdt.
\end{aligned}$$

The proof of Proposition 2 is complete. □

Now, based on Proposition 1 and 2, let us show Theorem 5.3.

Proof of Theorem 5.3. By (37), we get

$$\begin{cases} \frac{\partial^{\tilde{s}} p_k}{\partial t^{\tilde{s}}} = \frac{\partial^{\tilde{s}}\theta_k}{\partial t^{\tilde{s}}} - \frac{\partial^{\tilde{s}}\tilde{p}_k}{\partial t^{\tilde{s}}},\ k=1,2,\cdots,N_1, \\ \frac{\partial^{1+\tilde{s}} p_k}{\partial t^{1+\tilde{s}}} = \frac{\partial^{1+\tilde{s}}\theta_k}{\partial t^{1+\tilde{s}}} - \frac{\partial^{1+\tilde{s}}\tilde{p}_k}{\partial t^{1+\tilde{s}}},\ k=1,2,\cdots,N_1. \end{cases}$$

Hence,

$$\begin{aligned} &\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{1+\tilde{s}}p_k(x,t)}{\partial t^{1+\tilde{s}}})^2+(\frac{\partial^{\tilde{s}}p_k(x,t)}{\partial t^{\tilde{s}}})^2]dxdt \\ =\ &\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{1+\tilde{s}}\theta_k(x,t)}{\partial t^{1+\tilde{s}}}-\frac{\partial^{1+\tilde{s}}\tilde{p}_k(x,t)}{\partial t^{1+\tilde{s}}})^2+(\frac{\partial^{\tilde{s}}\theta_k(x,t)}{\partial t^{\tilde{s}}}-\frac{\partial^{\tilde{s}}\tilde{p}_k(x,t)}{\partial t^{\tilde{s}}})^2]dxdt \\ \le\ &2\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{1+\tilde{s}}\theta_k(x,t)}{\partial t^{1+\tilde{s}}})^2+(\frac{\partial^{1+\tilde{s}}\tilde{p}_k(x,t)}{\partial t^{1+\tilde{s}}})^2+(\frac{\partial^{\tilde{s}}\theta_k(x,t)}{\partial t^{\tilde{s}}})^2 \\ &+(\frac{\partial^{\tilde{s}}\tilde{p}_k(x,t)}{\partial t^{\tilde{s}}})^2]dxdt. \end{aligned}$$

Then by (46), there exist positive constants C and T such that

$$\begin{aligned} &\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{1+\tilde{s}}p_k(x,t)}{\partial t^{1+\tilde{s}}})^2+(\frac{\partial^{\tilde{s}}p_k(x,t)}{\partial t^{\tilde{s}}})^2]dxdt \\ \le\ &C\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{2+\tilde{s}}\theta_k(x,t)}{\partial t^{2+\tilde{s}}})^2+(\frac{\partial^{1+\tilde{s}}\theta_k(x,t)}{\partial t^{1+\tilde{s}}})^2+(\frac{\partial^{\tilde{s}}\theta_k(x,t)}{\partial t^{\tilde{s}}})^2]dxdt. \end{aligned} \tag{53}$$

Therefore, by Corollary 1, we get

$$\begin{aligned} &C\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\frac{\partial^{2+\tilde{s}}\theta_k(x,t)}{\partial t^{2+\tilde{s}}})^2+(\frac{\partial^{1+\tilde{s}}\theta_k(x,t)}{\partial t^{1+\tilde{s}}})^2+(\frac{\partial^{\tilde{s}}\theta_k(x,t)}{\partial t^{\tilde{s}}})^2]dxdt \\ \ge\ &\|(\theta^0,u^0,u^1)\|_{\mathcal{H}}^2. \end{aligned}$$

Thus, the proof of Theorem 5.3 is complete. □

5.3. **Polynomial decay rate (Proof of Theorem 5.4).** This subsection is devoted to deduce the polynomial decay rate for system (2) based on the derived observability inequality in Theorem 5.3. We need the following result from [31] (see also [26]).

Lemma 5.6. *Let $\mathcal{A}$ generate a bounded C_0-semigroup on a Banach space V. Then there is a constant $\widehat{C}>0$ such that for any $v\in\mathcal{D}(\mathcal{A}^2)$, one has*

$$\|\mathcal{A}v\|_V^2 \le \widehat{C}\|v\|_V\|\mathcal{A}^2v\|_V. \tag{54}$$

Based on this lemma, we can deduce the following result.

Corollary 2. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then for $W\in\mathcal{D}(\mathcal{A})$,*

$$\|W\|_{\mathcal{H}} \le \widehat{C}\|\mathcal{A}^{-1-\tilde{s}}W\|_{\mathcal{H}}^{\frac{1}{\tilde{s}+2}}\|\mathcal{A}W\|_{\mathcal{H}}^{\frac{\tilde{s}+1}{\tilde{s}+2}}. \tag{55}$$

Proof. The inductive method are used to show this result by the follow two steps:

Step 1). When $\tilde{s}=1$, that is,

$$\|W\|_{\mathcal{H}} \le \widehat{C}\|\mathcal{A}^{-2}W\|_{\mathcal{H}}^{\frac{1}{3}}\|\mathcal{A}W\|_{\mathcal{H}}^{\frac{2}{3}}. \tag{56}$$

In fact, from (54) we have

$$\|W\|_{\mathcal{H}}^2 \le \widehat{C}\|\mathcal{A}^{-1}W\|_{\mathcal{H}}\|\mathcal{A}W\|_{\mathcal{H}}, \quad \|\mathcal{A}^{-1}W\|_{\mathcal{H}}^2 \le \widehat{C}\|\mathcal{A}^{-2}W\|_{\mathcal{H}}\|W\|_{\mathcal{H}}. \tag{57}$$

Hence, $\|W\|_{\mathcal{H}}^2 \le \widehat{C}\|\mathcal{A}^{-2}W\|_{\mathcal{H}}^{1/2}\|W\|_{\mathcal{H}}^{1/2}\|\mathcal{A}W\|_{\mathcal{H}}$, which leads to (56).

Step 2). Assume that when $\widetilde{s} = s_0$, (55) holds, that is,

$$\|W\|_{\mathcal{H}} \le \widehat{C}\|\mathcal{A}^{-1-s_0}W\|_{\mathcal{H}}^{\frac{1}{s_0+2}}\|\mathcal{A}W\|_{\mathcal{H}}^{\frac{s_0+1}{s_0+2}}. \tag{58}$$

We will finish the proof by showing that when $\widetilde{s} = s_0+1$, (55) also holds, that is

$$\|W\|_{\mathcal{H}} \le \widehat{C}\|\mathcal{A}^{-2-s_0}W\|_{\mathcal{H}}^{\frac{1}{s_0+3}}\|\mathcal{A}W\|_{\mathcal{H}}^{\frac{s_0+2}{s_0+3}}. \tag{59}$$

Indeed, since $\mathcal{A}^{-1}W \in \mathcal{D}(\mathcal{A})$, from (58), we have

$$\|\mathcal{A}^{-1}W\|_{\mathcal{H}} \le \widehat{C}\|\mathcal{A}^{-2-s_0}W\|_{\mathcal{H}}^{\frac{1}{s_0+2}}\|W\|_{\mathcal{H}}^{\frac{s_0+1}{s_0+2}}.$$

Then by (57), we have

$$\|W\|_{\mathcal{H}}^2 \le \widehat{C}\|\mathcal{A}^{-1}W\|_{\mathcal{H}}\|\mathcal{A}W\|_{\mathcal{H}} \le \widehat{C}\|\mathcal{A}^{-2-s_0}W\|_{\mathcal{H}}^{\frac{1}{s_0+2}}\|W\|_{\mathcal{H}}^{\frac{s_0+1}{s_0+2}}\|\mathcal{A}W\|_{\mathcal{H}}.$$

Hence,

$$\|W\|_{\mathcal{H}}^{2-\frac{s_0+1}{s_0+2}} \le \widehat{C}\|\mathcal{A}^{-2-s_0}W\|_{\mathcal{H}}^{\frac{1}{s_0+2}}\|\mathcal{A}W\|_{\mathcal{H}},$$

which implies (59) holds. The proof is complete. □

Let $E(\theta,u,u_t)$ be the natural energy of system (2), which is defined as (4). Define

$$E_r(t) := E(\Theta^r, U^r, U_t^r), \tag{60}$$

where

$$(\Theta^r, U^r, U_t^r) = \mathcal{A}^r(\theta,u,u_t),\ r = -\widetilde{s}-2,\ -\widetilde{s}-1,\ -\widetilde{s}, \cdots, -1,\ 1. \tag{61}$$

Note that (Θ^r, U^r, U_t^r), $r = -\widetilde{s}-2, -\widetilde{s}-1, -\widetilde{s}, \cdots, -1, 1$ are also the solution to system (2) with initial state $\mathcal{A}^r(\theta^0,u^0,u^1)$. Set $\varepsilon(t) = \sum\limits_{r=-\widetilde{s}-2}^{-1} E_r(t) + E(t)$. It is easy to check that

$$E_r(T)-E_r(S) = -\sum_{k=1}^{N_1}\int_S^T\int_0^{\ell_k}(\Theta^r_{k,t}(x,t))^2dxdt, \quad r = -\widetilde{s}-2,\ -\widetilde{s}-1,\ -\widetilde{s}, \cdots, -1. \tag{62}$$

Hence,

$$\varepsilon(S)-\varepsilon(T) = \sum_{k=1}^{N_1}\int_S^T\int_0^{\ell_k}[\sum_{r=-\widetilde{s}-2}^{-1}(\Theta^r_{k,t}(x,t))^2 + (\theta_{k,t}(x,t))^2]dxdt. \tag{63}$$

Applying Theorem 5.3 to $(\Theta^{-\widetilde{s}-1}, U^{-\widetilde{s}-1}, U_t^{-\widetilde{s}-1})$, we have

$$\begin{aligned} C\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}\Big[\left(\frac{\partial^{2+\widetilde{s}}[\Theta_k^{-\widetilde{s}-1}(x,t)]}{\partial t^{2+\widetilde{s}}}\right)^2 + \left(\frac{\partial^{1+\widetilde{s}}[\Theta_k^{-\widetilde{s}-1}(x,t)]}{\partial t^{1+\widetilde{s}}}\right)^2 \\ + \left(\frac{\partial^{\widetilde{s}}[\Theta_k^{-\widetilde{s}-1}(x,t)]}{\partial t^{\widetilde{s}}}\right)^2\Big]dxdt \ge E_{-\widetilde{s}-1}(0). \end{aligned} \tag{64}$$

Note that $\frac{\partial^{2+\widetilde{s}}[\Theta_k^{-\widetilde{s}-1}]}{\partial t^{2+\widetilde{s}}} = \theta_{k,t},\ k = 1, 2, \cdots, N_1$. Thus, we have $C[\varepsilon(0) - \varepsilon(T)] \geq E_{-\widetilde{s}-1}(0)$. By (55) in Corollary 2, we have

$$\varepsilon(t) \leq \overline{C}E(t) \leq \overline{C}\widehat{C}^2(E_{-1-\widetilde{s}}(t))^{\frac{1}{\widetilde{s}+2}}(E_1(t))^{\frac{\widetilde{s}+1}{\widetilde{s}+2}}. \tag{65}$$

Thus, there exists a positive constant C such that $\frac{(\varepsilon(0))^{\widetilde{s}+2}}{(E_1(0))^{\widetilde{s}+1}} \leq C[\varepsilon(0)-\varepsilon(T)]$. Fixing $T > 0$, we get

$$C[\varepsilon(mT) - \varepsilon((m+1)T)] \geq \frac{(\varepsilon((m+1)T))^{\widetilde{s}+2}}{(E_1(0))^{\widetilde{s}+1}}.$$

Here we have used $\varepsilon((m+1)T) \leq \varepsilon(0)$ due to the dissipativity of operator $\mathcal{A}$. Then we get $C[\frac{\varepsilon(mT)}{E_1(0)} - \frac{\varepsilon((m+1)T)}{E_1(0)}] \geq [\frac{\varepsilon((m+1)T)}{E_1(0)}]^{\widetilde{s}+2}$ and hence

$$\frac{\varepsilon((m+1)T)}{E_1(0)} \leq \frac{\varepsilon(mT)}{E_1(0)} - \frac{1}{C}[\frac{\varepsilon((m+1)T)}{E_1(0)}]^{\widetilde{s}+2}. \tag{66}$$

In order to proceed to deduce the decay rate of the energy of system (2), we need the following lemma from [3].

