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Minimal controllability time for the heat equation under unilateral state or control constraints

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The heat equation with homogeneous Dirichlet boundary conditions is well known to preserve non-negativity. Besides, due to infinite velocity of propagation, the heat equation is null controllable within arbitrary small time, with controls supported in any arbitrarily open subset of the domain (or its boundary) where heat diffuses. The following question then arises naturally: can the heat dynamics be controlled from a positive initial steady state to a positive final one, requiring that the state remains non-negative along the controlled time-dependent trajectory? We show that this state-constrained controllability property can be achieved if the control time is large enough, but that it fails to be true in general if the control time is too short, thus showing the existence of a positive minimal controllability time. In other words, in spite of infinite velocity of propagation, realizing controllability under the unilateral non-negativity state constraint requires a

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positive minimal time. We establish similar results for unilateral control constraints. We give some explicit bounds on the minimal controllability time, first in 1D by using the sinusoidal spectral expansion of solutions, and then in the multi-dimensional case. We illustrate our results with numerical simulations, and we discuss similar issues for other control problems with various boundary conditions.

Keywords: Heat process; control; constraints; waiting time.

AMS Subject Classification: 35K05, 93B05, 93C20

1. Introduction

This work is devoted to analyze the controllability properties on the heat equation under natural unilateral constraints on the state or on the control.

The free heat equation, in the absence of control and, for instance, for homogeneous Dirichlet boundary conditions, is well known to preserve the non-negativity of the initial datum. Besides, due to infinite velocity of propagation, the heat equation is null controllable within arbitrary small time with controls acting on an arbitrarily small open subset of the domain (or its boundary) where heat propagates.

The following question then arises naturally: if the initial datum and the final target are positive steady states, can the heat dynamics be controlled under a non-negativity state (or control) constraint along the whole trajectory?

In this paper, we present two types of complementary results exhibiting a "waiting or minimal time phenomenon". Roughly, we first show that the system can be steered from any positive state to any positive steady state while preserving non-negativity of the state *provided that the time horizon* $[0,T]$ *be long enough.* This is not surprising: in fact, when the time interval is long, we expect the control property to be achieved with controls of small amplitude, thus ensuring small deformations of the state and, in particular, preserving its positivity. As we will see, it suffices that the target be positive, regardless of the sign of the initial datum, to ensure that the controls be positive as well. If, in addition, the initial datum is positive as well, the positivity of the control ensures positivity of the state everywhere.

More surprisingly, we then prove that, if the time interval is too short, then controllability fails under the non-negativity state constraint. In other words, even if the initial datum and final steady target are positive, non-negativity cannot be preserved along the controlled trajectory if the time horizon is too short: in spite of infinite velocity of propagation, if the time horizon is too short, controlling the system then requires to violate the natural non-negativity constraint on the state. This means, roughly speaking, that the necessary action of the control to avoid the state to cross the boundary established by the constraint is an impediment for the state to reach the target, unless the control time horizon is large enough.

This negative result, which is counterintuitive to some extent, is a serious warning for the practical use of existing controllability results, which are valid within arbitrarily small time since often in applications state constraints need to be preserved along controlled trajectories. This is the case, for instance, when the heat equation or, more generally, the diffusive system under consideration, models the propagation of a population density, as in the context of population dynamics.

We mainly focus on the heat equation with Dirichlet boundary control. To simplify the presentation, we first analyze the 1D heat equation although, as we shall see, the results can then be extended to the multi-dimensional case.

To begin with, we consider the 1D heat equation with Dirichlet boundary controls:

$$\partial_t y(t,x) = \partial_x^2 y(t,x) \quad (t > 0, x \in (0,1)), \tag{1.1a}$$

$$y(t,0) = u_0(t) \qquad (t > 0), \tag{1.1b}$$

$$y(t,1) = u_1(t) \qquad (t > 0), \tag{1.1c}$$

with *constant* initial condition

$$y(0,x) = \mathrm{y}^0 > 0 \quad \text{(steady state)}.$$

The controls are represented by the time-dependent functions $u_0(t)$ and $u_1(t)$ that act on the system at the boundary points $x = 0$ and $x = 1$, respectively. Here, we assume that the initial state is a positive constant $\mathrm{y}^0 > 0$, thus a steady state of the system that can be sustained with constant controls $u_0(t) \equiv u_1(t) \equiv \mathrm{y}^0$.

Given a constant steady state target $\mathrm{y}^1 > 0$ (steady state sustainable with the constant controls $u_0(t) \equiv u_1(t) \equiv \mathrm{y}^1$), as mentioned above, whatever the time of control $T > 0$ is, we know that there exist controls u_0 and u_1 in $L^2(0,T)$ steering the system (1.1) to y^1 in time T, i.e. such that

$$y(T,x) = \mathrm{y}^1 \quad (x \in [0,1] \text{ a.e.}). \tag{1.2}$$

The problem we analyze is whether the controls u_0 and u_1 can be chosen so that the solution remains non-negative along the time interval, i.e. whether the following unilateral state constraint can be satisfied:

$$y(t,x) \geq 0, \quad \forall\, t \in (0,T), \quad x \in (0,1). \tag{1.3}$$

Of course we are only interested in the nontrivial case where $\mathrm{y}^1 \neq \mathrm{y}^0$. Otherwise, when $\mathrm{y}^1 = \mathrm{y}^0$ the trivial trajectory $y \equiv \mathrm{y}^0 = \mathrm{y}^1$ solves the problem with constant controls $u_0 \equiv u_1 \equiv \mathrm{y}^0 = \mathrm{y}^1$. In this paper, we establish two types of results:

- Controllability can be achieved while preserving non-negativity (1.3) if $T > 0$ is large enough;
- Controllability fails if $T > 0$ is too small; more precisely, there exists $\underline{T}(\mathrm{y}^0, \mathrm{y}^1) > 0$ such that, for every $T \in (0, \underline{T}(\mathrm{y}^0, \mathrm{y}^1))$, there do not exist any controls u_0 and u_1 such that the solution y of the heat equation, with initial condition $y(0) = \mathrm{y}^0$, satisfies (1.2) and the state constraint (1.3). This means that there is a positive *minimal time* or *waiting time* for constrained control.

In the absence of state constraints it is well known that, given any $T > 0$, there exist controls such that the solution y of the heat equation reaches y^1 in time T

(see for instance Sec. 11.5 of Ref. 25 and the references therein), and this result is even valid for more general multi-dimensional heat equations with various types of controls and boundary conditions. But it is important to emphasize that usual results of the literature do not take into consideration the objective of preserving state constraints. Actually the numerical results in the existing literature show that the corresponding controls and controlled trajectories may enjoy significant oscillations (see Ref. 16), which are not compatible with the requirement of preserving the non-negativity of the state.

Our result ensuring that constrained controllability can be achieved if T is large enough can be roughly proved as follows. When T is large, the cost of control tends to zero. In fact in the present setting of Dirichlet boundary controls the cost of control can be made exponentially small and the control arbitrarily small in L^∞-norm as $T \to +\infty$. Moreover, the comparison or maximum principle for the heat equation guarantees that the controlled trajectory remains in a tubular neighborhood of y^0 and y^1 all along the time horizon $[0, T]$ and, in particular, preserves the non-negativity constraint when y^0 and y^1 are positive. Note that this argument fails in small time when the size of controls may become large.

To prove that, in general, if the control time T is too small then non-negativity of the controlled solution cannot be guaranteed (which ensures the positivity of the "minimal" or "waiting time"), we use the classical maximum or comparison principle for the heat equation. In fact, since $\mathrm{y}^0 > 0$, controllability under the state constraint (1.3) is equivalent to the apparently weaker property

$$u_0(t) = y(t, 0) \geq 0 \quad \text{and} \quad u_1(t) = y(t, 1) \geq 0 \quad (t \in (0, T) \text{ a.e.}). \tag{1.4}$$

Indeed, when the non-negativity state constraint (1.3) is satisfied, the boundary constraint (1.4) is satisfied too. On the other hand, the latter and the positivity of the initial state suffice to show that the state is non-negative everywhere.

This shows the equivalence between the problems of control under non-negative state and control constraints for the heat equation with Dirichlet controls.

