Minimal controllability time for the heat equation under state constraints

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The Problem

Consider the $1-D$ heat equation

$$\begin{aligned} \dot{y}(t,x) &= \partial_x^2 y(t,x) && (t\in\mathbb{R}_+^*\,,\ x\in(0,1))\,,\\ \partial_x y(t,0) &= v_0(t) && (t\in\mathbb{R}_+^*)\,,\\ \partial_x y(t,1) &= v_1(t) && (t\in\mathbb{R}_+^*)\,,\end{aligned}$$

with initial condition $\mathrm{y}^0 \geqslant 0$, given,

$$y(0,x) = \mathrm{y}^0(x) \qquad (x\in(0,1))\,.$$

The aim is to control this system to a constant steady state $\mathrm{y}^1 > 0$

$$y(T,x) = \mathrm{y}^1 \qquad (x\in[0,1] \text{ a.e.})\,,$$

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It is well known that

- for every time $T > 0$ there exists controls v_0 and $v_1 \in L^2(0,T)$ such that $y(T,\cdot) = \mathrm{y}^1$
- if $v_0 = v_1 = 0$, y is non-negative.

Is it possible to find $T > 0$ and controls v_0 and v_1 such that y satisfies $y(T,\cdot) = \mathrm{y}^1$ together with,

$$y(t,x) \geqslant 0 \qquad (t\geqslant 0\,,\ x\in(0,1)\ \text{ a.e.})\,?$$

First considerations I

If $\inf_{x\in(0,1)} \mathrm{y}^0(x) > \mathrm{y}^1$, then y^1 cannot be reached in arbitrarily small time T.

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- finally,

$$y(t,\tfrac{1}{2}) > \mathrm{y}^1 \qquad \text{for } t \in \left[0, \frac{1}{\pi^2} \ln \frac{\inf \mathrm{y}^0}{\mathrm{y}^1}\right).$$

First considerations II

Due to the comparison principle, the constraint

$$y(t,x) \geqslant 0$$

is equivalent to the constraint

$$y(t,0) \geqslant 0 \qquad \text{and} \qquad y(t,1) \geqslant 0\,.$$

Controllability and Observability I

Consider the dynamical system

$$\dot{y} = Ay + Bu\,, \qquad y(0) = \mathrm{y}^0\,,$$

with $y(t) \in X$ the state and $u \in U$ the control, X and U are assumed to be two Hilbert spaces identified with their dual. By Duhamel formula, the solution for $u \in L^2_{loc}(\mathbb{R}, U)$ is

$$y(t) = e^{tA}\mathrm{y}^0 + \Phi_t u\,,$$

with $\Phi_t u = \displaystyle\int_0^t e^{(t-s)A} Bu(s)\,\mathrm{d}s$.

We say that (A, B) is *null-controllable* in time $T > 0$ if for every y^0, there exist a control u such that

$$e^{TA}\mathrm{y}^0 + \Phi_T u = 0\,.$$

That is to say

$$\mathrm{Ran}\, e^{TA} \subset \mathrm{Ran}\, \Phi_T\,.$$

Controllability and Observability II

Using the closed graph theorem, this is equivalent to

$$\exists c(T) > 0 \;\text{ s.t. }\; \|e^{TA^*}\mathrm{z}^1\|_X^2 \leqslant c(T)\|\Phi_T^*\mathrm{z}^1\|_{L^2(\mathbb{R}_+,U)}^2 \qquad (\mathrm{z}^1 \in X)\,,$$

that is to say,

$$\|z(0)\|_X^2 \leqslant c(T)\int_0^T \|B^*z(t)\|_U^2\,\mathrm{d}t\,,$$

where z is solution of the adjoint system

$$-\dot{z} = A^*z\,, \qquad z(T) = \mathrm{z}^1\,.$$

We say that (A^*, B^*) is *final state observable* in time T.

Controllability and Observability III

One can look for a control of minimal norm,

$$\min \quad \frac{1}{2}\int_0^T \|u(t)\|_U^2 \,\mathrm{d}t$$
$$\Big| \qquad y(T) = 0\,.$$

Using Fenchel-Rockafellar duality, we obtain that the minimal control is givan by

$$u(t) = B^* z(t)\,,$$

where z is solution of the adjoint problem and is the minimizer of

$$\min \frac{1}{2}\int_0^T \|B^* z(t)\|_U^2 \,\mathrm{d}t + \langle z(0), \mathrm{y}^0\rangle_X \;:= J(\mathrm{z}^1)\,.$$