Lemma 5.7. *Let $\{a_m\}_{m=1}^{\infty}$ be a sequence of positive number satisfying*

$$a_{m+1} \leq a_m - C(a_{m+1})^{2+\alpha}, \quad \forall m \geq 1, \tag{67}$$

for some constants $C > 0$ and $\alpha > -1$. Then there exists a positive constant $M_{C,\alpha}$ such that

$$a_m \leq \frac{M_{C,\alpha}}{(m+1)^{\frac{1}{1+\alpha}}}.$$

By the above Lemma, together with (66), it is easy to get that $\frac{\varepsilon(mT)}{E_1(0)} \leq \frac{M_{C,\widetilde{s}}}{(m+1)^{\frac{1}{1+\widetilde{s}}}}$. Thus, $\varepsilon(mT) \leq \frac{M_{C,\widetilde{s}}}{(m+1)^{\frac{1}{1+\widetilde{s}}}}E_1(0)$. Therefore, there exists a positive constant C such that $E(t) \leq \varepsilon(t) \leq \frac{C}{t^{\frac{1}{1+\widetilde{s}}}}E_1(0)$. The proof of Theorem 5.4 is complete. □

5.4. **Logarithmic decay rate.** In the previous subsection, we have proved that if $\ell_j,\ j = N_1 + 1, N_1 + 2, \cdots, N$ satisfy the conditions in Lemma 5.5 (see also Theorem 5.3), then we always can find some $s > 0$ such that the weights in inequality (41) satisfy (42) and based on which, the polynomial decay rate of system (2) is derived. However, if all of these conditions are not fulfilled, there is no s satisfying $\gamma_n \geq \lambda_n^{-s},\ n \geq 1$. Thus system (2) can not achieve polynomial decay and some weaker decay rate may hold. In fact, the logarithmic decay rate can be derived when the weights γ_n decay exponentially, that is,

$$\gamma_n^2 \sim ce^{-an}, \quad n = 1, 2, \cdots, \tag{68}$$

where c, a are positive constants. Note that this kind of weights $\gamma_n,\ n \geq 1$ in (68) can not be deduced by the conditions in Lemma 5.5.

By Proposition 1, there exists a positive constant T such that

$$\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[p_{k,t}^2 + p_k^2]dxdt \geq \sum_{n\geq 1} ce^{-an}[\lambda_n^2 a_n^2 + b_n^2]. \tag{69}$$

From the above inequality, we have the following observability estimate.

Theorem 5.8. *For $b \in (0, \frac{1}{2})$, there exist positive constants T and C, such that for $(\theta^0, u^0, u^1) \in \mathcal{D}(\mathcal{A})$,*

$$C\left[a^{-1}\ln\left(\frac{c\|(\theta^0,u^0,u^1)\|^2_{\mathcal{H}}}{\sum\limits_{k=1}^{N_1}\|\theta_k\|^2_{H^2(0,T;L^2(0,\ell_k))}}\right)\right]^{-2b}\|\mathcal{A}(\theta^0,u^0,u^1)\|^2_{\mathcal{H}} \geq \|(\theta^0,u^0,u^1)\|^2_{\mathcal{H}}, \tag{70}$$

where a, c are positive constants given as in (68).

The proof of Theorem 5.8 will be given at the end of this subsection. Let us proceed to show the following logarithmic decay rate of system (2) based on the weakened observability inequality (70). We mainly employ some techniques from [31] to deduce the logarithmic decay.

Theorem 5.9. *Assume that the weights in (41) satisfy condition (68). Then for any $(\theta^0, u^0, u^1) \in \mathcal{D}(\mathcal{A})$ and $b \in (0, \frac{1}{2})$, there always exist positive constants a', c' such that the energy of system (2) satisfies*

$$E(t) \leq \frac{a'}{(\ln[c'(t+1)])^{2b}}\|(\theta^0,u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A})}, \ \forall t \geq 0. \tag{71}$$

Proof. Set $\varepsilon_1(t) = E_{-1}(t) + E(t) + E_1(t)$, where $E_j(t)$ is defined the same as in (60). It is easy to get that

$$\frac{dE(t)}{dt} = -\sum_{k=1}^{N_1}\int_0^{\ell_k}\theta^2_{k,t}dx, \quad \frac{dE_1(t)}{dt} = -\sum_{k=1}^{N_1}\int_0^{\ell_k}\theta^2_{k,tt}dx,$$

$$\frac{dE_{-1}(t)}{dt} = -\sum_{k=1}^{N_1}\int_0^{\ell_k}\theta^2_{k}dx$$

and hence $\varepsilon_1(0) - \varepsilon_1(T) = \sum\limits_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}(\theta^2_{k,tt} + \theta^2_{k,t} + \theta^2_k)dxdt$. From (70), we get

$$C\left(a^{-1}\ln\left[\frac{cE(0)}{\sum\limits_{k=1}^{N_1}\|\theta_k\|^2_{H^2(0,T;L^2(0,\ell_k))}}\right]\right)^{-2b} \geq \frac{E(0)}{E_1(0)}. \tag{72}$$

A direct calculation yields $\sum\limits_{k=1}^{N_1}\|\theta_k\|^2_{H^2(0,T;L^2(0,\ell_k))} \geq cE(0)e^{-a\left(\frac{CE_1(0)}{E(0)}\right)^{\frac{1}{2b}}}$, $b \in (0, \frac{1}{2})$. Hence,

$$\varepsilon_1(0) - \varepsilon_1(T) \geq cE(0)e^{-a\left(\frac{CE_1(0)}{E(0)}\right)^{\frac{1}{2b}}}, \quad \forall b \in (0, \frac{1}{2}). \tag{73}$$

Then replacing the initial date (θ^0, u^0, u^1) by $\mathcal{A}^{-1}(\theta^0, u^0, u^1)$, we get that (73) becomes

$$\varepsilon(0) - \varepsilon(T) \geq cE_{-1}(0)e^{-a\left(\frac{CE(0)}{E_{-1}(0)}\right)^{\frac{1}{2b}}}, \quad \forall b \in (0, \frac{1}{2}). \tag{74}$$

Note that $E(t) \le \varepsilon(t) \le \widetilde{C}E(t)$. By Corollary 2, we have $E_{-1}(0) \ge \frac{(E(0))^2}{\widetilde{C}\widehat{C}^2 E_1(0)}$, and hence $\widetilde{C}\widehat{C}^2\frac{E_1(0)}{E(0)} \ge \frac{E(0)}{E_{-1}(0)}$. Thus, there exist constants $C_1, \widetilde{C}_1 > 0$ such that

$$\varepsilon(0) - \varepsilon(T) \ge c\frac{(E(0))^2}{\widetilde{C}_1 E_1(0)} e^{-a\left(\frac{C_1 E_1(0)}{E(0)}\right)^{\frac{1}{2b}}}, \quad \forall b \in (0, \frac{1}{2}). \tag{75}$$

Set $\alpha_m = \frac{\varepsilon(mT)}{E_1(0)}$. We have that

$$\alpha_m - \alpha_{m+1} \ge \frac{c}{\widetilde{C}_1}\alpha_m^2 e^{-a(\frac{C_1}{\alpha_m})^{\frac{1}{2b}}}. \tag{76}$$

Here we have used that $E_1(t)$ is non-increasing. From (76), for any $n \in \mathbb{N}$, we get

$$nc\alpha_n^2 e^{-a(\frac{C_1}{\alpha_n})^{\frac{1}{2b}}} \le \sum_{m=1}^{n} \widetilde{C}_1(\alpha_m - \alpha_{m+1}) \le \widetilde{C}_1\alpha_1,$$

since α_m is non-increasing respective to m. Note that it is easy to check

$$\min_{\rho\in(0,\alpha_1)} \rho^2 e^{a(\frac{C_1}{\rho})^{\frac{1}{2b}}} > 0.$$

Thus, there exists a constant C_b such that

$$\alpha_n^2 \ge C_b e^{-a(\frac{C_1}{\alpha_n})^{\frac{1}{2b}}}, \quad \forall b \in (0, \frac{1}{2}).$$

Hence, $nce^{-2a(\frac{C_1}{\alpha_n})^{\frac{1}{2b}}} \le \frac{\widetilde{C}_1}{C_b}\alpha_1$. Then we calculate directly that

$$\alpha_n \le C_1\left[\frac{1}{2a}\ln(\frac{C_b nc}{\widetilde{C}_1\alpha_1})\right]^{-2b}, \ \forall b \in (0, \frac{1}{2}).$$

Therefore, by the definition of α_n, for sufficient large n,

$$\varepsilon(nT) \le C_1\left[\frac{1}{2a}\ln(\frac{C_b nc}{\widetilde{C}_1\alpha_1})\right]^{-2b} E_1(0), \quad \forall b \in (0, \frac{1}{2}), \tag{77}$$

which implies the logarithmic decay rate in Theorem 5.9. The proof is complete. □