In view of this, in some particular cases, it is easy to see that a positive minimal time is required for controlling the system under non-negativity state constraints. This occurs for instance if the final target is smaller than the initial datum, i.e. if $0 < \mathrm{y}^1 < \mathrm{y}^0$ (recall that y^0 and y^1 are constant). Indeed, from the maximum or comparison principle for the heat equation we infer that, whatever the controls u_0 and u_1 are, if the solution y satisfies the non-negativity state constraint (1.3), then, due to (1.4), we must have

$$y(t, x) \geq \tilde{y}(t, x) \quad (t \in (0, T), x \in (0, 1) \text{ a.e.}),$$

where $\tilde{y}$ is solution of

$$\begin{aligned} \partial_t \tilde{y}(t, x) &= \partial_x^2 \tilde{y}(t, x) && (t > 0, x \in (0, 1)), \\ \tilde{y}(t, 0) &= \tilde{y}(t, 1) = 0 && (t > 0), \\ \tilde{y}(0, x) &= \mathrm{y}^0 && (x \in (0, 1)), \end{aligned}$$

and then, since $\tilde{y}(0,x) \geq \mathrm{y}^0 \sin(\pi x)$, we deduce that

$$\tilde{y}(t,x) \geq \mathrm{y}^0 \exp(-\pi^2 t) \sin(\pi x),$$

$\sin(\pi\,\cdot)$ being the first eigenfunction of the Dirichlet–Laplacian in $(0,1)$. It follows that

$$\sup_{x\in[0,1]} y(t,x) \geq \mathrm{y}^0 \exp(-\pi^2 t).$$

Thus, if $0 < \mathrm{y}^1 < \mathrm{y}^0$, then there exists a positive minimal time $\underline{T}(\mathrm{y}^0, \mathrm{y}^1) > 0$ to steer the system from y^0 to y^1, and moreover we have

$$\underline{T}(\mathrm{y}^0, \mathrm{y}^1) \geq \frac{1}{\pi^2} \ln\left(\frac{\mathrm{y}^0}{\mathrm{y}^1}\right).$$

In this paper, we will also establish that a similar but much less obvious phenomenon occurs when $\mathrm{y}^1 > \mathrm{y}^0 > 0$. In contrast to the previous observation corresponding to the case where $0 < \mathrm{y}^1 < \mathrm{y}^0$, this fact is quite surprising because, at the first view, nothing prevents y from increasing as fast as needed under the action of non-negative controls. To handle this case, we will need to study more deeply the Dirichlet control problem, using spectral expansions of solutions. This will be done in Sec. 2 where, moreover, the initial datum y^0 can be any function in $L^2(0,1)$.

In Sec. 3, we extend these results to the multi-dimensional case. As mentioned above, the fact that constrained controllability can be achieved in large enough time is a consequence of classical results on the decay of the control cost as $T \to +\infty$, and thus it is easy to extend to the multi-dimensional setting. In contrast, the 1D proof of the necessity of a waiting time uses explicit properties of the 1D Laplacian. We first extend this result to the multi-dimensional ball, by using the explicit form of radially symmetric eigenfunctions and getting explicit lower bounds on the waiting time. Even if the eigenfunctions are easier to express in the square, they do not satisfy the sign properties needed to prove the positivity of the waiting time. This is why we first use a ball instead of a square. However, once the positivity of the waiting time has been established for the unit ball, we extend this result to any other domain by comparison with the largest ball included in the domain. In Sec. 3.3, we present an interesting remark done by Tucsnak (see Ref. 24), indicating that the waiting time phenomenon for the control of the linear heat equation under non-negativity state constraints is related to that of Ref. 17 on the control of the viscous Hamilton–Jacobi equations.

In Sec. 4, we consider similar issues in various other situations. First, in the case of Neumann boundary control, the same arguments lead to a waiting time principle under state constraints. The reason is the same: if the state remains positive, it can always be viewed as a controlled trajectory with positive Dirichlet controls. But we also consider, among others, the case of interior controls applied in a subdomain of the region where heat diffuses.

In Sec. 5, we present some numerical simulations that confirm our analytical results, shedding new light on the nature of the constrained controls in the minimal control time and raising new questions.

We conclude with a section devoted to open problems and some possible extensions.

2. Minimal Time for the Dirichlet-Controlled 1D Heat Equation under Non-Negativity Constraints

In this section, we consider the 1D heat equation (1.1) with Dirichlet boundary controls. We assume that the final target y^1 is a positive constant, which corresponds to a steady state of the system.

We have seen in Sec. 1 that, if $\mathrm{y}^0 > 0$, then controllability under the non-negativity state constraint (1.3) is equivalent to controllability under the non-negativity control constraints (1.4). This is why this section is devoted to consider unilateral non-negativity control constraints for arbitrary initial data y^0 in $L^2(0,1)$.

But all results hereafter are then valid as well for unilateral non-negativity state constraints as soon as the initial datum y^0 is non-negative.

2.1. *Main results*

We consider an arbitrary $\mathrm{y}^0 \in L^2(0,1)$ (without any sign requirement), and the problem of steering the heat equation (1.1) from y^0 to the positive steady state y^1, under the unilateral control constraints

$$u_0(t) \geq 0, \quad u_1(t) \geq 0, \quad \text{a.e. } t \in (0,T). \tag{2.1}$$

Our first main result states the existence of such controls for a large enough time T.

Theorem 2.1. *Let* $\mathrm{y}^0 \in L^2(0,1)$ *be arbitrary, and let* $\mathrm{y}^1 > 0$ *be a positive constant, such that* $\mathrm{y}^0 \neq \mathrm{y}^1$. *Then we have the following results*:

- *There exist* $T > 0$, *depending on the initial and final data* $\mathrm{y}^0, \mathrm{y}^1$, *and controls* $u_0 \in L^1(0,T)$ *and* $u_1 \in L^1(0,T)$, *satisfying the non-negativity constraints* (2.1), *such that the corresponding solution* y *of* (1.1), *with* $y(0) = \mathrm{y}^0$, *satisfies* $y(T) = \mathrm{y}^1$.
- *For the minimal control time defined by*

$$\begin{aligned}\underline{T}(\mathrm{y}^0,\mathrm{y}^1) = \inf\{T > 0, \exists\, u_0, u_1 \in L^1(0,T)\, s.t.\ u_0 \geq 0, u_1 \geq 0 \\ \text{and } y(0) = \mathrm{y}^0, y(T) = \mathrm{y}^1\},\end{aligned} \tag{2.2}$$

  we have that $\underline{T}(\mathrm{y}^0,\mathrm{y}^1) > 0$.
- *Moreover, there exist non-negative controls* $\underline{u}_0, \underline{u}_1 \in \mathcal{M}(0, \underline{T}(\mathrm{y}^0,\mathrm{y}^1))$ (*the space of Radon measures*) *steering the heat equation* (1.1) *from* y^0 *to* y^1 *in time* $\underline{T}(\mathrm{y}^0,\mathrm{y}^1)$.

Remark 2.1. As we will see in the proof, for $T > 0$ large enough the controls are actually positive due to positivity of the target at the final time, without any sign requirement on y^0.

Remark 2.2. This result states not only that the minimal time required to steer y^0 to y^1 in Theorem 2.1 is positive, but also that controllability can also be achieved at the minimal time $\underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ with controls that are Radon measures.

In fact, the numerical simulations we present further seem to indicate that controls at the minimal time cannot be more regular.

Remark 2.3. Given any $T > \underline{T}(\mathrm{y}^0, \mathrm{y}^1)$, by definition, there exist controls u_0 and u_1 in $L^1(0, T)$, satisfying the non-negativity constraints (2.1), steering the heat equation (1.1) from y^0 to y^1 in time T. If y^0 is symmetric with respect to $x = 1/2$, meaning that $\mathrm{y}^0(x) = \mathrm{y}^0(1 - x)$ for almost every $x \in [0, 1]$, then we can take $u_0 = u_1$.

Theorem 2.1 and the above remarks are proved in Sec. 2.3. As discussed previously, this result yields a positive minimal time as well for the control of the Dirichlet heat equation under non-negativity state constraints.

In Sec. 2.5, we give explicit lower estimates of the minimal time $\underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ when the initial datum y^0 is a positive constant.

When controls are Radon measures, the weak solution of the heat equation is defined by transposition as recalled in Sec. 2.2.

Remark 2.4. The previous statement can be extended to deal with the problem of controllability to trajectories:

> *Let* $\bar{\mathrm{y}}^0 \in L^2(0, 1)$ *and let* $\bar{u}_0$ *and* $\bar{u}_1$ *in* $L^1_{\mathrm{loc}}([0, +\infty))$ *be arbitrary. Let* $\bar{y}$ *be the solution of* (1.1) *corresponding to the initial condition* $\bar{y}(0) = \bar{\mathrm{y}}^0$ *and to the controls* $\bar{u}_0$ *and* $\bar{u}_1$. *We assume that there exists* $\nu > 0$ *such that* $\bar{u}_i(t) \geq \nu$, *for* $i = 0, 1$ *and for almost every* $t \in [0, +\infty)$.
>
> *Then, for every* $\mathrm{y}^0 \in L^2(0, 1)$, *there exist* $T > 0$ *and non-negative controls* $u_0 \in L^1(0, T)$ *and* $u_1 \in L^1(0, T)$ *such that the solution* y *of* (1.1) *corresponding to the initial condition* $y(0) = \mathrm{y}^0$ *and to the controls* u_0 *and* u_1 *satisfies* $y(T) = \bar{y}(T)$.