From which we obtain that there exist a null control u satisfying

$$\int_0^T \|u(t)\|_U^2 \,\mathrm{d}t \leqslant c(T)\|\mathrm{y}^0\|_X^2\,.$$

Controllability to steady states I

A *steady state* $\bar{\mathrm{y}} \in X$ for $\dot{y} = Ay + Bu$ is an element in X such that there exists $\bar{\mathrm{u}} \in U$ such that

$$A\bar{\mathrm{y}} + B\bar{\mathrm{u}} = 0\,.$$

Proving the controllability to a steady state is equivalent as proving the null-controllability. In fact setting $\tilde{y} = y - \bar{\mathrm{y}}$ and $\tilde{u} = u - \bar{\mathrm{u}}$, we have

$$\dot{\tilde{y}} = A\tilde{y} + B\tilde{u}\,, \qquad \tilde{y}(0) = \mathrm{y}^0 - \bar{\mathrm{y}}\,.$$

The constrained Dirichlet control problem

Consider the $1-D$ heat equation

$$\begin{aligned} \dot{y}(t,x) &= \partial_x^2 y(t,x) && (t>0,\ x\in(0,1)),\\ y(t,0) &= u_0(t) && (t>0),\\ y(t,1) &= u_1(t) && (t>0), \end{aligned}$$

with constant initial condition $\mathrm{y}^0 \in L^2(0,1)$, given,

$$y(0,x) = \mathrm{y}^0(x) \qquad (x\in(0,1)).$$

The aim is to control this system to a constant steady state $\mathrm{y}^1 > 0$

$$y(T,x) = \mathrm{y}^1 \qquad (x\in[0,1] \text{ a.e.}),$$

with the control constraint

$$u_0(t) \geqslant 0 \quad \text{and} \quad u_1(t) \geqslant 0 \qquad (t>0\ \text{ a.e.}).$$

Existence of controls I

The constrained Dirichlet control problem

Proposition

There exists a time T large enough and positive controls $u_0, u_1 \in H^1(0,T)$ such that $y(T,\cdot) = \mathrm{y}^1$.

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This allows us to define

$$\underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) = \inf\left\{T > 0\,,\ \exists u_0, u_1 \in L^1(0,T) \text{ s.t. } u_0 \geqslant 0\,,\ u_1 \geqslant 0 \text{ and } y(T,\cdot) = \mathrm{y}^1\right\} \geqslant 0\,,$$

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proof. (see also Schmidt 1980)
Setting $\tilde{y}(t,x) = y(t,x) - \mathrm{y}^1$, $\tilde{u}_0(t) = u_0(t) - \mathrm{y}^1$ and $\tilde{u}_1 = u_1 - \mathrm{y}^1$, we aim to prove (omitting the tildes) that there exists a time $T > 0$ and controls u_0 and u_1 satisfying,

$$u_0(t) > -\mathrm{y}^1 \quad \text{and} \quad u_1(t) > -\mathrm{y}^1$$

such that the solution y with initial condition

$$y(0,x) = \mathrm{y}^0(x) - \mathrm{y}^1 \qquad (x \in (0,1))\,,$$

satisfies $y(T,\cdot) = 0$.

Existence of controls II

The constrained Dirichlet control problem

For any $T>0$ the existence of controls $u_0, u_1 \in H^1(0,T)$ such that $y(T,\cdot)=0$ is ensured by Fattorini-Russel 1971.

In terms of the adjoint system,

$$\begin{aligned} -\dot z(t,x) &= \partial_x^2 z(t,x) && (t>0,\ x\in(0,1)),\\ z(t,0) &= z(t,1)=0 && (t>0),\\ z(T,x) &= \mathrm{z}^0(x) && (x\in(0,1)), \end{aligned}$$

there exists a constant $\tilde c(T)>0$ such that,

$$\|z(0,\cdot)\|^2_{L^2(0,1)} \leqslant \tilde c(T)\left(\|\partial_x z(\cdot,0)\|^2_{H^{-1}(0,T)} + \|\partial_x z(\cdot,1)\|^2_{H^{-1}(0,T)}\right) \qquad (\mathrm{z}^0\in L^2(0,1)).$$

This inequality being true in any time interval, we also have

$$\|z(\tfrac{T}{2},\cdot)\|^2_{L^2(0,1)} \leqslant \tilde c(\tfrac{T}{2})\left(\|\partial_x z(\cdot,0)\|^2_{H^{-1}(0,T)} + \|\partial_x z(\cdot,1)\|^2_{H^{-1}(0,T)}\right)$$

Using the dissipativity properties,

$$\|z(0,\cdot)\|^2_{L^2(0,1)} \leqslant e^{-C_0\frac{T}{2}}\|z(\tfrac{T}{2},\cdot)\|^2_{L^2(0,1)}.$$

Consequently,

$$\|z(0,\cdot)\|^2_{L^2(0,1)} \leqslant e^{-C_0\frac{T}{2}}\tilde c(\tfrac{T}{2})\left(\|\partial_x z(\cdot,0)\|^2_{H^{-1}(0,T)} + \|\partial_x z(\cdot,1)\|^2_{H^{-1}(0,T)}\right).$$