Proof of Theorem 5.8. Similarly to [27], for $b \in (0, \frac{1}{2})$, we construct a concave and increasing function in $t \in (0, 1)$,

$$\Phi_b(t) = \left[\frac{a}{\ln(\frac{c}{t})}\right]^{2b}. \tag{78}$$

It is easy to check that $\Phi_b(ce^{at})t^{2b} = 1$. By the inverse Jensen's inequality (see [27]), we get

$$1 \le \Phi_b\left(\frac{\sum\limits_{n\ge1} ce^{-an}(\lambda_n^2 a_n^2 + b_n^2)}{\sum\limits_{n\ge1}(\lambda_n^2 a_n^2 + b_n^2)}\right)\frac{\sum\limits_{n\ge1} n^{2b}(\lambda_n^2 a_n^2 + b_n^2)}{\sum\limits_{n\ge1}(\lambda_n^2 a_n^2 + b_n^2)}, \tag{79}$$

where λ_n are the eigenvalues of the corresponding operator A for system (38) and $a_n,\ b_n$ is the Fourier coefficients given as (40). Hence,

$$\Phi_b^{-1}\left(\frac{\sum\limits_{n\ge1}(\lambda_n^2 a_n^2 + b_n^2)}{\sum\limits_{n\ge1} n^{2b}(\lambda_n^2 a_n^2 + b_n^2)}\right)\sum_{n\ge1}(\lambda_n^2 a_n^2 + b_n^2) \le \sum_{n\ge1} ce^{-an}(\lambda_n^2 a_n^2 + b_n^2). \tag{80}$$

Then by (41) in Proposition 1,

$$\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[p_{k,t}^2+p_k^2]dxdt \geq \Phi_b^{-1}\left(\frac{\sum\limits_{n\geq 1}(\lambda_n^2a_n^2+b_n^2)}{\sum\limits_{n\geq 1}n^{2b}(\lambda_n^2a_n^2+b_n^2)}\right)\sum_{n\geq 1}(\lambda_n^2a_n^2+b_n^2),$$

which implies

$$\Phi_b\left(\frac{\sum\limits_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[p_{k,t}^2+p_k^2]dxdt}{\sum\limits_{n\geq 1}(\lambda_n^2a_n^2+b_n^2)}\right)\sum_{n\geq 1}n^{2b}(\lambda_n^2a_n^2+b_n^2)\geq \sum_{n\geq 1}(\lambda_n^2a_n^2+b_n^2).$$

Substituting (78) into the above inequality, we get

$$\left(a^{-1}\ln\left[\frac{c\sum\limits_{n\geq 1}(\lambda_n^2a_n^2+b_n^2)}{\sum\limits_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[p_{k,t}^2+p_k^2]dxdt}\right]\right)^{-2b}\sum_{n\geq 1}n^{2b}(\lambda_n^2a_n^2+b_n^2)\geq \sum_{n\geq 1}(\lambda_n^2a_n^2+b_n^2).$$

By proposition 2, there exist positive constants C and T such that

$$\begin{aligned}&\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\widetilde{p}_{k,t}(x,t))^2+(\widetilde{p}_k(x,t))^2]dxdt\\ \leq\ & C\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[(\theta_k(x,t))^2+(\theta_{k,t}(x,t))^2+(\theta_{k,tt}(x,t))^2]dxdt,\end{aligned}$$

where $(\widetilde{p},\ \widetilde{w})$ is the solution to (39). Note that $\theta_k=p_k+\widetilde{p}_k,\ k=1,2,\cdots,N_1$. Thus, there exist constants $C>0$ and $T>0$, such that

$$\begin{aligned}\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[p_{k,t}^2+p_k^2]dxdt &\leq 2\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[\theta_{k,t}^2+\widetilde{p}_{k,t}^2+\theta_k^2+\widetilde{p}_k^2]dxdt\\ &\leq C\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[\theta_k^2+\theta_{k,t}^2+\theta_{k,tt}^2]dxdt.\end{aligned}$$

Note that by Weyl's formula (see [21]), when n sufficient large,

$$\lambda_n\sim\frac{n\pi}{\sum\limits_{j=1}^{N}\ell_j}. \tag{81}$$

Then we get the following estimate: there exists a constant $\widetilde{C}>0$ such that

$$\widetilde{C}\left(a^{-1}\ln\left[\frac{c\sum\limits_{n\geq 1}(\lambda_n^2a_n^2+b_n^2)}{\sum\limits_{k=1}^{N_1}\|\theta_k\|^2_{H^2(0,T;L^2(0,\ell_k))}}\right]\right)^{-2b}\sum_{n\geq 1}\lambda_n^{2b}(\lambda_n^2a_n^2+b_n^2)\geq \sum_{n\geq 1}(\lambda_n^2a_n^2+b_n^2). \tag{82}$$

Note that for any $(\theta^0,u^0,u^1)\in\mathcal{D}(\mathcal{A})$ and $b\in(0,\frac{1}{2})$, there exists a positive constant C such that

$$\|\mathcal{A}(\theta^0,u^0,u^1)\|_{\mathcal{H}}^2 = \sum_{k=1}^{N_1}\int_0^{\ell_k}(\theta^0_{k,xxx})^2+\sum_{j=N_1+1}^{N}[(u^1_{j,x})^2+(u^0_{j,xx})^2]dx$$

$$\geq \quad C\| \left((\theta^0,u^0),(\theta^0_{xx},u^1)\right) \|^2_{\prod_{j=1}^N H_0^{1+b}\times \prod_{j=1}^N H_0^b}.$$

Since $\| \left((\theta^0,u^0),(\theta^0_{xx},u^1)\right) \|^2_{\prod_{j=1}^N H_0^{1+b}\times \prod_{j=1}^N H_0^b} \sim \sum_{n\geq 1}\lambda_n^{2b}(\lambda_n^2 a_n^2+b_n^2)$ and $\|(\theta^0,u^0,u^1)\|^2_{\mathcal{H}} \leq \sum_{n\geq 1}(\lambda_n^2 a_n^2+b_n^2)$, then we deduce from (82) that for $b\in(0,\frac{1}{2})$, there exists a constant $C>0$ such that

$$C\left(a^{-1}\ln\left[\frac{c\|(\theta^0,u^0,u^1)\|^2_{\mathcal{H}}}{\sum_{k=1}^{N_1}\|\theta_k\|^2_{H^2(0,T;L^2(0,\ell_k))}}\right]\right)^{-2b}\|\mathcal{A}(\theta^0,u^0,u^1)\|^2_{\mathcal{H}} \geq \|(\theta^0,u^0,u^1)\|^2_{\mathcal{H}}.$$

The proof of Theorem 5.8 is complete. □

6. **Decay rate for more general networks.** In this section, we present some results for general networks, which can be deduced similarly by the techniques proposed in this paper.

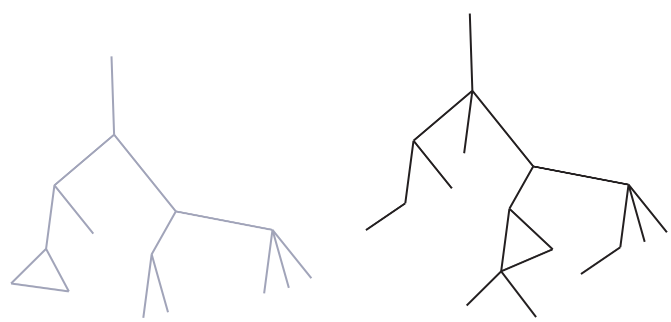

FIGURE 4. heat network (left); wave network (right)

• *Assume that the Dirichlet condition is satisfied at least at one exterior node of the network. Then:*

a). If the heat equation is fulfilled in all the edges (see Fig. 4 (left) for example), then the energy of the network decays exponentially. This can be easily seen by the energy identity and the Poincaré inequality along the network.

b). If the wave equation is satisfied in all edges (see Fig. 4 (right) for example), then the energy is conserved.

• *Assume that $\mathcal{M}$ is a tree-shaped wave network with Dirichlet condition at its root. Consider a network composed by $\mathcal{M}$ extended by heat equations, such that all the leaves in the resulting network are heat-like (see Fig. 5 for example). Then the total energy of the network decays polynomially with the following sharp decay rate:*

$$E(t)\leq t^{-4}\|(\theta^0,u^0,u^1)\|^2_{\mathcal{D}(A)}, \tag{83}$$

where (θ^0,u^0,u^1) is the initial condition.

Remark 6. This polynomial decay rate can be proved by combining the frequency domain method together with the multipliers, as in the analysis of star-shaped networks in Section 4 above.

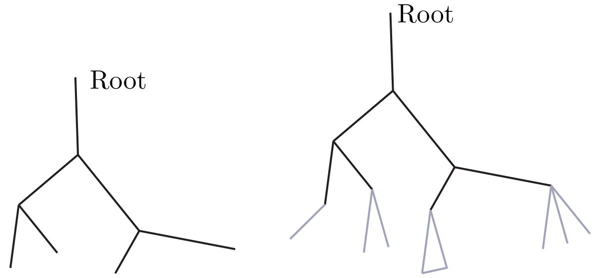

FIGURE 5. tree-shaped network $\mathcal{M}$ (left); the heat-wave network extended from $\mathcal{M}$ (right)

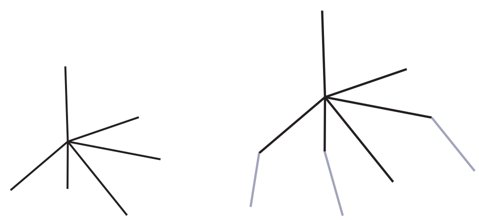

FIGURE 6. star-shaped wave network $\mathcal{M}$ (left); heat-wave network(right) extended from $\mathcal{M}$

Especially, if the network $\mathcal{M}$ in the above result is a star-shaped one, the following extended result holds.

- *If only part of the leaves in the resulting network are heat-like (see Fig. 6 for example), then we have the following polynomial decay rate:*

$$E(t) \leq C \frac{1}{t^{\frac{1}{\widetilde{s}+1}}} \|(\theta^0, u^0, u^1)\|^2_{\mathcal{D}(\mathcal{A})}, \ \forall t \geq 0,$$

in which $\widetilde{s}$ is given as follows:

1) $\widetilde{s} = 1 + [N - \widetilde{N} - 1 + \epsilon]$, if $\frac{\ell_i}{\ell_j} \in B_\epsilon$, $i, j \in \sharp\mathcal{M}_w$, $i \neq j$,

2) $\widetilde{s} = N - \widetilde{N}$, if $\frac{\ell_i}{\ell_j} \in \mathcal{S}$, $i, j \in \sharp\mathcal{M}_w$, $i \neq j$,

3) $\widetilde{s} = 1 + [1 + \epsilon]$, if ℓ_j, $j \in \sharp\mathcal{M}_w$ satisfy the condition (S),

where $\sharp\mathcal{M}_w$ stands for the set of edges described by the wave equations which do not connect heat-like edges directly. The sets B_ϵ, $\mathcal{S}$ and the condition (S) are given as in Definition 5.1 and 5.2.

Remark 7. The above decay rate can be shown by the observability estimate method similarly as the discussion in Section 5, in which the heat-wave network is replaced by pure wave network when deducing the observability inequality.

- *Based on the observability estimate method (see Section 5), we can get the decay rates for general heat-wave network G, as long as the weakened observability estimate can be obtained for the corresponding wave network, in which all heat equations are replaced by the wave ones. More precisely, assume that the following weakened observability inequality holds:*

$$\sum_{k \in \sharp G_h} \int_0^T \int_0^{\ell_k} [p_{k,t}^2 + p_k^2] dx dt \geq \sum_{n \geq 1} \gamma_n^2 [\lambda_n^2 a_n^2 + b_n^2], \tag{84}$$

where $(p_k, p_{k,t})$, $k \in \sharp G_h$ are the states of the wave equations which replace the heat equations in the heat-wave network G; λ_n denote the corresponding eigenvalues and a_n, b_n the Fourier coefficients of the initial state, similar as in (40). In here $\sharp G_h$ stands for the set of edges evolving according to the heat equation.