This result is an extension of Theorem 2.1 (where $\bar{y} \equiv \mathrm{y}^1$ and $\bar{u}_0 = \bar{u}_1 = \mathrm{y}^1$). The proof is similar to the one of Theorem 2.1: by subtracting $\bar{y}$ to y, we have to steer $\mathrm{y}^0 - \bar{\mathrm{y}}^0$ to 0 with control $v = u - \bar{u}$. We then prove that there exists a control $v \in H^1(0, T)$ (for T large enough) satisfying $v(t) \geq -\nu$, i.e. $u(t) \geq 0$. Then similarly, we can define a minimal reaching time $\underline{T}(\bar{\mathrm{y}}^0, \bar{y})$.

The minimal time is defined in (2.2) with controls in $L^1(0, T)$, but it could also be defined with other functional control spaces. But the arguments we shall develop, based on spectral expansions, lead to uniform bounds for the controls in $L^1(0, T)$ with respect to $T > \underline{T}(\mathrm{y}^0, \mathrm{y}^1)$. On the other hand, the following result shows that, if controllability can be achieved in time T with controls in $L^1(0,T)$, then it can also be achieved in time $T+\tau$ with $\tau > 0$ arbitrarily small, with controls that are arbitrarily more regular and this shows that, roughly, the minimal time is independent of the class of regularity of the controls that we employ in its definition.

Proposition 2.1. *Let* $\mathrm{y}^0 \in L^2(0,1)$ *be arbitrary, and let* $\mathrm{y}^1 > 0$ *be a positive constant. Let* $T > 0$ *be such that there exist controls* $u_0, u_1 \in L^1(0,T)$, *satisfying the non-negativity constraints* (2.1), *for which the corresponding solution* y *of* (1.1), *with* $y(0) = \mathrm{y}^0$, *satisfies* $y(T) = \mathrm{y}^1$.

Now, let $n \in \mathbb{N}^*$ *and let* $\tau > 0$ *be arbitrary. Then there exist controls* $\tilde{u}_0 \in C^n([0,T+\tau])$ *and* $\tilde{u}_1 \in C^n([0,T+\tau])$, *satisfying the non-negativity constraints* (2.1), *such that the corresponding solution* $\tilde{y}$ *of* (1.1), *with* $\tilde{y}(0) = \mathrm{y}^0$, *satisfies* $\tilde{y}(T+\tau) = \mathrm{y}^1$.

This result will be proved in Sec. 2.4.

2.2. *Weak solutions with Radon measures as controls*

We recall here the concept of solution y of the Dirichlet control problem (1.1) with controls in the (Banach) space of Radon measures $\mathcal{M}(0,T)$ endowed with the norm

$$\|\mu\|_{\mathcal{M}(0,T)} = \sup\left\{\int_{[0,T]} \varphi(t)\mathrm{d}\mu(t) \,|\, \varphi \in C^0([0,T],\mathbb{R}),\ \max_{[0,T]}|\varphi| = 1\right\}.$$

Solutions of the Dirichlet control problem (1.1) with controls in $\mathcal{M}(0,T)$ are defined by transposition: given $\mathrm{y}^0 \in L^2(0,1)$, u_0 and u_1 in $\mathcal{M}(0,T)$, we say that y is a weak solution of (1.1) if

$$\begin{aligned}\int_0^T\int_0^1 &\big(-\partial_t\varphi(t,x) - \partial_x^2\varphi(t,x)\big)y(t,x)\mathrm{d}x\,\mathrm{d}t - \int_0^1 \mathrm{y}^0(x)\varphi(0,x)\mathrm{d}x\\ &= \int_{[0,T]} \partial_x\varphi(t,0)\mathrm{d}u_0(t) - \int_{[0,T]} \partial_x\varphi(t,1)\mathrm{d}u_1(t), \qquad (2.3)\end{aligned}$$

for every $\varphi \in C^2([0,T]\times[0,1])$ satisfying $\varphi(t,0) = \varphi(t,1) = \varphi(T,x) = 0$ for all $(t,x) \in [0,T]\times[0,1]$.

When the controls are taken to be in $\mathcal{M}(0,T)$ the solution defined by transposition can be shown to be in $L^\infty(0,T;H^{-s}(0,1))$ for every $s > 3/2$. The trace of the solution at $t = 0$ and $t = T$ has to be understood in the sense of (2.3). This suffices to give a sense to the controllability problem.

To check this regularity property, it suffices to observe that the solutions of the forced adjoint problem

$$\begin{aligned}-\partial_t\varphi(t,x) &= \partial_x^2\varphi(t,x) + f(t,x) && (t \in (0,T), x \in (0,1)),\\ \varphi(t,0) &= \varphi(t,1) = 0 && (t \in (0,T)),\\ \varphi(T,x) &= 0 && (x \in (0,1)),\end{aligned}$$

satisfy $\varphi_x(0,t), \varphi_x(1,t) \in C^0([0,T])$ when $f \in L^1(0,T;H^s(0,1))$ with $s > 3/2$. This is due to the fact that $\varphi \in C^0([0,T];H^s(0,1))$ and that $H^s(0,1)$ is continuously embedded in $C^1(0,1)$ when $s > 3/2$.

When the control belongs to $L^1(0,T)$, by a density argument the solution can be proved to be in $y \in C^0([0,T];H^{-s}(0,1))$, for $s > 3/2$. And this regularity property suffices to give a sense to the trace of the solution at $t = T$.

The spectral expansion of the solutions of the controlled problem provides an alternative way of representing the solutions given by transposition as above. Indeed, consider the eigenbasis $(\sqrt{2}\sin(n\pi x))_{n\in\mathbb{N}^*}$ of the Dirichlet–Laplacian. For every $t \geq 0$ and every $n \in \mathbb{N}^*$, we set

$$y_n(t) = \int_0^1 y(t,x)\sin(n\pi x)\mathrm{d}x.$$

Then, by integrating by parts, we easily have

$$\dot{y}_n(t) = n\pi(u_0(t) - (-1)^n u_1(t)) - n^2\pi^2 y_n(t),$$

with $y_n(0) = \int_0^1 \mathrm{y}^0(x)\sin(n\pi x)\mathrm{d}x = \mathrm{y}_n^0$, and thus,

$$y_n(t) = e^{-n^2\pi^2 t}\mathrm{y}_n^0 + n\pi\int_{[0,t]} e^{-n^2\pi^2(t-s)}\mathrm{d}(u_0 - (-1)^n u_1)(s).$$

Then, when the controls u_0 and u_1 belong to $\mathcal{M}(0,T)$, at the final time $t = T$ we have

$$y_n(T) = e^{-n^2\pi^2 T}\mathrm{y}_n^0 + n\pi\int_{[0,T]} e^{-n^2\pi^2(T-s)}\mathrm{d}(u_0 - (-1)^n u_1)(s),$$

which is well defined in view of the fact that $e^{-n^2\pi^2(T-s)}$ depends continuously on $s \in (0,T)$. This gives a sense to the trace of the transposition solution at the final time $t = T$.

2.3. *Proof of Theorem* 2.1

Existence of positive controls in time *T*. Let us first mention that this result directly follows from Theorem 4.1(b) of Ref. 18. However, we provide a sketch of the proof, showing moreover that the controls can even be chosen in $H^1(0,T)$.

First of all, observe that, by subtracting y^1, it suffices to prove that there exist a time $T > 0$ and controls u_0 and u_1 in $H^1(0,T)$ satisfying $u_0(t) > -\mathrm{y}^1$ and $u_1(t) > -\mathrm{y}^1$ on $[0,T]$, such that the corresponding solution y of (1.1), with $y(0) = \mathrm{y}^0 - \mathrm{y}^1$, satisfies $y(T) = 0$.

According to Theorem 3.3 of Ref. 8, for any $T > 0$ there exist controls u_0 and u_1 in $H^1(0,T)$ such that the solution y of (1.1) for any initial condition in $L^2(0,1)$ satisfies $y(T,\cdot) = 0$.