Existence of controls III

The constrained Dirichlet control problem

By duality this means that the controls u_0 and u_1 can be chosen such that

$$\|u_i\|^2_{H^1(0,T)} \leqslant e^{-C_0\frac{T}{2}}\tilde{c}(\tfrac{T}{2})\,\|\mathrm{y}^0-\mathrm{y}^1\|^2_{L^2(0,1)} \qquad (i\in\{0,1\})$$

Using the embedding $H^1(0,T)\subset L^\infty(0,T)$,

$$\|u_i\|^2_{L^\infty(0,T)} \leqslant Ce^{-C_0\frac{T}{2}}\tilde{c}(\tfrac{T}{2})\,\|\mathrm{y}^0-\mathrm{y}^1\|^2_{L^2(0,1)} \qquad (i\in\{0,1\})$$

Thus, for T large enough,

$$\|u_0\|_{L^\infty(0,T)},\ \|u_1\|_{L^\infty(0,T)} < \mathrm{y}^1$$

and hence,

$$u_0(t) > -\mathrm{y}^1 \quad \text{and} \quad u_1(t) > -\mathrm{y}^1 \qquad (t\in[0,T]\ \text{a.e.})\,.$$

□

Minimal control time I

The constrained Dirichlet control problem

Theorem

Let $\mathrm{y}_0 \in L^2(0,1)$ *and* $\mathrm{y}_1 \in \mathbb{R}_+^*$ *with* $\mathrm{y}_0 \neq \mathrm{y}_1$. *Then,*

1. $\underline{T} := \underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) > 0$,
2. *there exist non-negative controls* $\underline{u}_0, \underline{u}_1 \in \mathcal{M}(0, \underline{T})$ *such that the solution* y *with controls* $\underline{u}_0$ *and* $\underline{u}_1$ *satisfies* $y(T, \cdot) = \mathrm{y}^1$.

The solution y, of the Dirichlet control problem with controls in the set of Radon measures, is defined by transposition.

Remark

$\underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) > 0$ *even if* $\mathrm{y}^0 < \mathrm{y}^1$.

Minimal control time II

The constrained Dirichlet control problem

Proof.

- $\underline{T} > 0$:

Define $y_n(t) = \int_0^1 y(t,x)\sin(n\pi x)\,\mathrm{d}x$. y being solution of the heat equation, we have

$$\dot{y}_n(t) = \int_0^1 \partial_x^2 y(t,x)\sin(n\pi x)\,\mathrm{d}x = -n\pi\int_0^1 \partial_x y(t,x)\cos(n\pi x)\,\mathrm{d}x$$
$$= n\pi\left(u_0(t) - (-1)^n u_1(t)\right) - (n\pi)^2 y_n(t)$$

with $y_n(0) = \int_0^1 \mathrm{y}^0(x)\sin(n\pi x)\,\mathrm{d}x := \mathrm{y}_n^0$. Thus,

$$y_n(T) = e^{-(n\pi)^2 T}\mathrm{y}_n^0 + n\pi\int_0^T e^{-(n\pi)^2(T-t)}\left(u_0(t) - (-1)^n u_1(t)\right)\,\mathrm{d}t\,.$$

On the other hand, if $y(T,x) \equiv \mathrm{y}_1$, we have $y_n(T) = \int_0^1 \mathrm{y}_1\sin(n\pi x)\,\mathrm{d}x = \frac{1-(-1)^n}{n\pi}\mathrm{y}_1$.

Consequently,

$$\frac{1-(-1)^n}{n\pi}\mathrm{y}^1 - e^{-(n\pi)^2 T}\mathrm{y}_n^0 = n\pi\int_0^T e^{-(n\pi)^2(T-t)}\left(u_0(t) - (-1)^n u_1(t)\right)\,\mathrm{d}t\,.$$

Minimal control time III

The constrained Dirichlet control problem

For $n = 2p$,

$$\int_0^T e^{(2p\pi)^2 t} \left(u_0(t) - u_1(t)\right) \mathrm{d}t = \frac{\mathrm{y}^0_{2p}}{2p\pi} \,,$$

For $n = 2p + 1$,

$$\frac{2\,\mathrm{y}^1}{(2p+1)\pi} - e^{-(2p+1)^2\pi^2 T}\, \mathrm{y}^0_{2p+1} = (2p+1)\pi \int_0^T e^{-(2p+1)^2\pi^2(T-t)} \left(u_0(t) + u_1(t)\right) \mathrm{d}t \,.$$