Then the heat-wave network G decays as follows:

1). If the weights in (84) satisfy $\gamma_n \geq \lambda_n^{-s}$, $n \geq 1$, then network G decays polynomially, that is,

$$E(t) \leq C \frac{1}{t^{\frac{1}{\widetilde{s}+1}}} \|(\theta^0, u^0, u^1)\|^2_{\mathcal{D}(\mathcal{A})}, \ \forall t \geq 0,$$

where $\widetilde{s} := [s] + 1$.

2). If $\gamma_n^2 \sim ce^{-an}$, $n \geq 1$, where c, a are positive constants, then network G can achieve logarithmic decay rate, that is, for any $b \in (0, \frac{1}{2})$, there always exists positive constants a', c' such that

$$E(t) \leq \frac{a'}{(\ln[c'(t+1)])^{2b}} \|(\theta^0, u^0, u^1)\|^2_{\mathcal{D}(\mathcal{A})}, \ \forall t \geq 0.$$

Remark 8. Following the proof in Section 5, the decay rate of star-shaped heat-wave networks is deduced from the corresponding star-shaped wave networks. Hence, employing a similar analysis, we can extend these results to more general heat-wave networks. This approach is useful since there have been several methods and results on the observability for wave networks (see [9], [15]). Note however that, in general, this approach does not lead to optimal decay rates.

Remark 9. Despite the contributions of the present paper, the obtention of sharp decay rates for all possible planar heat-wave networks is still a widely open subject of research.

7. Appendix.

7.1. Appendix: Construction of ray-like approximate solutions of heat-wave networks .

This appendix is devoted to build ray-like approximate solutions of (7) by the WKB approach of asymptotic expansion (see [25], [5]).

Step 1). The approximate solutions for the wave equations

We rewrite (8) as follows

$$u^\epsilon_{\widehat{j}}(x,t) = e^{i\tau t/\epsilon}\left[\sum_{k=0}^{\infty}\epsilon^k[e^{i\xi x/\epsilon}a^k_{\widehat{j}}(x,t)+e^{-i\xi x/\epsilon}b^{2k}_{\widehat{j}}(x,t)]+\sum_{n=0}^{\infty}\epsilon^{n+\frac{1}{2}}e^{-i\xi x/\epsilon}b^{2n+1}_{\widehat{j}}(x,t)\right], \quad (85)$$

$$u^\epsilon_{j}(x,t) = e^{i\tau t/\epsilon}\left[\sum_{k=0}^{\infty}\epsilon^k[e^{-i\xi x/\epsilon}b^{2k}_{j}(x,t)]+\sum_{n=0}^{\infty}\epsilon^{n+\frac{1}{2}}e^{-i\xi x/\epsilon}b^{2n+1}_{j}(x,t)\right], \quad N_1+1 \leq j \leq N, \ j \neq \widehat{j}. \quad (86)$$

A direct calculation yields that

$$\begin{aligned}
&\partial_{tt}u^\epsilon_{\widehat{j}}(x,t)\\
= &-\frac{\tau^2}{\epsilon^2}e^{i\tau t/\epsilon}\left[\sum_{k=0}^{\infty}\epsilon^k[e^{i\xi x/\epsilon}a^k_{\widehat{j}}(x,t)+e^{-i\xi x/\epsilon}b^{2k}_{\widehat{j}}(x,t)]+\sum_{n=0}^{\infty}\epsilon^{n+\frac{1}{2}}e^{-i\xi x/\epsilon}b^{2n+1}_{\widehat{j}}(x,t)\right]\\
&+2\frac{i\tau}{\epsilon}e^{i\tau t/\epsilon}\left[\sum_{k=0}^{\infty}\epsilon^k[e^{i\xi x/\epsilon}\partial_t a^k_{\widehat{j}}(x,t)+e^{-i\xi x/\epsilon}\partial_t b^{2k}_{\widehat{j}}(x,t)]+\sum_{n=0}^{\infty}\epsilon^{n+\frac{1}{2}}e^{-i\xi x/\epsilon}\partial_t b^{2n+1}_{\widehat{j}}(x,t)\right]\\
&+e^{i\tau t/\epsilon}\left[\sum_{k=0}^{\infty}\epsilon^k[e^{i\xi x/\epsilon}\partial_{tt} a^k_{\widehat{j}}(x,t)+e^{-i\xi x/\epsilon}\partial_{tt} b^{2k}_{\widehat{j}}(x,t)]+\sum_{n=0}^{\infty}\epsilon^{n+\frac{1}{2}}e^{-i\xi x/\epsilon}\partial_{tt} b^{2n+1}_{\widehat{j}}(x,t)\right]
\end{aligned}$$

and

$$\begin{aligned}&\partial_{xx}u^{\epsilon}_{\hat{j}}(x,t)\\&=-\frac{\xi^2}{\epsilon^2}e^{i\tau t/\epsilon}\left[\sum_{k=0}^{\infty}\epsilon^k[e^{i\xi x/\epsilon}a^k_{\hat{j}}(x,t)+e^{-i\xi x/\epsilon}b^{2k}_{\hat{j}}(x,t)]+\sum_{n=0}^{\infty}\epsilon^{n+\frac{1}{2}}e^{-i\xi x/\epsilon}b^{2n+1}_{\hat{j}}(x,t)\right]\\&\quad+2\frac{i\xi}{\epsilon}e^{i\tau t/\epsilon}\left[\sum_{k=0}^{\infty}\epsilon^k[e^{i\xi x/\epsilon}\partial_x a^k_{\hat{j}}(x,t)-e^{-i\xi x/\epsilon}\partial_x b^{2k}_{\hat{j}}(x,t)]-\sum_{n=0}^{\infty}\epsilon^{n+\frac{1}{2}}e^{-i\xi x/\epsilon}\partial_x b^{2n+1}_{\hat{j}}(x,t)\right]\\&\quad+e^{i\tau t/\epsilon}\left[\sum_{k=0}^{\infty}\epsilon^k[e^{i\xi x/\epsilon}\partial_{xx}a^k_{\hat{j}}(x,t)+e^{-i\xi x/\epsilon}\partial_{xx}b^{2k}_{\hat{j}}(x,t)]+\sum_{n=0}^{\infty}\epsilon^{n+\frac{1}{2}}e^{-i\xi x/\epsilon}\partial_{xx}b^{2n+1}_{\hat{j}}(x,t)\right].\end{aligned}$$

Let $u^{\epsilon}_{\hat{j}}(x,t)$ satisfy $u^{\epsilon}_{\hat{j},tt}(x,t)-u^{\epsilon}_{\hat{j},xx}(x,t)=O(\epsilon^{\infty})$. According to the term of $O(\epsilon^{-1})$, we get that

$$2\frac{i\tau}{\epsilon}e^{i\tau t/\epsilon}[e^{i\xi x/\epsilon}\partial_t a^0_{\hat{j}}(x,t)+e^{-i\xi x/\epsilon}\partial_t b^0_{\hat{j}}(x,t)]-2\frac{i\xi}{\epsilon}e^{i\tau t/\epsilon}[e^{i\xi x/\epsilon}\partial_x a^0_{\hat{j}}(x,t)-e^{-i\xi x/\epsilon}\partial_x b^0_{\hat{j}}(x,t)]=0.$$

Thus, we have $\tau\partial_t a^0_{\hat{j}}-\xi\partial_x a^0_{\hat{j}}=0,\quad \tau\partial_t b^0_{\hat{j}}+\xi\partial_x b^0_{\hat{j}}=0$. Using the similar argument on other terms of $O((\sqrt{\epsilon})^n),\ n=-1,0,1,2,\cdots$, we deduce that

$$\begin{cases}\tau\partial_t a^0_{\hat{j}}-\xi\partial_x a^0_{\hat{j}}=0,\quad i2\tau\partial_t a^n_{\hat{j}}-i2\xi\partial_x a^n_{\hat{j}}+\partial_{tt}a^{n-1}_{\hat{j}}-\partial_{xx}a^{n-1}_{\hat{j}}=0,\\ \tau\partial_t b^0_j+\xi\partial_x b^0_j=0,\quad \tau\partial_t b^1_j+\xi\partial_x b^1_j=0,\\ i2\tau\partial_t b^{2n}_j+i2\xi\partial_x b^{2n}_j+\partial_{tt}b^{2(n-1)}_j-\partial_{xx}b^{2(n-1)}_j=0,\\ i2\tau\partial_t b^{2n+1}_j+i2\xi\partial_x b^{2n+1}_j+\partial_{tt}b^{2n-1}_j-\partial_{xx}b^{2n-1}_j=0,\\ j=N_1+1,N_1+2,\cdots,N,\ n=1,2,\cdots.\end{cases}\tag{87}$$

Let us consider the equations (87) with initial conditions imposed at the joint node $x=0$, that is,

$$a^n_{\hat{j}}(0,t)=a^{n,0}_{\hat{j}}(t),\ b^n_j(0,t)=b^{n,0}_j(t),\ j=N_1+1,N_1+2,\cdots,N,\ n=0,1,2,\cdots.\tag{88}$$

Note that they are all transport equations in (88). We get the unique ray-like solutions to the above problem as follows:

$$\begin{cases}a^0_{\hat{j}}(x,t)=a^{0,0}_{\hat{j}}(t+\frac{\tau}{\xi}x),\\ a^n_{\hat{j}}(x,t)=a^{n,0}_{\hat{j}}(t+\frac{\tau}{\xi}x)-i(2\xi)^{-1}\int_0^x(\partial_{tt}-\partial_{xx})a^{n-1}_{\hat{j}}(\theta,t-\frac{\tau}{\xi}(\theta-x))d\theta,\\ b^0_j(x,t)=b^{0,0}_j(t-\frac{\tau}{\xi}x),\quad b^1_j(x,t)=b^{1,0}_j(t-\frac{\tau}{\xi}x),\\ b^{2n}_j(x,t)=b^{2n,0}_j(t-\frac{\tau}{\xi}x)+i(2\xi)^{-1}\int_0^x(\partial_{tt}-\partial_{xx})b^{2(n-1)}_j(\theta,t+\frac{\tau}{\xi}(\theta-x)d\theta,\\ b^{2n+1}_j(x,t)=b^{2n+1,0}_j(t-\frac{\tau}{\xi}x)+i(2\xi)^{-1}\int_0^x(\partial_{tt}-\partial_{xx})b^{2n-1}_j(\theta,t+\frac{\tau}{\xi}(\theta-x)d\theta,\\ j=N_1+1,N_1+2,\cdots,N,\ n=1,2,\cdots.\end{cases}\tag{89}$$

Thus, the ray-like solutions (8) have been uniquely gotten from the initial conditions $a^{n,0}_{\hat{j}}(t),\ b^{n,0}_j(t),\ j=N_1+1,N_1+2,\cdots,N,\ n=0,1,2,\cdots$.