Note that, at this point, we use the classical controllability property for the 1D heat equation without any constraint. Our goal is to show that, if the control time

$T > 0$ is large enough, then the controls can be taken such that $u_0(t) > -\mathrm{y}^1$ and $u_1(t) > -\mathrm{y}^1$ for almost every $t \in [0, T]$.

It is well known (see Ref. 13) that controllability with controls in $H^1(0, T)$, without any constraint, is equivalent to an observability inequality, namely to the existence of an observability constant $c(T) > 0$ such that

$$\|z(0,\cdot)\|^2_{L^2(0,1)} \le c(T)(\|\partial_x z(\cdot,0)\|^2_{H^{-1}(0,T)} + \|\partial_x z(\cdot,1)\|^2_{H^{-1}(0,T)}),$$

for any solution of the adjoint system

$$-\partial_t z(t,x) = \partial_x^2 z(t,x) \qquad (t > 0, x \in (0,1)), \tag{2.4a}$$

$$z(t,0) = z(t,1) = 0 \quad (t > 0), \tag{2.4b}$$

such that $z(T) \in L^2(0,1)$. Actually, the controllability (and equivalently, the observability of the adjoint system) being true on any time interval (τ, T), we also have

$$\|z(\tau,\cdot)\|^2_{L^2(0,1)} \le c(T-\tau)(\|\partial_x z(\cdot,0)\|^2_{H^{-1}(\tau,T)} + \|\partial_x z(\cdot,1)\|^2_{H^{-1}(\tau,T)}).$$

Using the spectral expansion and the Parseval equality, we have

$$\|z(0,\cdot)\|^2_{L^2(0,1)} \le e^{-2\pi^2\tau}\|z(\tau,\cdot)\|^2_{L^2(0,1)},$$

for every $0 < \tau < T$, and hence

$$\|z(0,\cdot)\|^2_{L^2(0,1)} \le e^{-2\pi^2\tau}c(T-\tau)\big(\|\partial_x z(\cdot,0)\|^2_{H^{-1}(0,T)} + \|\partial_x z(\cdot,1)\|^2_{H^{-1}(0,T)}\big).$$

By duality, this means that the controls u_0 and u_1 can be chosen such that

$$\|u_i\|^2_{H^1(0,T)} \le e^{-2\pi^2\tau}c(T-\tau)\|\mathrm{y}^0 - \mathrm{y}^1\|^2_{L^2(0,1)}, \quad i = 0,1,$$

for any $0 < \tau < T$. By the continuous embedding of $H^1(0,T)$ into $L^\infty(0,T)$,

$$\|u_i\|^2_{L^\infty(0,T)} \le C\|u_i\|_{H^1(0,T)} \le C(T)e^{-2\pi^2\tau}c_1(T-\tau)\|\mathrm{y}^0 - \mathrm{y}^1\|^2_{L^2(0,1)}, \quad i = 0,1,$$

with $C(T)$ the constant of the continuous embedding. Hence, taking $\tau = T/2$ and observing that $c_1(T-\tau) = c_1(T/2)$ is monotonic decreasing with respect to T and observing that $C(T)$ is bounded by a polynomial function, we have, for T large enough,

$$\|u_0\|_{L^\infty(0,T)}, \quad \|u_1\|_{L^\infty(0,T)} < \mathrm{y}^1$$

and hence $u_0(t) > -\mathrm{y}^1$ and $u_1(t) > -\mathrm{y}^1$ on $[0,T]$. This ends the proof of Theorem 2.1.

Positivity of $\underline{T}(\mathbf{y^0}, \mathbf{y^1})$. Due to the previous point, $\underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ given by (2.2) is well defined.

In view of the spectral decomposition given in Sec. 2.2, the fact that controls u_0 and u_1 steer the solution y from y^0 to y^1 in time T is equivalent to

$$y_n(T) = \int_0^1 \mathrm{y}^1 \sin(n\pi x)\mathrm{d}x = \frac{1-(-1)^n}{n\pi}\mathrm{y}^1, \quad \forall\, n \geq 1,$$

and thus,

$$\frac{1-(-1)^n}{n\pi}\mathrm{y}^1 - e^{-n^2\pi^2 T}\mathrm{y}^0_n = n\pi \int_{[0,T]} e^{-n^2\pi^2(T-t)}\mathrm{d}(u_0 - (-1)^n u_1)(t), \quad \forall\, n \geq 1.$$

In particular, for $n = 2p$,

$$\int_{[0,T]} e^{(2p)^2\pi^2 t}\mathrm{d}(u_0 - u_1)(t) = -\frac{\mathrm{y}^0_{2p}}{2p\pi}, \tag{2.5a}$$

and for $n = 2p+1$,

$$\begin{aligned}\frac{2\mathrm{y}^1}{(2p+1)\pi} &- e^{-(2p+1)^2\pi^2 T}\mathrm{y}^0_{2p+1} \\ &= (2p+1)\pi \int_{[0,T]} e^{-(2p+1)^2\pi^2(T-t)}\mathrm{d}(u_0 + u_1)(t). \end{aligned} \tag{2.5b}$$

But, for every $t \in [0,T]$, we have $e^{-(2p+1)^2\pi^2 T} \leq e^{-(2p+1)^2\pi^2(T-t)} \leq 1$. Consequently, assuming that u_0 and u_1 are non-negative controls, we obtain, for every $p \in \mathbb{N}$,

$$\begin{aligned} e^{-(2p+1)^2\pi^2 T}\int_{[0,T]} \mathrm{d}(u_0+u_1)(t) &\leq \int_{[0,T]} e^{-(2p+1)^2\pi^2(T-t)}\mathrm{d}(u_0+u_1)(t) \\ &\leq \int_{[0,T]} \mathrm{d}(u_0+u_1)(t), \end{aligned}$$

that is,

$$\begin{aligned} &(2p+1)\pi e^{-(2p+1)^2\pi^2 T}\int_{[0,T]} \mathrm{d}(u_0+u_1)(t) \\ &\quad \leq \frac{2\mathrm{y}^1}{(2p+1)\pi} - e^{-(2p+1)^2\pi^2 T}\mathrm{y}^0_{2p+1} \leq (2p+1)\pi \int_{[0,T]} \mathrm{d}(u_0+u_1)(t) \end{aligned}$$

and hence,

$$\begin{aligned} \frac{2\mathrm{y}^1}{(2p+1)^2\pi^2} - e^{-(2p+1)^2\pi^2 T}\frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi} &\leq \int_{[0,T]} \mathrm{d}(u_0+u_1)(t) \\ &\leq e^{(2p+1)^2\pi^2 T}\frac{2\mathrm{y}^1}{(2p+1)^2\pi^2} - \frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi}. \end{aligned} \tag{2.6}$$

Now, assume by contradiction that for every $T > 0$ there exist non-negative controls u_0^T and u_1^T steering y^0 to y^1 in time T. Then, (2.6) ensures that

$\lim_{T\to 0}\int_{[0,T]}(u_0^T+u_1^T)(t)\mathrm{d}t$ exists and we have

$$\lim_{T\to 0}\int_{[0,T]}(u_0^T+u_1^T)(t)\mathrm{d}t=\frac{2\mathrm{y}^1}{(2p+1)^2\pi^2}-\frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi}\quad(p\in\mathbb{N}).$$

Then, by uniqueness of the limit, necessarily, there exists $\gamma\in\mathbb{R}$ such that

$$\gamma=\frac{2\mathrm{y}^1}{(2p+1)^2\pi^2}-\frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi},$$

and therefore

$$\mathrm{y}^0_{2p+1}=\frac{2\mathrm{y}^1}{(2p+1)\pi}-(2p+1)\pi\gamma,$$

for every $p\in\mathbb{N}$. Since $\mathrm{y}^0\in L^2(0,1)$ and thus $\sum_{n=0}^{+\infty}|\mathrm{y}^0_n|^2<+\infty$, we must have $\gamma=0$, and hence

$$\mathrm{y}^0_{2p+1}=\frac{2\mathrm{y}^1}{(2p+1)\pi}\quad(p\in\mathbb{N}),\quad\text{and}\quad\lim_{T\to 0}\int_{[0,T]}(u_0^T+u_1^T)\mathrm{d}t=0.$$

Since u_0^T and u_1^T are non-negative, we also conclude that $\int_{[0,T]}u_0^T\,\mathrm{d}t\to 0$ and $\int_{[0,T]}u_1^T\,\mathrm{d}t\to 0$ as $T\to 0$. Letting T tend to 0 in (2.5a), we obtain $\mathrm{y}^0_{2p}=0$ for every $p\in\mathbb{N}^*$.