But,

$$e^{-(2p+1)^2\pi^2 T} \leqslant e^{-(2p+1)^2\pi^2(T-t)} \leqslant 1 \qquad (t \in [0, T]) \,.$$

u_0 and u_1 being non-negative,

$$e^{-(2p+1)^2\pi^2 T} \int_0^T \left(u_0(t) + u_1(t)\right) \mathrm{d}t \leqslant \int_0^T e^{-(2p+1)^2\pi^2(T-t)} \left(u_0(t) + u_1(t)\right) \mathrm{d}t$$

$$\leqslant \int_0^T \left(u_0(t) + u_1(t)\right) \mathrm{d}t \,,$$

Minimal control time IV

The constrained Dirichlet control problem

We have obtained,

$$\frac{2\,\mathrm{y}^1}{(2p+1)^2\pi^2} - e^{-(2p+1)^2\pi^2 T}\,\frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi} \leqslant \int_0^T (u_0(t)+u_1(t))\,\mathrm{d}t$$
$$\leqslant e^{(2p+1)^2\pi^2 T}\,\frac{2\,\mathrm{y}^1}{(2p+1)^2\pi^2} - \frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi}\,.$$

If for every $T>0$ there exists non-negative controls u_0^T and u_1^T steering y_0 to y_1 in time T, then

$$\lim_{T\to 0}\int_0^T \left(u_0^T(t)+u_1^T(t)\right)\,\mathrm{d}t = \frac{2\,\mathrm{y}^1}{(2p+1)^2\pi^2} - \frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi} := \gamma \in \mathbb{R} \qquad (p\in\mathbb{N})\,.$$

Hence,

$$\mathrm{y}^0_{2p+1} = \frac{2\mathrm{y}^1}{(2p+1)\pi} - (2p+1)\pi\gamma \qquad (p\in\mathbb{N})\,.$$

$\mathrm{y}^0 \in L^2(0,1)$, ensures that $\sum_{n=0}^{\infty}\left|\mathrm{y}^0_n\right|^2 < \infty$ and hence $\gamma=0$, $\mathrm{y}^0_{2p+1} = \dfrac{2\mathrm{y}^1}{(2p+1)\pi}$ and

$$\lim_{T\to 0}\int_0^T \left(u_0^T(t)+u_1^T(t)\right)\,\mathrm{d}t = 0\,.$$

Minimal control time V

The constrained Dirichlet control problem

Since $u_0^T \geqslant 0$ and $u_1^T \geqslant 0$, we can also conclude

$$\lim_{T\to 0}\int_0^T u_0^T(t)\,\mathrm{d}t = \lim_{T\to 0}\int_0^T u_1(t)\,\mathrm{d}t = 0\,.$$

consequently passing to the limit $T \to 0$ in

$$\int_0^T e^{(2p\pi)^2 t}\left(u_0^T(t) - u_1^T(t)\right)\,\mathrm{d}t = \frac{\mathrm{y}_{2p}^0}{2p\pi}\,,$$

we obtain

$$\mathrm{y}_{2p}^0 = 0 \qquad (p \in \mathbb{N}^*)\,.$$

All in all, since the family $\left\{\sqrt{2}\sin(n\pi\,\cdot)\right\}_{n\in\mathbb{N}^*}$ is an orthonormal basis of $L^2(0,1)$, we conclude that y^0 can be steered to y^1 in arbitrarily small time with non-negative controls if and only if

$$\mathrm{y}^0(x) = \mathrm{y}^1 \qquad (x \in (0,1))\,.$$

Minimal control time VI

The constrained Dirichlet control problem

- *Controllability in the minimal time $\underline{T}$*:

Define $(\varepsilon_k)_{k\in\mathbb{N}}$ a sequence of positive numbers converging to 0.
For every $k \in \mathbb{N}$, there exist non-negative controls $u_0^k, u_1^k \in L^1(0, \underline{T} + \varepsilon_k)$, so that the solution y satisfies $y(\underline{T} + \varepsilon_k, \cdot) = \mathrm{y}^1$.
Define $\bar{\varepsilon} = \sup_{k\in\mathbb{N}} \varepsilon_k$.

Minimal control time VI

The constrained Dirichlet control problem

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Define $(\varepsilon_k)_{k\in\mathbb{N}}$ a sequence of positive numbers converging to 0.
For every $k\in\mathbb{N}$, there exist non-negative controls $u_0^k, u_1^k \in L^1(0,\underline{T}+\varepsilon_k)$, so that the solution y satisfies $y(\underline{T}+\varepsilon_k,\cdot)=\mathrm{y}^1$.
Define $\bar{\varepsilon}=\sup\limits_{k\in\mathbb{N}}\varepsilon_k$.