Step 2). The approximate solutions for the heat equations

Let us build the solutions of the form (9) for the heat equations in (7). It can be calculated directly that: for $k=1,2,\cdots,N_1$,

$$\partial_t\theta^{\epsilon}_k(x,t)\quad\sim\quad\frac{i\tau}{\epsilon}e^{i(\tau t/\epsilon+x\hat{\xi}/\sqrt{\epsilon})}\sum_{n=0}^{\infty}(\sqrt{\epsilon})^n B^n_k(x,t)+e^{i(\tau t/\epsilon+x\hat{\xi}/\sqrt{\epsilon})}\sum_{n=0}^{\infty}(\sqrt{\epsilon})^n\partial_t B^n_k(x,t),$$

$$\partial_{xx}\theta_k^\epsilon(x,t) \sim -\frac{\widehat{\xi}^2}{\epsilon}e^{i(\tau t/\epsilon+x\widehat{\xi}/\sqrt{\epsilon})}\sum_{n=0}^{\infty}(\sqrt{\epsilon})^n B_k^n(x,t)+2\frac{i\widehat{\xi}}{\sqrt{\epsilon}}e^{i(\tau t/\epsilon+x\widehat{\xi}/\sqrt{\epsilon})}\sum_{n=0}^{\infty}(\sqrt{\epsilon})^n\partial_x B_k^n(x,t)$$
$$+e^{i(\tau t/\epsilon+x\widehat{\xi}/\sqrt{\epsilon})}\sum_{n=0}^{\infty}(\sqrt{\epsilon})^n\partial_{xx}B_k^n(x,t).$$

Since $\theta_k^\epsilon,\ k=1,2,\cdots,N_1$ satisfy $\theta_{k,t}^\epsilon(x,t)-\theta_{k,xx}^\epsilon(x,t)=O(\epsilon^\infty)$, similar to the discussion for the wave equations, according to the term of $O((\sqrt{\epsilon})^n)$, we get

$$\begin{cases} \partial_x B_k^0(x,t)=0, \\ -2i\widehat{\xi}\partial_x B_k^n(x,t)+(\partial_t-\partial_{xx})B_k^{n-1}(x,t)=0,\ n=1,2,3,\cdots \end{cases} \tag{90}$$

From (9), we have

$$B_k^n(x,t)|_{x=0}=f^n(t),\ \ n=0,1,2,\cdots,\ k=1,2,\cdots,N_1. \tag{91}$$

Then by (90) and (91), it is easy to get the unique functions $B_k^n(x,t),\ n=0,1,2,\cdots$ from $f^n(t)$ as follows:

$$\begin{cases} B_k^0(x,t)=f^0(t), \quad B_k^1(x,t)=f^1(t)-\frac{i\frac{df^0(t)}{dt}}{2\widehat{\xi}}x, \\ B_k^n(x,t)=f^n(t)-\frac{i}{2\widehat{\xi}}\int_0^x(\partial_t-\partial_{xx})B_k^{n-1}(x,t)dx,\ n=1,2,\cdots, \\ k=1,2,\cdots,N_1. \end{cases} \tag{92}$$

Thus, we have identified $\theta_k^\epsilon(x,t),k=1,2,\cdots,N_1$ uniquely from $f^n,n=0,1,2,\cdots$.

Step 3). Gluing $u_j^\epsilon(x,t)$ and $\theta_k^\epsilon(x,t)$ by the transmission conditions

By (9) and (91), we obtain

$$\theta_k^\epsilon(0,t) \sim e^{i\tau t/\epsilon}[\sum_{n=0}^{\infty}\epsilon^n f^{2n}(t)+\sum_{n=0}^{\infty}\epsilon^{n+\frac{1}{2}}f^{2n+1}(t)],\ k=1,2,\cdots,N_1,$$

$$\partial_x\theta_k^\epsilon(0,t) \sim \frac{1}{\sqrt{\epsilon}}e^{i\tau t/\epsilon}\Big[i\widehat{\xi}f^0(t)+\sum_{n=1}^{\infty}\epsilon^n(i\widehat{\xi}f^{2n}(t)+\partial_x B_k^{2n-1}(0,t))$$
$$+\sum_{n=1}^{\infty}\epsilon^{n-\frac{1}{2}}(i\widehat{\xi}f^{2n-1}(t)+\partial_x B_k^{2n-2}(0,t))\Big].$$

On the other hand, from (8), we get that

$$u_{\widehat{j}}^\epsilon(0,t) \sim e^{i\tau t/\epsilon}\Big[\sum_{n=0}^{\infty}\epsilon^n(a_{\widehat{j}}^{n,0}(t)+b_{\widehat{j}}^{2n,0}(t))+\sum_{n=0}^{\infty}\epsilon^{n+\frac{1}{2}}b_{\widehat{j}}^{2n+1,0}(t)\Big],$$
$$u_j^\epsilon(0,t) \sim e^{i\tau t/\epsilon}\Big[\sum_{n=0}^{\infty}\epsilon^n b_j^{2n,0}(t)+\sum_{n=0}^{\infty}\epsilon^{n+\frac{1}{2}}b_j^{2n+1,0}(t)\Big],$$
$$j=N_1+1,N_1+2,\cdots,N,j\neq\widehat{j},$$

where $a_{\widehat{j}}^{n,0}(t)$ and $b_j^{n,0}(t)$ are the same as in (88). Note that from the transmission conditions in (7), we have $\theta_k^\epsilon(0,t)=u_j^\epsilon(0,t),\ k=1,2,\cdots,N_1,\ j=N_1+1,N_1+2,\cdots,N$. Hence,

$$\begin{cases} f^0(t)=a_{\widehat{j}}^{0,0}(t)+b_{\widehat{j}}^{0,0}(t)=b_j^{0,0}(t),\ j=N_1+1,N_1+2,\cdots,N,j\neq\widehat{j}, \\ f^{2n}(t)=a_{\widehat{j}}^{n,0}(t)+b_{\widehat{j}}^{2n,0}(t)=b_j^{2n,0}(t),\ j=N_1+1,N_1+2,\cdots,N,\ j\neq\widehat{j}, \\ f^{2n+1}(t)=b_j^{2n+1,0}(t),\ j=N_1+1,N_1+2,\cdots,N,\ n=0,1,2,\cdots. \end{cases} \tag{93}$$

We further calculate that

$$\begin{aligned}\partial_x u^\epsilon_{\widehat{j}}(0,t) \sim\ & \frac{1}{\sqrt{\epsilon}}e^{i\tau t/\epsilon}\Bigg[\frac{i\xi}{\sqrt{\epsilon}}(a^{0,0}_{\widehat{j}}(t)-b^{0,0}_{\widehat{j}}(t))-i\xi b^{1,0}_{\widehat{j}}(t)\\ &+\sum_{n=1}^{\infty}\epsilon^n[-i\xi b^{2n+1,0}_{\widehat{j}}(t)+\partial_x b^{2n-1}_{\widehat{j}}(0,t)]\\ &+\sum_{n=1}^{\infty}\epsilon^{n-\frac12}[i\xi(a^{n,0}_{\widehat{j}}(t)-b^{2n,0}_{\widehat{j}}(t))+\partial_x a^{n-1}_{\widehat{j}}(0,t)+\partial_x b^{2(n-1)}_{\widehat{j}}(0,t)]\Bigg];\\ \partial_x u^\epsilon_j(0,t) \sim\ & \frac{1}{\sqrt{\epsilon}}e^{i\tau t/\epsilon}\Bigg[-\frac{i\xi}{\sqrt{\epsilon}}b^{0,0}_j(t)-i\xi b^{1,0}_j(t)+\sum_{n=1}^{\infty}\epsilon^n[-i\xi b^{2n+1,0}_j(t)+\partial_x b^{2n-1}_j(0,t)]\\ &+\sum_{n=1}^{\infty}\epsilon^{n-\frac12}[-i\xi b^{2n,0}_j(t)+\partial_x b^{2(n-1)}_j(0,t)]\Bigg],\\ & j=N_1+1,N_1+2,\cdots,N,\ j\neq\widehat{j}.\end{aligned}$$

Thus, from $\sum\limits_{j=N_1+1}^{N} u^\epsilon_{j,x}(0,t)+\sum\limits_{k=1}^{N_1}\theta^\epsilon_{k,x}(0,t)=0$,

$$\begin{aligned}&0\sim\frac{1}{\sqrt{\epsilon}}e^{i\tau t/\epsilon}\left[\frac{i\xi}{\sqrt{\epsilon}}a^{0,0}_{\widehat{j}}(t)+\sum_{n=1}^{\infty}\epsilon^{n-\frac12}(i\xi a^{n,0}_{\widehat{j}}(t)+\partial_x a^{n-1}_{\widehat{j}}(0,t))\right]\\ &+\sum_{j=N_1+1}^{N}\Bigg\{\frac{1}{\sqrt{\epsilon}}e^{i\tau t/\epsilon}\Bigg[-\frac{i\xi}{\sqrt{\epsilon}}b^{0,0}_j(t)-i\xi b^{1,0}_j(t)+\sum_{n=1}^{\infty}\epsilon^n[-i\xi b^{2n+1,0}_j(t)+\partial_x b^{2n-1}_j(0,t)]\\ &+\sum_{n=1}^{\infty}\epsilon^{n-\frac12}[-i\xi b^{2n,0}_j(t)+\partial_x b^{2n-2}_j(0,t)]\Bigg]\Bigg\}\\ &+\sum_{k=1}^{N_1}\Bigg\{\frac{1}{\sqrt{\epsilon}}e^{i\tau t/\epsilon}\Big[i\widehat{\xi}f^0(t)+\sum_{n=1}^{\infty}\epsilon^n(i\widehat{\xi}f^{2n}(t)+\partial_x B^{2n-1}_k(0,t))\\ &\quad+\sum_{n=1}^{\infty}\epsilon^{n-\frac12}(i\widehat{\xi}f^{2n-1}+\partial_x B^{2n-2}_k(0,t))\Big]\Bigg\}.\end{aligned}$$

Based on the above equation, in order to identify $u^\epsilon_j(x,t)$ and $\theta^\epsilon_j(x,t)$, we will get f^n and $b^{n,0}_j(t),\ j=N_1+1,N_1+2,\cdots,N$ from $a^{n,0}_{\widehat{j}}(t)$. From the term of $O(\epsilon^{-1})$, we have

$$a^{0,0}_{\widehat{j}}(t)=\sum_{j=N_1+1}^{N} b^{0,0}_j(t). \tag{94}$$

From the term of $O(\epsilon^{-\frac12})$, we get

$$-\sum_{j=N_1+1}^{N} i\xi b^{1,0}_j(t)+N_1 i\widehat{\xi}f^0(t)=0. \tag{95}$$

Note that from (93),

$$b^{0,0}_{\widehat{j}}(t)=f^0(t)-a^{0,0}_{\widehat{j}}(t),\qquad b^{0,0}_j(t)=f^0(t),\ j=N_1+1,N_1+2,\cdots,N,j\neq\widehat{j}, \tag{96}$$

which together with (94) implies that $2a^{0,0}_{\widehat{j}}(t)=(N-N_1)f^0(t)$. Hence,

$$f^0=\frac{2a^{0,0}_{\widehat{j}}(t)}{N-N_1}. \tag{97}$$

Then, from the third equation in (93) and (95), we have

$$-\sum_{j=N_1+1}^{N} i\xi f^1(t) + \frac{2iN_1\widehat{\xi}a_{\widehat{j}}^{0,0}(t)}{N-N_1} = 0.$$