All in all, since the family $(\sqrt{2}\sin(n\pi\cdot))_{n\in\mathbb{N}^*}$ is an orthonormal basis of $L^2(0,1)$, taking into account that $\mathrm{y}^0_{2p+1}=2\mathrm{y}^1/((2p+1)\pi)$ and $\mathrm{y}^0_{2p}=0$ for every p, we conclude that y^0 can be steered to y^1 in arbitrarily small time with non-negative controls if and only if $\mathrm{y}^0=\mathrm{y}^1$. This proves the first part of the theorem.

Constrained controllability at the minimal time $\underline{T}=\underline{T}(\mathrm{y}^0,\mathrm{y}^1)$. Let us prove the existence of measure-valued non-negative controls realizing the controllability exactly in time $\underline{T}$.

In view of the definition of the minimal control time as the infimum of positive times T for which there exist controls u_0 and u_1 in $L^1(0,T)$, for every $n\in\mathbb{N}$, there exist controls u_0^n and u_1^n in $L^1(0,\underline{T}+1/n)$ satisfying the constraints (2.1), such that the corresponding solution y of (1.1), with $y(0)=\mathrm{y}^0$, satisfies $y(\underline{T}+1/n)=\mathrm{y}^1$. We extend the controls u_0^n and u_1^n by 0 on $(T+1/n,T+1)$. According to (2.6), we have, for every $n\in\mathbb{N}$,

$$\begin{aligned}\|u_0^n\|_{L^1(0,\underline{T}+1)}+\|u_1^n\|_{L^1(0,\underline{T}+1)}&=\int_0^{\underline{T}+1/n}(u_0(t)+u_1(t))\mathrm{d}t\\&\leq\inf_{p\in\mathbb{N}}\left(e^{(2p+1)^2\pi^2\underline{T}}\frac{2\mathrm{y}^1}{(2p+1)^2\pi^2}-\frac{\mathrm{y}^0_{2p+1}}{(2p+1)\pi}\right)\\&\leq\frac{2e^{\pi^2\underline{T}}\mathrm{y}^1}{\pi^2}-\frac{\mathrm{y}^0_1}{\pi}<+\infty.\end{aligned}$$

Then, the sequences $(u_0^n)_n$ and $(u_1^n)_n$ are bounded in $L^1(0,\underline{T}+1)$ and therefore, by weak compactness of $\mathcal{M}(0,T)$, taking a subsequence if necessary, we may assume that they converge in the weak sense of Radon measures to some controls $\underline{u}_i$ in $\mathcal{M}(0,T)$.

Thanks to the well-posedness results recalled at the beginning of the section, ensuring that the corresponding solutions are bounded in $C^0([0,\underline{T}],H^{-s}(0,1))$ for any $s>3/2$, we can pass to limit (taking subsequences if necessary) so that the controls converge weakly in the sense of measures in $\mathcal{M}(0,T)$ and the corresponding solutions in the weak-* topology of $L^\infty([0,T],H^{-s}(0,1))$ for any $s>3/2$. Clearly, the limit controls satisfy the non-negativity constraint and the limit solution solves the limit non-homogeneous Dirichlet problem (1.1) in the sense of transposition (2.3). The limit solution reaches the target y^1 in time $\underline{T}$.

Proof of the statement in Remark 2.3. Assuming that $\mathrm{y}^0(x)=\mathrm{y}^0(1-x)$ for almost every $x\in[0,1]$, we claim that we can take $u_0=u_1$. Indeed, it is easy to see that, given any pair (u_0,u_1) of controls realizing the controllability at some arbitrary time $T\geq\underline{T}$, the pair $((u_0+u_1)/2,(u_0+u_1)/2)$ satisfies the same conclusion. The statement follows. □

2.4. *Proof of Proposition* 2.1

First of all, according to Theorem 2.1, for T large enough there exist controls $u_0,u_1\in L^1(0,T)$, satisfying the constraints (2.1), such that the corresponding solution y of (1.1), with $y(0)=\mathrm{y}^0$, satisfies $y(T)=\mathrm{y}^1$.

As in the proof of Theorem 2.1, it suffices to prove that there exist controls $\hat{v}_0$ and $\hat{v}_1$ in $C^n([0,T+\tau])$ satisfying $\hat{v}_0(t)>-\mathrm{y}^1$ and $\hat{v}_1(t)>-\mathrm{y}^1$ on $[0,T+\tau]$, such that the corresponding solution $\hat{y}$ of (1.1), with $\hat{y}(0)=\mathrm{y}^0-\mathrm{y}^1$, satisfies $\hat{y}(T+\tau)=0$. Note first that the controls $v_0=u_0-\mathrm{y}^1$ and $v_1=u_1-\mathrm{y}^1$ satisfy $v_0(t)>-\mathrm{y}^1$ and $v_1(t)>-\mathrm{y}^1$ on $[0,T]$, and the corresponding solution y of (1.1), with $y(0)=\mathrm{y}^0-\mathrm{y}^1$, satisfies $y(T)=0$.

The idea of the proof is the following. First, we will smoothen the controls v_0 and v_1. Then the solution of the heat equation at time T, obtained with these smooth controls, will be some $\bar{\mathrm{y}}^1$, close to 0. Second, we will steer $\bar{\mathrm{y}}^1$ to the target 0 in time τ with smooth controls. Of course, in order to ensure that the controls steering $\bar{\mathrm{y}}^1$ to 0 in time τ are greater than $-\mathrm{y}^1$, the smaller τ is, the more $\bar{\mathrm{y}}^1$ must be close to 0.

For every $\varepsilon>0$, by a density argument, there exist $\bar{v}_0^\varepsilon$ and $\bar{v}_1^\varepsilon$ in $C^\infty(0,T)$ such that $\|v_i-\bar{v}_i^\varepsilon\|_{L^1(0,T)}<\varepsilon$, and such that all time derivatives of $\bar{v}_0^\varepsilon$ and $\bar{v}_1^\varepsilon$ vanish at time T. Since $v_0\geq-\mathrm{y}^1$ and $v_1\geq-\mathrm{y}^1$, we assume moreover that $\bar{v}_0^\varepsilon\geq-\mathrm{y}^1$ and $\bar{v}_1^\varepsilon\geq-\mathrm{y}^1$.

The well-posedness of the heat equation for L^1-Dirichlet boundary conditions (see Sec. 2.2) ensures that the solution $\bar{y}$ of (1.1), with $\bar{y}(0)=\mathrm{y}^0-\mathrm{y}^1$ and boundary controls $\bar{v}_0^\varepsilon$ and $\bar{v}_1^\varepsilon$, satisfies $\|\bar{y}(T)\|_{H^{-2}(0,1)}\leq C_1\varepsilon$, for some $C_1>0$.

Set $\bar{\mathrm{y}}^1=\bar{y}(T)\in H^{-2}(0,1)$ and $\bar{\mathrm{y}}^0=y_0(\tau/2)\in L^2(0,1)$, where y_0 is the solution of (1.1) corresponding to the initial condition $y_0(0)=\bar{\mathrm{y}}^1$ and to null boundary controls. Then there exists $C_2(\tau) > 0$ such that $\|\bar{y}^0\|_{L^2(0,1)} \leq C_2(\tau)\|\bar{y}^1\|_{H^{-2}(0,1)}$, i.e. $\|\bar{y}^0\|_{L^2(0,1)} \leq C_1 C_2(\tau)\varepsilon$.

The aim is now to steer $\bar{y}^0$ to 0 with controls $\tilde{v}_0$ and $\tilde{v}_1$ in $C^n([0,\tau/2])$ such that

$$\tilde{v}_i(t) \geq -y^1 \quad (t \in [0,\tau/2]) \quad \text{and} \quad v_i(0) = \cdots = v_i^{(n)}(0) = 0 \quad (i = 0,1).$$

This is a consequence of the following lemma.

Lemma 2.1. *Let* $y^0 \in L^2(0,1)$, *let* $n \in \mathbb{N}^*$ *and let* $\tau > 0$ *be arbitrary. There exist controls* v_0 *and* v_1 *in* $C^n([0,\tau])$ *satisfying* $v_i(0) = \cdots = v_i^{(n)}(0) = 0$, *for* $i = 1, 2$, *such that the corresponding solution of* (1.1) (*with controls* v_0 *and* v_1) *with initial condition* $y(0) = y^0$ *satisfies* $y(\tau) = 0$. *Moreover*, v_0 *and* v_1 *can be chosen such that*

$$\max(\|v_0\|_{L^\infty(0,\tau)}, \|v_1\|_{L^\infty(0,\tau)}) \leq \kappa_n(\tau)\|y^0\|_{L^2(0,1)},$$

for some $\kappa_n(\tau) > 0$ *only depending on* τ *and* n.