According to

$$\frac{2\,\mathrm{y}^1}{(2p+1)\pi}-e^{-(2p+1)^2\pi^2 T}\,\mathrm{y}^0_{2p+1}=(2p+1)\pi\int_0^T e^{-(2p+1)^2\pi^2(T-t)}\left(u_0^k(t)+u_1^k(t)\right)\,\mathrm{d}t\,,$$

we obtain,

$$\|u_0^k\|_{L^1(0,\underline{T}+\bar{\varepsilon})}+\|u_1^k\|_{L^1(0,\underline{T}+\bar{\varepsilon})}=\int_0^{\underline{T}+\varepsilon_k}\left(u_0^k(t)+u_1^k(t)\right)\,\mathrm{d}t$$

$$\leqslant\inf_{p\in\mathbb{N}}\left(e^{(2p+1)^2\pi^2(\underline{T}+\varepsilon_k)}\frac{2\,\mathrm{y}^1}{(2p+1)^2\pi^2}-\frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi}\right)$$

$$\leqslant\frac{2e^{\pi^2(\underline{T}+\bar{\varepsilon})}\,|\mathrm{y}^1|}{\pi^2}+\frac{|\mathrm{y}^0_1|}{\pi}\leqslant\infty\,.$$

Minimal control time VII

The constrained Dirichlet control problem

In conclusion,

- The sequences $(u_0^k)_k$ and $(u_1^k)_k$ are bounded in $L^1(0, \underline{T} + \bar{\varepsilon})$,
- $(u_0^k)_k$ and $(u_1^k)_k$ have their support contained in $[0, \underline{T} + \varepsilon_k]$, with $\varepsilon_k \to 0$,
- Thus, they are (up to a subsequence) weakly convergent in the sense of measures to some non-negative controls $\underline{u}_i$ in $\mathcal{M}([0, \underline{T}])$,
- These limits ensure the control requirements in the minimal control time $\underline{T}$.

□

Minimal control time VIII

The constrained Dirichlet control problem

When y^0 is a constant initial condition, $\underline{T} := \underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ satisfies

1. if $\mathrm{y}^1 < \mathrm{y}^0$,

$$\underline{T} > \frac{1}{\pi^2} \log \frac{\mathrm{y}^0}{\mathrm{y}^1} \qquad \text{and} \qquad \sup_{p \in \mathbb{N}^*} \frac{1}{(2p+1)^2} \left(\frac{\mathrm{y}^1}{\mathrm{y}^0} - e^{-(2p+1)^2 \pi^2 \underline{T}} \right) \leqslant \frac{\mathrm{y}^1}{\mathrm{y}^0} e^{\pi^2 \underline{T}} - 1 \,;$$

2. if $\mathrm{y}^1 > \mathrm{y}^0$,

$$\frac{\mathrm{y}^1}{\mathrm{y}^0} - e^{-\pi^2 \underline{T}} \leqslant \inf_{p \in \mathbb{N}^*} \frac{1}{(2p+1)^2} \left(\frac{\mathrm{y}^1}{\mathrm{y}^0} e^{(2p+1)^2 \pi^2 \underline{T}} - 1 \right) \,,$$

Numerical examples

The constrained Dirichlet control problem

- From $y^0 = 5$ to $y^1 = 1$, $\underline{T}\left(y^0, y^1\right) \simeq 0.1931$.
- From $y^0 = 1$ to $y^1 = 5$, $\underline{T}\left(y^0, y^1\right) \simeq 0.0438$.

Consequences for the $1-D$ heat equation with non-negative state constraints I

Consider the $1-D$ heat equation

$$\begin{aligned} \dot{y}(t,x) &= \partial_x^2 y(t,x) + \mathbf{1}_\omega(x) w(t,x) && (t>0\,,\ x\in(0,1))\,,\\ \partial_x y(t,0) &= v_0(t) && (t>0)\,,\\ \partial_x y(t,1) &= v_1(t) && (t>0)\,,\end{aligned}$$

with initial condition $\mathrm{y}^0 > 0$, given,

$$y(0,\cdot) = \mathrm{y}^0 \in L^2(0,1) \qquad (x\in(0,1))\,.$$

The aim is to control this system to a constant steady state $\mathrm{y}^1 > 0$

$$y(T,x) = \mathrm{y}^1 \qquad (x\in(0,1) \text{ a.e.})\,,$$

with the state constraint,

$$y(t,x) \geqslant 0 \qquad (t\geqslant 0\,,\ x\in(0,1)\ \text{ a.e.}).$$

We assume $\omega \subset (0,1)$ is such that there exists an interval $(a,b) \subset (0,1)\setminus\omega$.