Thus,

$$b_j^{1,0}(t) = f^1(t) = \frac{2N_1\widehat{\xi}a_{\widehat{j}}^{0,0}(t)}{(N-N_1)^2\xi},\ j = N_1+1, N_1+2, \cdots, N. \tag{98}$$

From the term of $O(\epsilon^{n-\frac{1}{2}})$ $(n \geq 1)$,

$$\sum_{k=1}^{N_1}(i\widehat{\xi}f^{2n}(t) + \partial_x B_k^{2n-1}(0,t)) + \sum_{j=N_1+1}^{N}(-i\xi b_j^{2n+1,0}(t) + \partial_x b_j^{2n-1}(0,t)) = 0. \tag{99}$$

Similarly, from the term of $O(\epsilon^{n-1}),\ n = 1, 2, \cdots,$

$$\begin{aligned} &i\xi a_{\widehat{j}}^{n,0}(t) + \partial_x a_{\widehat{j}}^{n-1}(0,t) + \sum_{j=1}^{N_1}(i\widehat{\xi}f^{2n-1}(t) + \partial_x B_j^{2n-2}(0,t)) \\ &+ \sum_{j=N_1+1}^{N}\Big[-i\xi b_j^{2n,0}(t) + \partial_x b_j^{2n-2}(0,t)\Big] = 0. \end{aligned} \tag{100}$$

When $n = 1$, from (100), we get

$$i\xi a_{\widehat{j}}^{1,0}(t) + \partial_x a_{\widehat{j}}^{0}(0,t) + \sum_{j=1}^{N_1}(i\widehat{\xi}f^1(t) + \partial_x B_j^0(0,t)) + \sum_{j=N_1+1}^{N}\Big[-i\xi b_j^{2,0}(t) + \partial_x b_j^0(0,t)\Big] = 0.$$

Then by (93),

$$i2\xi a_{\widehat{j}}^{1,0}(t) + \partial_x a_{\widehat{j}}^{0}(0,t) + \sum_{j=1}^{N_1}(i\widehat{\xi}f^1(t) + \partial_x B_j^0(0,t)) + \sum_{j=N_1+1}^{N}\Big[-i\xi f^2(t) + \partial_x b_j^0(0,t)\Big] = 0.$$

Thus, a direct calculation yields

$$f^2(t) = \frac{1}{(N-N_1)i\xi}\left[i2\xi a_{\widehat{j}}^{1,0}(t) + \partial_x a_{\widehat{j}}^{0}(0,t) + \sum_{j=1}^{N_1}(i\widehat{\xi}f^1(t) + \partial_x B_j^0(0,t)) + \sum_{j=N_1+1}^{N}\partial_x b_j^0(0,t)\right],$$

in which $b_j^{0,0}$, f^0 are given as (96), (97); $a_{\widehat{j}}^0(x,t)$, $b_j^0(x,t)$ and $B_j^0(x,t)$, can be uniquely determined by (89) and (92), respectively. Thus, by (93), we get all $b_j^{2,0}(t),\ j = N_1+1, N_1+2, \cdots, N$.

Continuing the similar argument by induction, we can obtain all the $b_j^{n,0}(t)$, $f^n(t)$ uniquely from $a_{\widehat{j}}^{n,0}(t)$ and hence $a_{\widehat{j}}^n(x,t)$, $b_j^n(x,t)$ and $B_k^n(x,t)$ can also be identified from $a_{\widehat{j}}^{n,0}(t)$. It means that by the transmission conditions in (7), the reflected waves in (8) and the heat-like solutions (9) can be determined uniquely from the constructed incoming wave in the wave domain.

7.2. **Appendix: Proof of Lemma 4.3.** This appendix is devoted to prove Lemma 4.3. We mainly construct a counterexample by the trick from [4] to show this Lemma.

Let us consider the resolvent system for $\mathcal{A}$ as follows

$$i\sigma W - \mathcal{A}W = F, \tag{101}$$

where $W = ((\theta_k)_{k=1}^{N-1}, u, z)$, $F = ((f_k^0)_{k=1}^{N-1}, f^1, f^2)$. Thus system (101) can be rewritten as follows:

$$i\sigma\theta_k - \theta_{k,xx} = f_k^0, \quad k = 1, 2, \cdots, N-1, \tag{102}$$

$$i\sigma u - z = f^1, \tag{103}$$

$$i\sigma z - u_{xx} = f^2, \tag{104}$$

with boundary and transmission conditions

$$\begin{array}{l} u(\ell_N) = \theta_k(\ell_k) = 0, \quad k = 1, 2, \cdots, N-1, \\ \theta_k(0) = u(0), \quad \forall k = 1, 2, 3, \cdots, N-1, \\ \sum\limits_{k=1}^{N-1} \theta_{k,x}(0) + u_x(0) = 0. \end{array} \tag{105}$$

Set $f_k^0 = f^1 = 0$, $k = 1, 2, \cdots, N-1$, and $f^2 = g$. Thus, (102) can be rewritten as follows:

$$i\sigma\theta_k - \theta_{k,xx} = 0, \ k = 1, 2, \cdots, N-1, \tag{106}$$

$$-\sigma^2 u - u_{xx} = g. \tag{107}$$

We calculate directly that

$$\left\{ \begin{array}{l} \theta_k(x) = \frac{\theta_k(0)}{\sinh\sqrt{i\sigma}\ell_k} \sinh\sqrt{i\sigma}(\ell_k - x), \ k = 1, 2, \cdots, N-1, \\ u(x) = \frac{u(0)}{\sin\sigma\ell_N} \sin\sigma(\ell_N - x) + \frac{\sin\sigma(\ell_N - x)}{\sigma\sin\sigma\ell_N} \int_0^{\ell_N} g(\ell_N - s)\sin\sigma(\ell_N - s)ds \\ \qquad -\frac{1}{\sigma}\int_0^{\ell_N - x} g(\ell_N - s)\sin\sigma(\ell_N - x - s)ds. \end{array} \right. \tag{108}$$

Note that $\sum\limits_{k=1}^{N-1} \theta_{k,x}(0) + u_x(0) = 0$ and $\theta_k(0) = u(0)$, $k = 1, 2, \cdots, N-1$. Hence, we get

$$\begin{aligned} &\sigma u(0)\left[-\sum_{k=1}^{N-1} \frac{\sqrt{i}\cosh\sqrt{i\sigma}\ell_k}{\sqrt{\sigma}\sinh\sqrt{i\sigma}\ell_k} - \frac{\cos\sigma\ell_N}{\sin\sigma\ell_N}\right] \\ = \ &\frac{\cos\sigma\ell_N}{\sin\sigma\ell_N}\int_0^{\ell_N} g(\ell_N - s)\sin\sigma(\ell_N - s)ds \\ &-\int_0^{\ell_N} g(\ell_N - s)\cos\sigma(\ell_N - s)ds, \end{aligned} \tag{109}$$

which implies that

$$\sigma u(0) = \frac{\frac{\cos\sigma\ell_N}{\sin\sigma\ell_N}\int_0^{\ell_N} g(\ell_N - s)\sin\sigma(\ell_N - s)ds - \int_0^{\ell_N} g(\ell_N - s)\cos\sigma(\ell_N - s)ds}{-\sum\limits_{k=1}^{N-1} \frac{\sqrt{i}\cosh\sqrt{i\sigma}\ell_k}{\sqrt{\sigma}\sinh\sqrt{i\sigma}\ell_k} - \frac{\cos\sigma\ell_N}{\sin\sigma\ell_N}}.$$

Note that $-\sum\limits_{k=1}^{N-1} \frac{\sqrt{i}\cosh\sqrt{i\sigma}\ell_k}{\sqrt{\sigma}\sinh\sqrt{i\sigma}\ell_k} \sim -\sum\limits_{k=1}^{N-1} \frac{\sqrt{i}}{\sqrt{\sigma}}$, $\sigma \to +\infty$. Now, set $g(s) = \sin(\sigma(\ell_N - s))$, where $\sigma\ell_N = 2n\pi + \frac{\pi}{2} + n^{-\frac{1}{2}}$. Then we have

$$\cos(\sigma\ell_N) \sim n^{-\frac{1}{2}}, \quad \sin(\sigma\ell_N) \sim 1, \quad \sigma \to +\infty. \tag{110}$$

Note that $\int_0^x \sin(\sigma s)\sin\sigma(x - s)ds = -\frac{x\cos(\sigma x)}{2} - \frac{\sin^3(\sigma x)}{2\sigma\ell_0} + \frac{\cos(\sigma x)\sin(2\sigma x)}{2\sigma}$. A direct calculation yields

$$\int_0^{\ell_N} \sin(\sigma s)\sin\sigma(\ell_N - s)ds \to 0, \quad \sigma \to +\infty. \tag{111}$$

Similarly, we get

$$\int_0^{\ell_N} \sin(\sigma s)\cos\sigma(\ell_N - s)ds \to \frac{\ell_N}{2}, \quad \sigma \to +\infty. \tag{112}$$

Thus, when $\sigma \to +\infty$,

$$\begin{aligned}\sigma u(0) \quad &\sim \quad \frac{\frac{\cos\sigma\ell_N}{\sin\sigma\ell_N}\int_0^{\ell_N}\sin(\sigma s)\sin\sigma(\ell_N - s)ds - \int_0^{\ell_N}\sin(\sigma s)\cos\sigma(\ell_N - s)ds}{-(N-1)\frac{\sqrt{i}}{\sqrt{\sigma}} - \frac{\cos\sigma\ell_N}{\sin\sigma\ell_N}} \\ &\sim \quad \frac{-\frac{\ell_N}{2}}{-(N-1)\frac{\sqrt{i}}{\sqrt{\sigma}} - \frac{1}{\sqrt{\sigma}}} = \sqrt{\sigma}\frac{\ell_N}{2(N-1)\sqrt{i}+2}. \end{aligned} \tag{113}$$

Hence, when $\sigma \to +\infty$,

$$\begin{aligned}\sigma u(x) \quad &= \quad \frac{\sigma u(0)}{\sin\sigma\ell_N}\sin\sigma(\ell_N - x) + \frac{\sin\sigma(\ell_N - x)}{\sin\sigma\ell_N}\int_0^{\ell_N}\sin(\sigma s)\sin\sigma(\ell_N - s)ds \\ &\quad - \int_0^{\ell_N - x}\sin(\sigma s)\sin\sigma(\ell_N - x - s)ds \\ &\sim \quad \sqrt{\sigma}\frac{\ell_N}{2(N-1)\sqrt{i}+2}\sin\sigma(\ell_N - x) - \int_0^{\ell_N - x}\sin(\sigma s)\sin\sigma(\ell_N - x - s)ds. \end{aligned} \tag{114}$$