Proof. Pick a function $\rho \in C^\infty([0,T])$ such that $\rho(t) \in [0,1]$, $\rho(t) = 1$ for $t \in [\tau/2,\tau]$ and all derivatives of ρ vanish at 0. We will prove that there exist two functions u_0 and u_1 in $H^{n+1}(0,\tau)$ such that the solution y of (1.1) with initial condition $y(0) = y^0$ and controls $v_0 = \rho u_0$ and $v_1 = \rho u_1$ satisfies $y(\tau) = 0$. Consequently, we will have v_0 and v_1 in $C^n(0,\tau)$ and $v_i(0) = \cdots = v_i^{(n)}(0) = 0$, for $i = 0,1$.

In order to prove this fact, we will establish an observability inequality for the adjoint problem, namely, the existence of $c_n(\tau) > 0$ only depending on τ and n such that

$$\|z(0,\cdot)\|_{L^2(0,1)} \leq c_n(\tau)(\|\rho^2\partial_x z(\cdot,0)\|_{H^{-(n+1)}(0,\tau)} + \|\rho^2\partial_x z(\cdot,1)\|_{H^{-(n+1)}(0,\tau)}),$$

where z is solution of the adjoint problem (2.4).

But, according to Theorem 3.3 of Ref. 8, for any $T > 0$ there exist controls u_0 and u_1 in $H^{n+1}(0,T)$ such that the solution y of (1.1) for any initial condition in $L^2(0,1)$ satisfies $y(T) = 0$. This ensures the existence of a constant $c_n(\tau)$ such that

$$\|z(\tau/2,\cdot)\|_{L^2(0,1)} \leq c_n(\tau)(\|\partial_x z(\cdot,0)\|_{H^{-(n+1)}(\tau/2,\tau)} + \|\partial_x z(\cdot,1)\|_{H^{-(n+1)}(\tau/2,\tau)}).$$

But we have

$$\|\partial_x z(\cdot,x)\|_{H^{-(n+1)}(\tau/2,\tau)} \leq \|\rho^2\partial_x z(\cdot,x)\|_{H^{-(n+1)}(0,\tau)} \quad (x \in \{0,1\})$$

and hence,

$$\|z(\tau/2,\cdot)\|_{L^2(0,1)} \leq c_n(\tau)(\|\rho^2\partial_x z(\cdot,0)\|_{H^{-(n+1)}(0,\tau)} + \|\rho^2\partial_x z(\cdot,1)\|_{H^{-(n+1)}(0,\tau)}).$$

Due to the dissipativity properties of the heat equation, we have

$$\|z(0,\cdot)\|_{L^2(0,1)} \leq e^{-\pi^2\tau/2}\|z(\tau/2,\cdot)\|_{L^2(0,1)}$$

and hence,

$$\|z(0,\cdot)\|_{L^2(0,1)} \le e^{-\pi^2\tau/2} c_n(\tau)(\|\rho^2\partial_x z(\cdot,0)\|_{H^{-(n+1)}(0,\tau)} + \|\rho^2\partial_x z(\cdot,1)\|_{H^{-(n+1)}(0,\tau)}).$$

The latter inequality ensures that there exist two functions u_0 and u_1 in $H^{n+1}(0,\tau)$ such that the solution y of (1.1) with initial condition $y(0) = \mathrm{y}^0$ and controls $v_0 = \rho u_0$ and $v_1 = \rho u_1$ satisfies $y(\tau) = 0$. By duality, this also means that u_0 and u_1 can be chosen such that

$$\|u_i\|_{H^{n+1}(0,\tau)} \le e^{-\pi^2\tau/2} c_n(\tau)\|\mathrm{y}^0\|_{L^2(0,1)} \quad (i = 0,1).$$

Since $H^{n+1}(0,\tau)$ is continuously embedded in $L^\infty(0,\tau)$, with embedding constant $C_n(\tau)$, for any $n \ge 0$, we infer that

$$\|u_i\|_{L^\infty(0,\tau)} \le C_n(\tau)\|u_i\|_{H^{n+1}(0,\tau)} \le e^{-\pi^2\tau/2} C_n(\tau)c_n(\tau)\|\mathrm{y}^0\|_{L^2(0,1)} \quad (i = 0,1).$$

Since $\rho(t) \in [0,1]$, we get $\|v_i\|_{L^\infty(0,\tau)} \le \|u_i\|_{L^\infty(0,\tau)}$. Lemma 2.1 is proved. □

We are now in a position to conclude the proof of Proposition 2.1. According to Lemma 2.1, given any $\tau > 0$, and $\bar{\mathrm{y}}^0 \in L^2(0,1)$, there exist controls $\tilde{v}_0$ and $\tilde{v}_0$ in $C^n([0,\tau/2])$ satisfying $\tilde{v}_0(0) = \tilde{v}_1(0) = \cdots = \tilde{v}_0^{(n)}(0) = \tilde{v}_1^{(n)}(0) = 0$, such that the solution $\tilde{y}$ of the heat process (1.1) with initial condition $\tilde{y}(0) = \bar{\mathrm{y}}^0$ and Dirichlet boundary controls $\tilde{v}_0$ and $\tilde{v}_1$ satisfies $\tilde{y}(\tau/2) = 0$. Moreover, we have

$$\inf_{(0,\tau/2)} \tilde{v}_i \ge -\kappa_n(\tau/2)\|\bar{\mathrm{y}}^0\|_{L^2(0,1)} \quad (i = 0,1).$$

But, at the beginning of this proof, it has been shown that $\|\bar{\mathrm{y}}^0\|_{L^2(0,1)} \le C_1C_2(\tau)\varepsilon$, where $\varepsilon > 0$ can be chosen arbitrary small. Consequently, for $\varepsilon < \varepsilon(n,\tau) = \mathrm{y}^1/C_1C_2(\tau)\kappa_n(\tau)$, we have $\tilde{v}_i(t) \ge -\mathrm{y}^1$ for $i = 0,1$ and for almost $t \in (0,\tau/2)$.

All in all, for this small enough $\varepsilon > 0$, we set

$$\hat{v}_i(t) = \begin{cases} \bar{v}_i^\varepsilon(t) & \text{if } t \in (0,T), \\ 0 & \text{if } t \in (T, T+\tau/2) \qquad (i = 0,1) \\ \tilde{v}_i(t - T - \tau/2) & \text{if } t \in (T+\tau/2, T+\tau). \end{cases}$$

Consequently, we have $\hat{v}_0$ and $\hat{v}_1$ in $C^n([0,T+\tau])$, $\hat{v}_0(t) \ge -\mathrm{y}^1$ and $\hat{v}_1(t) \ge -\mathrm{y}^1$ and the solution $\hat{y}$ of (1.1) with Dirichlet controls $\hat{v}_0$ and $\hat{v}_1$ and initial condition $\hat{y}(0) = \mathrm{y}^0 - \mathrm{y}^1$ satisfies $\hat{y}(T+\tau) = 0$. Proposition 2.1 is proved.

2.5. *Lower estimates of the minimal time for a constant initial datum*

In this section, we assume moreover that $\mathrm{y}^0 > 0$ is a positive constant. The arguments of the proof given in the previous section allow us to derive lower estimates of the waiting time $\underline{T}(\mathrm{y}^0,\mathrm{y}^1)$.

Lemma 2.2. *Let* y^0 *and* y^1 *be positive real numbers such that* $\mathrm{y}^0 \ne \mathrm{y}^1$. *We set* $\underline{T} = \underline{T}(\mathrm{y}^0,\mathrm{y}^1)$.