Consequences for the $1-D$ heat equation with non-negative state constraints II

For $v_0, v_1 \in L^2(0,T)$ and $w \in L^2((0,T)\times\omega)$, define

$$u_a(t) := y(t,a) \quad \text{and} \quad u_b(t) := y(t,b)\,.$$

We have (see Lions-Magenes 1968), $u_a, u_b \in L^2(0,T)$.
Furthermore, $y|_{(a,b)}$ is solution of

$$\begin{aligned} \dot{y}(t,x) &= \partial_x^2 y(t,x) && (t>0\,,\ x\in(a,b))\,,\\ y(t,a) &= u_a(t) && (t>0)\,,\\ y(t,b) &= u_b(t) && (t>0)\,,\end{aligned}$$

Consequently, if v_0, v_1 and w are controls in time $T>0$ such that

$$y(t,x) \geqslant 0 \qquad \text{and} \qquad y(T,x) = \mathrm{y}^1\,,$$

then we have

$$u_a(t) \geqslant 0 \quad \text{and} \quad u_b(t) \geqslant 0 \qquad (t\in[0,T]\ \text{ a.e.})$$

and hence T cannot be arbitrarily small.

Numerical example I

Consequences for the $1-D$ heat equation with non-negative state constraints

Consider the $1-D$ heat equation with Neumann controls

$$\begin{aligned} \dot{y}(t,x) &= \partial_x^2 y(t,x) && (t>0\,,\ x\in(0,1))\,,\\ \partial_x y(t,0) &= v_0(t) && (t>0)\,,\\ \partial_x y(t,1) &= v_1(t) && (t>0)\,, \end{aligned}$$

with the state constraint,

$$y(t,x) \geqslant 0 \qquad (t\geqslant 0\,,\ x\in(0,1)\ \text{ a.e.}).$$

- From $\mathrm{y}^0 = 5$ to $\mathrm{y}^1 = 1$, $\underline{T}(\mathrm{y}^0,\mathrm{y}^1) \simeq 0.1938$.

  Remind that with Diriclet controls, we had,

The constrained Dirichlet control problem in a ball

Set $D = B(0,1) \subset \mathbb{R}^d$. We consider the control system

$$\begin{aligned} \dot{y}(t,x) &= \Delta y(t,x) && (t>0\,,\ x\in D)\,,\\ y(t,x) &= u(t,x) && (t>0\,,\ x\in \partial D)\,, \end{aligned}$$

with the initial condition given in $L^2(D)$,

$$y(0,x) = \mathrm{y}^0(x) \qquad (x\in D)\,.$$

The aim is to steer y to a constant target $\mathrm{y}^1 \in \mathbb{R}_+^*$ with non-negative controls $u \in L^2(0,T;L^2(\partial D))$, i.e.

$$u(t,x) \geqslant 0 \qquad (t>0\,,\ x\in\partial D \ \text{ a.e.})\,.$$

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Proposition (Existence of a control in long time)

Set $\mathrm{y}^0 \in L^2(D)$ and $\mathrm{y}^1 \in \mathbb{R}_+^*$ with $\mathrm{y}^0 \neq \mathrm{y}^1$.
Then there exists $T>0$ and a strictly positive control $u \in L^2(0,T,L^2(\partial D))$, such that y satisfies $y(T,\cdot) = \mathrm{y}^1$.

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Set $D = B(0,1) \subset \mathbb{R}^d$. We consider the control system

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Then there exists $T>0$ and a strictly positive control $u \in L^2(0,T,L^2(\partial D))$, such that y satisfies $y(T,\cdot) = \mathrm{y}^1$.

Thus we can define,

$$\underline{T}\left(\mathrm{y}^0,\mathrm{y}^1\right) = \inf\left\{T>0\,,\ \exists u\in L^1((0,T)\times\partial D) \text{ s.t. } u\geqslant 0 \text{ and } y(T,\cdot)=\mathrm{y}^1\right\} \geqslant 0\,.$$

Minimal control time I

The constrained Dirichlet control problem in a ball

Theorem

Given $\mathrm{y}^0 \in L^2(D)$ *and* $\mathrm{y}^1 \in \mathbb{R}_+^*$, *with* $\mathrm{y}^0 \neq \mathrm{y}^1$, *there exists a time* $\underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) > 0$ *such that if there exists a time* $T > 0$ *and a control* $u \in L^1([0, T] \times \partial D)$ *so that*

$$u(t, x) \geqslant 0 \qquad ((t, x) \in [0, T] \times \partial D \text{ a.e.})$$

and so that y *satisfies* $y(T, \cdot) = \mathrm{y}^1$, *then we have:*

$$T \geqslant \underline{T}\left(\mathrm{y}^0, \mathrm{y}^1\right) > 0 \,.$$

Minimal control time I

The constrained Dirichlet control problem in a ball

Theorem

Given $y^0 \in L^2(D)$ and $y^1 \in \mathbb{R}_+^*$, with $y^0 \neq y^1$, there exists a time $\underline{T}(y^0, y^1) > 0$ such that if there exists a time $T > 0$ and a control $u \in L^1([0,T] \times \partial D)$ so that

$$u(t,x) \geqslant 0 \qquad ((t,x) \in [0,T] \times \partial D \text{ a.e.})$$

and so that y satisfies $y(T,\cdot) = y^1$, then we have:

$$T \geqslant \underline{T}\left(y^0, y^1\right) > 0\,.$$

Proof (for $y^0 \in \mathbb{R}$).
Define $(\lambda_n)_{n\in\mathbb{N}^*}$ and $(p_n)_{n\in\mathbb{N}^*}$ solutions of the Sturm-Liouville problems:

$$p_n''(r) + \frac{d-1}{r} p_n'(r) = -\lambda_n p_n(r) \qquad (r \in (0,1))\,,$$
$$p_n(1) = p_n'(0) = 0\,,$$

in order to fix p_n, we enforce:

$$p_n(0) = 1 \qquad \text{and define} \qquad \alpha_n = p_n'(1)\,.$$

Minimal control time II

The constrained Dirichlet control problem in a ball

We have $\lambda_n > 0$ and λ_n two by two distinct (Pöschel-Trubowitz 1987).
Let us then define:

$$\varphi_n(x) = p_n(|x|) \qquad (x \in D),$$

so that we have,

$$\begin{aligned}
\Delta\varphi_n(x) &= -\lambda_n \varphi_n(x) && (x \in D), \\
\varphi_n(0) &= 1, && \\
\varphi_n(x) &= 0 && (x \in \partial D), \\
\nabla\varphi_n(x) \cdot \mathrm{n}(x) &= \alpha_n && (x \in \partial D).
\end{aligned}$$

Minimal control time III

The constrained Dirichlet control problem in a ball

Let $T>0$ and $u^T \in L^1((0,T)\times\partial D)$ be a non-negative control such that y (with initial condition $\mathrm{y}^0 \in \mathbb{R}_+^*$) satisfies $y(T,\cdot)=\mathrm{y}^1$.

For every $n \in \mathbb{N}^*$, we define $y_n(t) = \displaystyle\int_D y(t,x)\varphi_n(x)\,\mathrm{d}x$. Integrating by parts, we obtain,

$$\begin{aligned}\dot{y}_n(t) &= \int_D \Delta y(t,x)\varphi_n(x)\,\mathrm{d}x \\ &= -\int_{\partial D} y(t,x)\nabla\varphi_n(x)\cdot \mathrm{n}(x)\,\mathrm{d}\Gamma_x + \int_D y(t,x)\Delta\varphi_n(x)\,\mathrm{d}x \\ &= -\lambda_n y_n(t) - \alpha_n \int_{\partial D} u^T(t,x)\,\mathrm{d}\Gamma_x\end{aligned}$$

and hence,

$$y_n(T) = e^{-\lambda_n T} y_n(0) - \alpha_n \int_0^T e^{-\lambda_n(T-t)} \int_{\partial D} u^T(t,x)\,\mathrm{d}\Gamma_x\,\mathrm{d}t\,.$$

Setting $\mathrm{y}_n^i = \mathrm{y}^i \displaystyle\int_D \varphi_n(x)\,\mathrm{d}x = -\omega_{d-1}\frac{\alpha_n}{\lambda_n}\mathrm{y}^i$ for $i\in\{0,1\}$, we obtain

$$\frac{\omega_{d-1}}{\lambda_n}\left(\mathrm{y}^1 - e^{-\lambda_n T}\mathrm{y}^0\right) = \int_0^T e^{-\lambda_n(T-t)} \int_{\partial D} u^T(t,x)\,\mathrm{d}\Gamma_x\,\mathrm{d}t\,.$$

Minimal control time III

The constrained Dirichlet control problem in a ball

Since $u^T \geqslant 0$ and $\lambda_n > 0$, we obtain:

$$e^{-\lambda_n T} \int_0^T \int_{\partial D} u^T(t,x) \, \mathrm{d}\Gamma_x \, \mathrm{d}t \leqslant \frac{\omega_{d-1}}{\lambda_n} \left(\mathrm{y}^1 - e^{-\lambda_n T} \mathrm{y}^0 \right) \leqslant \int_0^T \int_{\partial D} u^T(t,x) \, \mathrm{d}\Gamma_x \, \mathrm{d}t \,,$$

that is to say,

$$\frac{\omega_{d-1}}{\lambda_n} \left(\mathrm{y}^1 - e^{-\lambda_n T} \mathrm{y}^0 \right) \leqslant \int_0^T \int_{\partial D} u^T(t,x) \, \mathrm{d}\Gamma_x \, \mathrm{d}t \leqslant \frac{\omega_{d-1}}{\lambda_n} \left(e^{\lambda_n T} \mathrm{y}^1 - \mathrm{y}^0 \right) .$$

Thus, if, for every $T > 0$, such a non-negative control u^T exists, we have

$$\lim_{T \to 0} \int_0^T \int_{\partial D} u^T(t,x) \, \mathrm{d}\Gamma_x \, \mathrm{d}t = \frac{\omega_{d-1}}{\lambda_n} \left(\mathrm{y}^1 - \mathrm{y}^0 \right) := \gamma \in \mathbb{R} \qquad (n \in \mathbb{N}^*) .$$