Here we have used (110) and (111) and the boundedness of $\sin\sigma(\ell_N - x)$. Therefore,

$$\begin{aligned}\|\sigma u(x)\|^2 \sim \int_0^{\ell_N}\Bigg| &\sqrt{\sigma}\frac{\ell_N}{2(N-1)\sqrt{i}+2}\sin\sigma(\ell_N - x) \\ &- \int_0^{\ell_N - x}\sin(\sigma s)\sin\sigma(\ell_N - x - s)ds\Bigg|^2 dx. \end{aligned} \tag{115}$$

Note that $\int_0^{\ell_N - x}\sin(\sigma s)\sin\sigma(\ell_N - x - s)ds$ is bounded. Thus we have

$$\begin{aligned}\|\sigma u(x)\|^2 \quad &\geq \quad C_1\int_0^{\ell_N}\left|\sqrt{\sigma}\frac{\ell_N}{2(N-1)\sqrt{i}+2}\sin\sigma(\ell_N - x)\right|^2 dx + C_2 \\ &\sim \quad \widetilde{C}_1\sigma + \widetilde{C}_2. \end{aligned} \tag{116}$$

Therefore, $\|W\|_{\mathcal{H}}^2 \geq \|\sigma u(x)\|^2 \geq \widetilde{C}_1\sigma + \widetilde{C}_2$. Hence, there at least exists a sequence (σ_n, F_n) satisfying $\|(i\sigma_n I - \mathcal{A})^{-1}F_n\|_{\mathcal{H}}^2 \geq \widetilde{C}_1\sigma_n + \widetilde{C}_2,\ \sigma_n \to +\infty$ as $n \to \infty$.

7.3. **Appendix: Proof of Proposition 1.** This appendix is devoted to show Proposition 1. We mainly prove this proposition by estimating some inequalities together with the known observability results in [9].

Lemma 7.1. *Let $t_1, t_2 > 0$. Then for each $(\omega_1, \omega_2) \subset (0, \ell_k)$,*

$$\int_{t_1}^{t_2}\int_{\omega_1}^{\omega_2}(p_{k,x}^2(x,t) + p_{k,t}^2(x,t))dxdt \geq \frac{\omega_2 - \omega_1}{\ell_k}\int_{\widehat{t}_1}^{\widehat{t}_2}\int_0^{\ell_k}(p_{k,x}^2(x,t) + p_{k,t}^2(x,t))dxdt, \tag{117}$$

where $k = 1, 2, \cdots, N_1$ and

$$t_2 - t_1 > 2\max_k\{\ell_k - \omega_2, \omega_1\}, \widehat{t}_1 = t_1 + \max_k\{\ell_k - \omega_2, \omega_1\}, \widehat{t}_2 = t_2 - \max_k\{\ell_k - \omega_2, \omega_1\}. \tag{118}$$

Proof. By D'Alembert formula, we have that for $x_0 \in (0, \ell_k)$, $k = 1, 2, \cdots, N_1$,

$$\int_{t_1}^{t_2} (p_{k,x}^2(x_0,t) + p_{k,t}^2(x_0,t))dt \geq \int_{t_1+x_0-x}^{t_2+x-x_0} (p_{k,x}^2(x,t) + p_{k,t}^2(x,t))dt, \quad 0 \leq x < x_0$$

and

$$\int_{t_1}^{t_2} (p_{k,x}^2(x_0,t) + p_{k,t}^2(x_0,t))dt \geq \int_{t_1-x_0+x}^{t_2-x+x_0} (p_{k,x}^2(x,t) + p_{k,t}^2(x,t))dt, \quad \ell_k \geq x > x_0.$$

So,

$$\int_{t_1}^{t_2} (p_{k,x}^2(x_0,t) + p_{k,t}^2(x_0,t))dt \geq \int_{\widehat{t}_1}^{\widehat{t}_2} (p_{k,x}^2(x,t) + p_{k,t}^2(x,t))dt, \ k = 1, 2, \cdots, N_1,$$

where $\widehat{t}_1, \widehat{t}_2$ are given as (118). Thus, integrating respect to x from 0 to ℓ_k, we have

$$\ell_k \int_{t_1}^{t_2} (p_{k,x}^2(x_0,t) + p_{k,t}^2(x_0,t))dt \geq \int_{\widehat{t}_1}^{\widehat{t}_2} \int_0^{\ell_k} (p_{k,x}^2(x,t) + p_{k,t}^2(x,t))dxdt.$$

Finally, integrating respect to x_0 from ω_1 to ω_2, we obtain

$$\begin{aligned} & \ell_k \int_{t_1}^{t_2} \int_{\omega_1}^{\omega_2} (p_{k,x}^2(x_0,t) + p_{k,t}^2(x_0,t))dx_0dt \\ \geq \quad & (\omega_2 - \omega_1) \int_{\widehat{t}_1}^{\widehat{t}_2} \int_0^{\ell_k} (p_{k,x}^2(x,t) + p_{k,t}^2(x,t))dxdt. \end{aligned}$$

Therefore, (117) holds. □

Lemma 7.2. *Let $t_1, t_2, \varrho, \tau > 0$. Then there exists a constant C_3 such that for $k = 1, 2, \cdots, N_1$,*

$$C_3 \int_{t_1}^{t_2} \int_0^{\ell_k} (p_{k,t}^2 + p_k^2)dxdt \geq \int_{t_1+\varrho+\tau}^{t_2-\varrho-\tau} \int_0^{\ell_k} (p_{k,x}^2 + p_{k,t}^2)dxdt, \tag{119}$$

where ϱ, τ are some positive constants and

$$t_2 - t_1 > 2\varrho + 2\tau. \tag{120}$$

Proof. Multiplying the first equation in (38) by $x^2h(t)p_k$, where $h = h(t)$ is a non-negative smooth function such that

$$h = 1 \text{ in } [\mathrm{t}_1 + \varrho, \mathrm{t}_2 - \varrho], \quad \mathrm{h}(\mathrm{t}_1) = \mathrm{h}(\mathrm{t}_2) = 0,$$

and integrating in $(t_1, t_2) \times (0, \ell_k)$ we obtain

$$\int_{t_1}^{t_2} \int_0^{\ell_k} x^2h(t)p_kp_{k,tt}dxdt - \int_{t_1}^{t_2} \int_0^{\ell_k} x^2h(t)p_kp_{k,xx}dxdt = 0, \ k = 1, 2, \cdots, N_1. \tag{121}$$

Integrating by parts, we get

$$\begin{aligned} 0 \quad = \quad & \int_0^{\ell_k} x^2h(t)p_kp_{k,t}dx|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^{\ell_k} x^2(h'(t)p_k + h(t)p_{k,t})p_{k,t}dxdt \\ & - \int_{t_1}^{t_2} x^2h(t)p_kp_{k,x}dt|_0^{\ell_k} + \int_{t_1}^{t_2} \int_0^{\ell_k} h(t)(2xp_k + x^2p_{k,x})p_{k,x}dxdt, \\ & k = 1, 2, \cdots, N_1. \end{aligned}$$

Hence,

$$\int_{t_1}^{t_2} \int_0^{\ell_k} x^2h(t)p_{k,t}^2dxdt + \int_{t_1}^{t_2} \int_0^{\ell_k} x^2h'(t)p_{k,t}p_kdxdt$$

$$= \int_{t_1}^{t_2}\int_0^{\ell_k} 2xh(t)p_kp_{k,x}dxdt + \int_{t_1}^{t_2}\int_0^{\ell_k} x^2h(t)p_{k,x}^2dxdt.$$

Note that

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^{\ell_k} xh(t)p_kp_{k,x}dxdt &= \int_{t_1}^{t_2} h(t)xp_k^2dt|_0^{\ell_k} - \int_{t_1}^{t_2}\int_0^{\ell_k} h(t)(p_k+xp_{k,x})p_kdxdt \\ &= -\int_{t_1}^{t_2}\int_0^{\ell_k} h(t)(p_k+xp_{k,x})p_kdxdt.\end{aligned}$$

Hence, $\int_{t_1}^{t_2}\int_0^{\ell_k} xh(t)p_kp_{k,x}dxdt = -\frac{1}{2}\int_{t_1}^{t_2}\int_0^{\ell_k} h(t)p_k^2dxdt,\ k=1,2,\cdots,N_1$. So,

$$\begin{aligned}&\int_{t_1}^{t_2}\int_0^{\ell_k} x^2h(t)p_{k,t}^2dxdt + \int_{t_1}^{t_2}\int_0^{\ell_k} x^2h'(t)p_{k,t}p_kdxdt \\ =& -\int_{t_1}^{t_2}\int_0^{\ell_k} h(t)p_k^2dxdt + \int_{t_1}^{t_2}\int_0^{\ell_k} x^2h(t)p_{k,x}^2dxdt.\end{aligned}$$

Thus, using Cauchy-Schwarz inequality, we have that there exist $\tau>0$ and $\widetilde{C}_3$ such that

$$\widetilde{C}_3\int_{t_1}^{t_2}\int_0^{\ell_k}(p_{k,t}^2+p_k^2)dxdt \geq \int_{t_1}^{t_2}\int_\tau^{\ell_k} h(t)p_{k,x}^2dxdt,\ k=1,2,\cdots,N_1.$$

By the definition of $h(t)$, we obtain

$$\widetilde{C}_3\int_{t_1}^{t_2}\int_0^{\ell_k}(p_{k,t}^2+p_k^2)dxdt \geq \int_{t_1+\varrho}^{t_2-\varrho}\int_\tau^{\ell_k} p_{k,x}^2dxdt,\ k=1,2,\cdots,N_1.$$

On the other hand, from Lemma 7.1,

$$\int_{t_1+\varrho}^{t_2-\varrho}\int_\tau^{\ell_k}(p_{k,x}^2+p_{k,t}^2)dxdt \geq \frac{\ell_k-\tau}{\ell_k}\int_{t_1+\varrho+\tau}^{t_2-\varrho-\tau}\int_0^{\ell_k}(p_{k,x}^2+p_{k,t}^2)dxdt, k=1,2,\cdots,N_1.$$

Therefore, there exists constant $C_3>0$ such that

$$C_3\int_{t_1}^{t_2}\int_0^{\ell_k}(p_{k,t}^2+p_k^2)dxdt \geq \int_{t_1+\varrho+\tau}^{t_2-\varrho-\tau}\int_0^{\ell_k}(p_{k,x}^2+p_{k,t}^2)dxdt,\ k=1,2,\cdots,N_1.$$

The proof is complete. □

Note that using sidewise estimate, we can get

$$\int_{t_1}^{t_2}\int_0^{\ell_k}(p_{k,x}^2+p_{k,t}^2)dxdt \geq \ell_k\int_{t_1+\ell_k}^{t_2-\ell_k}[p_{k,x}^2(0,t)+p_{k,t}^2(0,t)]dt,\ k=1,2,\cdots,N_1. \tag{122}$$