(1) *If* $\mathrm{y}^1 < \mathrm{y}^0$ *then* $\underline{T} > \frac{1}{\pi^2} \ln \frac{\mathrm{y}^0}{\mathrm{y}^1}$ *and*

$$\sup_{p\in\mathbb{N}^*} \frac{1}{(2p+1)^2} \left(\frac{\mathrm{y}^1}{\mathrm{y}^0} - e^{-(2p+1)^2\pi^2\underline{T}} \right) \leq \frac{\mathrm{y}^1}{\mathrm{y}^0} e^{\pi^2\underline{T}} - 1. \tag{2.7a}$$

(2) *If* $\mathrm{y}^1 > \mathrm{y}^0$ *then*

$$\frac{\mathrm{y}^1}{\mathrm{y}^0} - e^{-\pi^2\underline{T}} \leq \inf_{p\in\mathbb{N}^*} \frac{1}{(2p+1)^2} \left(\frac{\mathrm{y}^1}{\mathrm{y}^0} e^{(2p+1)^2\pi^2\underline{T}} - 1 \right). \tag{2.7b}$$

Proof. By Remark 2.3, we can take $u_0 = u_1$. Using the notations of the proof of Theorem 2.1, since y^0 is constant, we have $\mathrm{y}^0_n = \frac{1-(-1)^n}{n\pi}\mathrm{y}^0$, for every $n \in \mathbb{N}^*$, and then (2.6) gives

$$\frac{2(\mathrm{y}^1 - e^{-(2p+1)^2\pi^2 T}\mathrm{y}^0)}{(2p+1)^2\pi^2} \leq 2\int_0^T u_0(t)\mathrm{d}t \leq \frac{2(e^{(2p+1)^2\pi^2 T}\mathrm{y}^1 - \mathrm{y}^0)}{(2p+1)^2\pi^2},$$

for every $p \in \mathbb{N}$, which yields

$$\sup_{p\in\mathbb{N}} \frac{\mathrm{y}^1 - e^{-(2p+1)^2\pi^2 T}\mathrm{y}^0}{(2p+1)^2\pi^2} \leq \inf_{p\in\mathbb{N}} \frac{e^{(2p+1)^2\pi^2 T}\mathrm{y}^1 - \mathrm{y}^0}{(2p+1)^2\pi^2}$$

and

$$\inf_{p\in\mathbb{N}} \frac{e^{(2p+1)^2\pi^2 T}\mathrm{y}^1 - \mathrm{y}^0}{(2p+1)^2\pi^2} \geq 0.$$

The lemma then follows by simple computations. □

Providing more explicit lower estimates is a bit technical. Let us do it however. For all $\delta > 1$, $\mu > 1$ and $Z \in [0, +\infty)$, we define

$$f_{\delta,\mu}(Z) = \frac{\delta}{\mu} Z^{\mu+1} - \left(\delta + \frac{1}{\mu}\right) Z + 1.$$

Noting that

$$f'_{\delta,\mu}(Z) = \frac{\mu+1}{\mu}\delta Z^{\mu} - \left(\delta + \frac{1}{\mu}\right) \quad \text{and} \quad f''_{\delta,\mu}(Z) = \delta(\mu+1)Z^{\mu-1} > 0 \quad (Z > 0),$$

it follows that $f_{\delta,\mu}$ is a strictly convex function on $[0, +\infty)$. Since

$$f_{\delta,\mu}(0) = 1 > 0, \quad f_{\delta,\mu}\left(\frac{1}{\delta}\right) = \frac{1}{\mu}\left(\frac{1}{\delta^{\mu}} - \frac{1}{\delta}\right) < 0,$$

$$f_{\delta,\mu}(1) = \frac{1-\mu}{\mu}(\delta - 1) < 0 \quad \text{and} \quad \lim_{Z\to+\infty} f_{\delta,\mu}(Z) = +\infty,$$

we infer that there exist $Z_{1,\delta,\mu} \in (0, 1/\delta)$ and $Z_{2,\delta,\mu} \in (1, +\infty)$ such that $f_{\delta,\mu}(Z_{1,\delta,\mu}) = f_{\delta,\mu}(Z_{2,\delta,\mu}) = 0$, and by convexity, $Z_{1,\delta,\mu}$ and $Z_{2,\delta,\mu}$ are the only roots of $f_{\delta,\mu}$.

We set $\mu_p = (2p+1)^2 \geq 1$, for $p \in \mathbb{N}$. The graphs of f_{δ,μ_p} for $p \in \{1, \ldots, 5\}$ and $\delta = 5$ are drawn in Fig. 1. Numerically, we obtain $Z_{1,\delta,\mu_1} = 0.195652$ and $Z_{2,\delta,\mu_1} = 1.255783$.

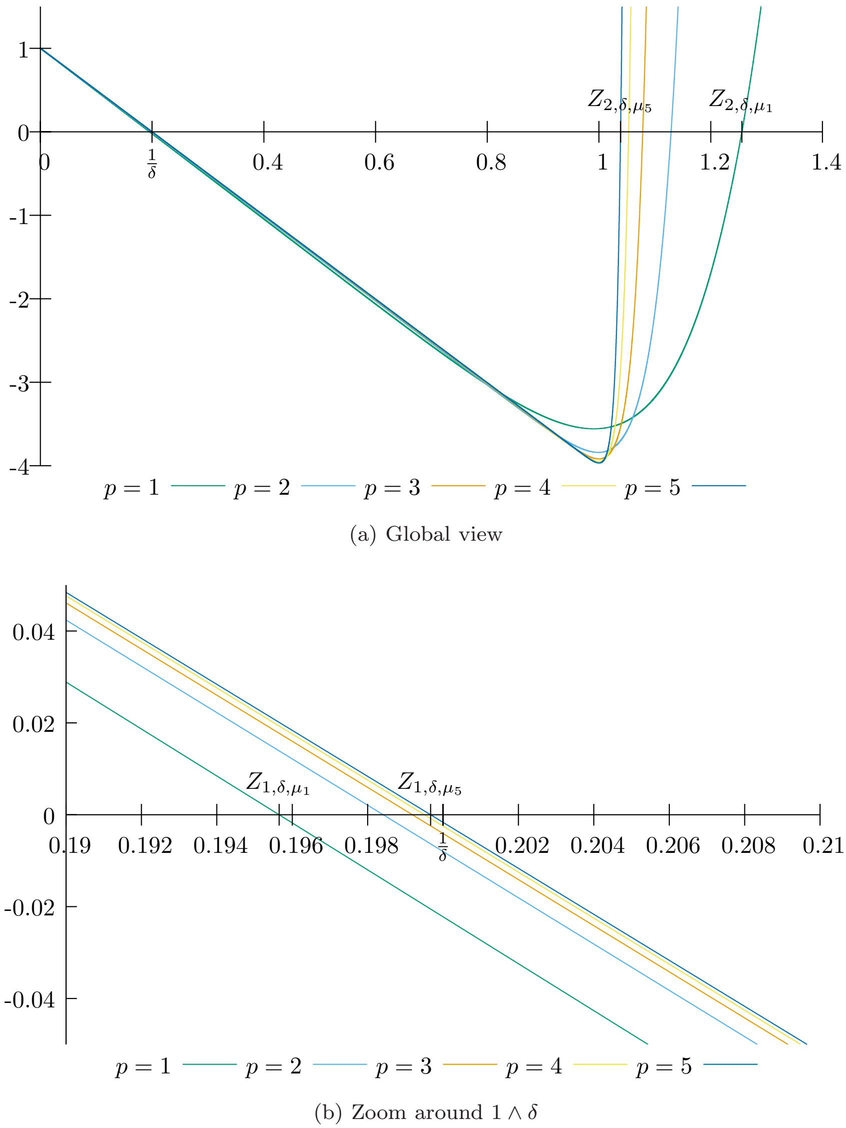


Fig. 1. Graphs of f_{δ,μ_p} for $p \in \{1, \dots, 5\}$ and $\delta = 5$.

Case $\mathbf{y^1 < y^0}$. Setting $\eta = \mathrm{y}^0/\mathrm{y}^1 > 1$ and $Z = \exp(-\pi^2 \underline{T}) \in (0, 1)$, we have $0 < Z < 1/\eta$ and condition (2.7a) becomes

$$\sup_{p \in \mathbb{N}^*} \frac{1}{\mu_p} \left(\frac{1}{\eta} - Z^{\mu_p} \right) \leq \frac{1}{\eta Z} - 1,$$

which gives (by multiplying by $-\eta Z$) $\sup_{p \in \mathbb{N}^*} f_{\eta,\mu_p}(Z) \geq 0$. Using the properties of $f_{\delta,\mu}$, this leads to $0 < Z \leq Z^* = \inf_{p \in \mathbb{N}^*} Z_{1,\eta,\mu_p}$.

We claim that $Z^* > 0$. Indeed, for $Z \in [0,1)$, we have $f_{\delta,\mu}(Z) \geq 1-(\delta+1/\mu)Z$. By definition, $f_{\delta,\mu}(Z_{1,\delta,\mu}) = 0$ and, consequently, $1-(\delta+1/\mu)Z_{1,\delta,\mu} \leq 0$. Since $Z \mapsto 1-(\delta+1/\mu)Z$ is decreasing, we conclude that $Z_{1,\delta,\mu} \geq (\delta+1/\mu)^{-1}$. Since $(\delta+1/\mu)^{-1} \to 1/\delta$ as $\mu \to +\infty$, the claim follows.