This is impossible since the λ_n are two by two distinct and $\mathrm{y}^0 \neq \mathrm{y}^1$. □

Consequences for the $d-D$ heat equation with non-negative state constraints I

Consider the control problem:

$$\begin{aligned} \dot{y}(t,x) &= \mathrm{div}\,(A\nabla y(t,x)) + \mathbf{1}_\omega(x) w(t,x) && (t>0\,,\ x\in\Omega)\,,\\ \nabla y(t,x)\cdot n(x) &= v(t,x) && (t>0\,,\ x\in\partial\Omega)\,, \end{aligned}$$

with the constant and non-negative initial condition,

$$y(0,x) = \mathrm{y}^0 \in \mathbb{R}_+^* \qquad (x\in\Omega)\,,$$

where Ω is an open bounded and regular set of $\mathbb{R}^d$, $A\in\mathbb{R}^{d\times d}$ is a positive matrix, and $\omega\subset\Omega$.

Given $\mathrm{y}^1\in\mathbb{R}_+^*$, the aim is to find controls v and w such that

$$y(T,\cdot) = \mathrm{y}^1 \quad \text{and} \quad y(t,x)\geqslant 0\,.$$

Assume there exists $x_0\in\Omega$ and $\varepsilon>0$ such that $B(x_0,\varepsilon)\subset\Omega\setminus\omega$.

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Assume there exists $x_0\in\Omega$ and $\varepsilon>0$ such that $B(x_0,\varepsilon)\subset\Omega\setminus\omega$.
Setting $A=P^\top P$, $\tilde{x}=Px$ and $\tilde{y}(P^\top x)=y(x)$, it is enough to control the system

$$\begin{aligned} \dot{\tilde{y}}(t,\tilde{x}) &= \Delta\tilde{y}(t,\tilde{x}) + \mathbf{1}_{P\omega}(\tilde{x})\tilde{w}(t,\tilde{x}) && (t>0\,,\ \tilde{x}\in P\Omega)\,,\\ \nabla\tilde{y}(t,\tilde{x})\cdot\tilde{n}(\tilde{x}) &= \tilde{v}(t,\tilde{x}) && (t>0\,,\ \tilde{x}\in P\partial\Omega)\,,\end{aligned}$$

$$\tilde{y}(0,\tilde{x}) = \mathrm{y}^0\in\mathbb{R}_+^* \qquad (\tilde{x}\in P\Omega)\,,$$

with the constraints $\quad \tilde{y}(T,\tilde{x}) = \mathrm{y}^1 \quad$ and $\quad \tilde{y}(t,\tilde{x})\geqslant 0.$

Consequences for the $d-D$ heat equation with non-negative state constraints II

Hence, we have to control,

$$\begin{aligned} \dot{y}(t,x) &= \Delta y(t,x) + \mathbf{1}_\omega(x) w(t,x) && (t>0\,,\ x\in\Omega)\,,\\ \nabla y(t,x)\cdot n(x) &= v(t,x) && (t>0\,,\ x\in\partial\Omega)\,, \end{aligned}$$

$$y(0,x) = \mathrm{y}^0 \in \mathbb{R}_+^* \qquad (x\in\Omega)\,,$$

with the constraints,

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where there exists $x_0 \in \Omega$ and $\varepsilon > 0$ such that $B(x_0,\varepsilon) \subset \Omega \setminus \omega$.

Set $T > 0$ and assume the exists such controls v and w with $v \in L^2((0,T)\times\partial\Omega)$ and $w \in L^2((0,T)\times\omega)$, due to regularity results (see Lions-Magenes 1968), we have $u_0 \in L^2((0,T)\times\partial B(x_0,\varepsilon))$, with

$$u_0(t,\cdot) = y(t,\cdot)|_{\partial B(x_0,\varepsilon)}\,.$$

Further more, $y \geqslant 0$ ensures that $u_0 \geqslant 0$ and consequently, T cannot be arbitrarily small.

Our proofs are based on spectral decomposition and this can be used to prove similar results for:

- Parabolic equation of the form $\dot{y} = \partial_x \left(a(x)\partial_x y\right) - p(x)\partial_x y$ with internal and/or boundary control;
- Finite dimensional systems with the particular structure

$$\dot{y} = \begin{pmatrix} A_0 & A_1 \\ \tilde{A}_0 & \tilde{A}_1 \end{pmatrix} y + \begin{pmatrix} 0 \\ \tilde{B}_1 \end{pmatrix} u\,.$$

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- $d-D$ heat equations with space dependent coefficients.
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THANK YOU FOR YOUR ATTENTION!