Thus, we have the following estimate.

$$C_4\sum_{k=1}^{N_1}\int_{t_1}^{t_2}\int_0^{\ell_k}[p_{k,t}^2+p_k^2]dxdt \geq \sum_{k=1}^{N_1}\int_{t_1+\rho+\tau+\ell_k}^{t_2-\rho-\tau-\ell_k}[p_{k,x}^2(0,t)+p_{k,t}^2(0,t)]dt.$$

Then by the above inequality and the transmission conditions in (38), together with the observability results on the exterior node controls for star-sharped network system (see [9]), we get that there exists positive constant T such that

$$\sum_{k=1}^{N_1}\int_0^T\int_0^{\ell_k}[p_{k,t}^2+p_k^2]dxdt \geq \sum_{n\geq 1}\gamma_n^2[\lambda_n^2a_n^2+b_n^2],$$

where $\gamma_n^2 > 0$ given as in Remark 5, which is dependent on the lengths of the strings involved in the network; λ_n are the eigenvalues of the operator A corresponding to system (38); a_n, b_n are the Fourier coefficients of the initial condition given as (40). The proof of Proposition 1 is complete. □

7.4. **Appendix: Polynomial decay rate of system (11) (based on observability estimate) .** In this appendix, the observability estimate method in Section 5 is used to discuss the decay rate of system (11) in which only one wave equation enters in the network. Similar to the discussion in subsection 5.2, we get the following observability estimate.

$$C\sum_{k=1}^{N-1}\int_0^T\int_0^{\ell_k}[\theta_{k,tt}^2+\theta_{k,t}^2+\theta_k^2]dxdt \geq \|(\theta^0,u^0,u^1)\|_{\mathcal{H}}^2, \tag{123}$$

where (θ^0,u^0,u^1) is the initial condition in system (11). Based on (123) and Lemma 5.7, we deduce that the $(S(t))_{t\geq 0}$ associated with the system (2) decays polynomially as

$$\|S(t)W_0\|_{\mathcal{H}} \leq \frac{C}{t^{\frac{1}{2}}}\|W_0\|_{\mathcal{D}(\mathcal{A})}. \tag{124}$$

Indeed, let (Θ^r, U^r, U_t^r), $E_r(t)$, $r=-2,-1,1$ be defined as (60) and (61), respectively. Then

$$E(T)-E(S) = -\sum_{k=1}^{N-1}\int_S^T\int_0^{\ell_k}\theta_{k,t}^2dxdt,$$

$$E_{-1}(T)-E_{-1}(S) = -\sum_{k=1}^{N-1}\int_S^T\int_0^{\ell_k}(\Theta_{k,t}^{-1})^2dxdt,$$

and

$$E_{-2}(T)-E_{-2}(S) = -\sum_{k=1}^{N-1}\int_S^T\int_0^{\ell_k}(\Theta_{k,t}^{-2})^2dxdt.$$

Set $\varepsilon(t)=E(t)+E_{-1}(t)+E_{-2}(t)$. We get that for $0\leq S\leq T<\infty$,

$$\begin{aligned}\varepsilon(T)-\varepsilon(S) &= -\sum_{k=1}^{N-1}\int_S^T\int_0^{\ell_k}[\theta_{k,t}^2+(\Theta_{k,t}^{-1})^2+(\Theta_{k,t}^{-2})^2]dxdt\\ &= -\sum_{k=1}^{N-1}\int_S^T\int_0^{\ell_k}[(\Theta_{k,tt}^{-1})^2+(\Theta_{k,t}^{-1})^2+(\Theta_k^{-1})^2]dxdt.\end{aligned} \tag{125}$$

By (123), we deduce that $C\sum\limits_{k=1}^{N-1}\int_S^T\int_0^{\ell_k}[(\Theta_{k,tt}^{-1})^2+(\Theta_{k,t}^{-1})^2+(\Theta_k^{-1})^2]dxdt \geq E_{-1}(S)$. Note that $\varepsilon(t)\sim E(t)$. Hence,

$$C(E(S)-E(T)) \geq E_{-1}(S). \tag{126}$$

From Corollary 2, we get $E(t)\leq \widehat{C}^2\sqrt{E_-(t)E_1(t)}$. Hence, $E_-(t)\geq \frac{(E(t))^2}{\widehat{C}^4E_1(t)}$, which together with (126), implies that there exists a constant $C>0$ such that $C(E(mT)-E((m+1)T))\geq \frac{(E(mT))^2}{E_1(0)}$. Here we have used $E_1(t)\leq E_1(0)$, $t\geq 0$, because of the dissipativity of system (11). Thus, $C(\frac{E(mT)}{E_1(0)}-\frac{E((m+1)T)}{E_1(0)})\leq \left(\frac{E(mT)}{E_1(0)}\right)^2$. Finally, according to Lemma 5.7, set $\alpha=0$, we get (124).

Remark 10. Compared to the decay rate obtained in Section 4, it is easy to find that the decay rate obtained from the observability estimate method is not the sharp one, which also implies that the weakened observability inequality (123) is not sharp.

REFERENCES

[1] K. Ammari and M. Jellouli, Stabilization of star-shaped networks of strings, *Differential and Integral Equations,* **17** (2004), 1395–1410.

[2] K. Ammari and M. Jellouli, Remark on stabilization of tree-shaped networks of strings, *Applications of Mathematics*, **52** (2007), 327–343.

[3] K. Ammari and M. Tucsnak, Stabilization of Bernoulli-Euler beams by means of a pointwise feedback force, *SIAM J. Control Optim.*, **39** (2000), 1160–1181.

[4] M. Alves, J. Muñoz Rivera, M. Sepúlveda and O. V. Villagrán, The lack of exponential stability in certain transmission problems with localized Kelvin–Voigt dissipation, *SIAM J. Appl. Math.*, **74** (2014), 345–365.

[5] V. M. Babich, The higher-dimensional WKB method or ray method. Its analogues and generalizations, in *Partial Differential Equations V, Encyclopedia of Mathematical Sciences*, Springer-Verlag, Berlin/New York, **34** (1999), 91–131, 241–247.

[6] J. von Below, A characteristic equation associated to an eigenvalue problem on C^2-networks, *Linear Algebra Appl.,* **71** (1985), 309–325.

[7] J. von Below, Classical solvability of linear parabolic equations on networks, *J. Differ. Equations*, **72** (1988), 316–337.

[8] A. Borichev and Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, *Math. Ann.*, **347** (2010), 455–478.

[9] R. Dager and E. Zuazua, *Wave Propagation, Observation and Control in 1-d Flexible Multi-structures*, Mathématiques et Applications 50, Springer-Verlag, Berlin, 2006.

[10] C. Farhat, M. Lesoinne and P. LeTallec, Load and motion transfer algorithms for fluid/structure interaction problems with non-matching discrete interfaces: Momentum and energy conservation, optimal discretization and application to aeroelasticity, *Comp. Meth. Appl. Mech. Eng.*, **157** (1998), 95–114.

[11] Z. J. Han and L. Wang, Riesz basis property and stability of planar networks of controlled strings, *Acta Appl. Math.*, **110** (2010), 511–533.

[12] Z. J. Han and G. Q. Xu, Spectrum and dynamical behavior of a kind of planar network of non-uniform strings with non-collocated feedbacks, *Networks and Heterogeneous Media*, **5** (2010), 315–334.

[13] Z. J. Han and G. Q. Xu, Dynamical behavior of networks of non-uniform Timoshenko beams system with boundary time-delay inputs, *Netw. Heterog. Media*, **6** (2011), 297–327.

[14] J. H. Hao and Z. Liu, Stability of an abstract system of coupled hyperbolic and parabolic equations, *Z. Angew. Math. Phys.*, **64** (2013), 1145–1159.

[15] J. Lagnese, G. Leugering and E. J. P. G. Schmidt, *Modeling, Analysis and Control of Dynamic Elastic Multi-link Structures*, in Systems & Control: Foundations & Applications, Birkhäuser, Boston, 1994.

[16] Z. Liu and R. Rao, Characterization of polynomial decay rate for the solution of linear evolution equation, *Z. Angew. Math. Phys.*, **56** (2005), 630–644.

[17] Z. Liu and S. Zheng, *Semigroups Associated with Dissipative Systems*, CRC Research Notes in Mathematics, vol. 398, Chapman and Hall/CRC, Boca Raton, 1999.

[18] R. von Loon, P. D. Anderson, J. de Hart and F. P. T. Baaijens, A combined fictitious domain/adaptive meshing method for fluid-structure interaction in heart valves, *Int. J. Numer. Meth. Fluids*, **46** (2004), 533–544.

[19] Yu. I. Lyubich and V. Q. Phóng, Asymptotic stability of linear differential equations in Banach spaces, *Studia Math.*, **88** (1988), 37–42.

[20] D. Mercier and V. Régnier, Spectrum of a network of Euler-Bernoulli beams, *Journal of Mathematical Analysis and Applications,* **337** (2008), 174–196.

[21] F. Ali Mehmeti, A characterization of a generalized C^∞-notion on nets, *Integr. Equat. Oper. Th,* **9** (1986), 753–766.

[22] H. Morand and R. Ohayon, *Fluid Structure Interaction: Applied Numerical Methods*, Wiley, New York, 1995.

[23] S. Nicaise and J. Valein, Stabilization of the wave equation on 1-D networks with a delay term in the nodal feedbacks, *Networks and Heterogeneous Media,* **2** (2007), 425–479.

[24] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, 1983.

[25] J. Rauch, X. Zhang and E. Zuazua, Polynomial decay for a hyperbolic-parabolic coupled system, *J. Math. Pures Appl.*, **84** (2005), 407–470.

[26] M. E. Taylor, *Pseudodifferential Operators*, Princeton Mathematical Series, 34, Princeton University Press, Princeton, N.J., 1981.

[27] J. Valein and E. Zuazua, Stabilization of the wave equation on 1-d networks, *SIAM J. Contr. Optim,* **48** (2009), 2771–2797.

[28] G. Q. Xu, D. Y. Liu and Y. Q. Liu, Abstract second order hyperbolic system and applications to controlled networks of strings, *SIAM J. Control Optim.*, **47** (2008), 1762–1784.

[29] X. Zhang and E. Zuazua, Polynomial decay and control of a 1-d hyperbolic-parabolic coupled system, *J. Differ. Equations*, **204** (2004), 380–438.

[30] X. Zhang ang E. Zuazua, Control, observation and polynomial decay for a coupled heat-wave system, *C. R. Acad. Sci. Paris, Ser. I,* **336** (2003), 823–828.

[31] X. Zhang and E. Zuazua, Long-time behavior of a coupled heat-wave system arising in fluid-structure interaction, *Arch. Ration. Mech. An.*, **184** (2007), 49–120.

[32] E. Zuazua, Null control of a 1-d model of mixed hyperbolic-parabolic type, in *Optimal Control and Partial Differential Equations,* J. L. Menaldi et al., eds., IOS Press, 2001, 198–210.

Received December 2015; revised May 16.

E-mail address: zjhan@tju.edu.cn
E-mail address: enrique.zuazua@uam.es