Since $Z = \exp(-\pi^2 \underline{T})$, we have thus obtained the lower bound

$$\underline{T} \geq \frac{1}{\pi^2}|\ln(Z^*)|.$$

Case $\mathrm{y}^1 > \mathrm{y}^0$. Similarly, setting $\delta = \mathrm{y}^1/\mathrm{y}^0 > 1$ and $Z = \exp(\pi^2 \underline{T}) > 1$, condition (2.7b) becomes

$$\delta - \frac{1}{Z} \leq \inf_{p\in\mathbb{N}^*} \frac{1}{\mu_p}(\delta Z^{\mu_p} - 1),$$

that is, $0 \leq \inf_{p\in\mathbb{N}^*} f_{\delta,\mu_p}(Z)$, and hence Z must be such that $Z \geq Z_* = \sup_{p\in\mathbb{N}^*} Z_{2,\delta,\mu_p}$. We prove similarly that $Z_* > 0$, and we obtain for $\underline{T}$ the same lower bound as above, replacing Z^* with Z_*.

Example 2.1. If $\mathrm{y}^0 = 5$ and $\mathrm{y}^1 = 1$, we have $\eta = 5$ and we observe (see Fig. 1(a)) that $Z^* = \inf_{p\in\mathbb{N}^*} Z_{1,\eta,\mu_p} = Z_{1,\eta,\mu_1}$. Therefore,

$$\underline{T}(\mathrm{y}^0,\mathrm{y}^1) \geq |\ln Z_{1,\eta,\mu_1}|/\pi^2 \simeq 0.165297.$$

If $\mathrm{y}^0 = 1$ and $\mathrm{y}^1 = 5$, we have $\delta = 5$ and we observe (see Fig. 1(a)) that $\sup_{p\in\mathbb{N}^*} Z_{2,\delta,\mu_p} = Z_{2,\delta,\mu_1}$. Therefore,

$$\underline{T}(\mathrm{y}^0,\mathrm{y}^1) \geq |\ln Z_{2,\delta,\mu_1}|/\pi^2 \simeq 0.023076.$$

3. Minimal Time for Dirichlet-Controlled Multi-Dimensional Heat Equations under Non-Negativity Constraints

3.1. *Heat equation in a ball with non-negative Dirichlet controls*

Let $D = B(0,1)$ be the unit ball of $\mathbb{R}^d$ and consider the Dirichlet control problem

$$\partial_t y(t,x) = \Delta y(t,x) \quad (t > 0, x \in D), \tag{3.1a}$$

$$y(t,x) = u(t,x) \qquad (t > 0, x \in \partial D), \tag{3.1b}$$

with the initial condition given in $L^2(D)$,

$$y(0,x) = \mathrm{y}^0(x) \quad (x \in D). \tag{3.1c}$$

The aim is to steer the solution y of (3.1) to a constant target $\mathrm{y}^1 > 0$ with non-negative controls $u \in L^2(0,T;L^2(\partial D))$, i.e. under the unilateral control constraint

$$u(t,x) \geq 0 \quad (t \geq 0, x \in \partial D \text{ a.e.}). \tag{3.2}$$

Remark 3.1. Under the assumption $\mathrm{y}^0 \geq 0$, the control problem (3.1) with the non-negativity control constraint (3.2) is equivalent to the control problem (3.1) with the non-negativity state constraint $y(t,x) \geq 0$.

Indeed, due to the (already-employed) comparison principle, if the control u is non-negative and y^0 is non-negative, then the solution of (3.1) is non-negative as well. Conversely, if the solution y of (3.1) is non-negative, then its trace on ∂D is non-negative as well.

Theorem 3.1. *Let $\mathrm{y}^0 \in L^2(D)$ be arbitrary and let $\mathrm{y}^1 \in L^2(D)$ be a positive steady state, i.e. satisfying $\Delta \mathrm{y}^1 = 0$ in D and $\mathrm{y}^1 = \bar{u}$ on ∂D for some $\bar{u} \in L^2(\partial D)$. We assume that $\mathrm{y}^0 \neq \mathrm{y}^1$ and that there exists $\varepsilon > 0$ such that $\mathrm{y}^1(x) \geq \varepsilon$ for every $x \in D$. Then, we have the following results*:

- *There exist $T > 0$ and a non-negative control $u \in L^2(0, T, L^2(\partial D))$ such that the corresponding solution y of* (3.1) *satisfies $y(T) = \mathrm{y}^1$.*
- *Defining the minimal time by*

$$\underline{T}(\mathrm{y}^0, \mathrm{y}^1) = \inf\{T > 0, \exists\, u \in L^1((0,T) \times \partial D) \text{ s.t. } u \geq 0 \text{ and } y(T) = \mathrm{y}^1\},$$

  we have $\underline{T}(\mathrm{y}^0, \mathrm{y}^1) > 0$.
- *Given any $T > \underline{T}(\mathrm{y}^0, \mathrm{y}^1)$*:
  - — *For every $\tau > 0$, there exists a non-negative control $u \in L^2(0, T, L^2(\partial D))$ satisfying $u \in C^\infty((0, T - \tau) \times \partial D)$, such that the corresponding solution y of* (3.1) *satisfies $y(T) = \mathrm{y}^1$.*
  - — *If $\mathrm{y}^1 \in C^\infty(\overline{\Omega})$, then there exists a non-negative control $u \in C^\infty((0,T) \times \partial D)$ such that the corresponding solution y of* (3.1) *satisfies $y(T) = \mathrm{y}^1$.*
- *For $T = \underline{T}(\mathrm{y}^0, \mathrm{y}^1)$, there exists a non-negative control $\underline{u} \in \mathcal{M}((0,T) \times \partial D)$* (*the space of Radon measures*) *such that the corresponding solution y of* (3.1) *satisfies $y(T) = \mathrm{y}^1$.*

As in the 1D case, the minimal control time could as well be defined using other functional control spaces than L^1. But the regularizing properties of the heat equation ensure that, in fact, the resulting minimal time is the same even if the controls under consideration are restricted to be smooth.

Remark 3.2. In order to prove controllability with a non-negative control in large time, it is enough that all the eigenvalues of the Dirichlet–Laplacian operator be negative. Notice that this is the case whatever the spatial domain is.

In order to prove the positivity of the minimal time for $\mathrm{y}^0 \neq \mathrm{y}^1$, we need an infinite number of eigenvalues of the Dirichlet–Laplacian operator for which the sign of their normal derivative on the boundary of the spatial domain is constant. For this reason, we use the unit ball of $\mathbb{R}^d$, for which all the radially symmetric eigenfunctions satisfy this property. Notice that for a square domain no such family of eigenfunctions exist.

By a classical extension–restriction argument we will be able to show the positivity of the waiting time $\underline{T} = \underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ for all spatial domains.

On the other hand, since the first eigenfunction of the Dirichlet–Laplacian operator is of constant sign on any domain and its normal derivative has a constant sign, we shall also show that in the minimal time, there exist non-negative controls in the space of Radon measures $\mathcal{M}([0,\underline{T}] \times \partial\Omega)$ steering y^0 to y^1 in time $\underline{T}$.

Proof of Theorem 3.1. The existence of a large enough time T for which there exists a non-negative control u steering the solution of (3.1) from y^0 to y^1 can be easily proved by adapting the proof of Theorem 2.1 or can be directly obtained from Theorem 4.1(b) of Ref. 18.

Proving that for every $T > \underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ there exists a non-negative and smooth control steering the solution of (3.1) from y^0 to y^1 is similar to the proof of Proposition 2.1.

Now, let us prove that $\underline{T}(\mathrm{y}^0, \mathrm{y}^1) > 0$. We proceed as in the proof of Theorem 2.1, but we now consider the (nondecreasing sequence of positive) eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ and eigenfunctions $(p_n)_{n\in\mathbb{N}^*}$ of the Sturm–Liouville problem, defined by

$$\frac{\mathrm{d}^2 p_n}{\mathrm{d}r^2}(r) + \frac{d-1}{r}\frac{\mathrm{d}p_n}{\mathrm{d}r}(r) = -\lambda_n p_n(r) \quad (r \in (0,1)), \tag{3.3a}$$

$$p_n(0) = 1, \quad p_n(1) = \frac{\mathrm{d}p_n}{\mathrm{d}r}(0) = 0. \tag{3.3b}$$

Defining $\alpha_n = \mathrm{d}p_n(1)/\mathrm{d}r$ and $\varphi_n(x) = p_n(\|x\|)$ for $x \in D$, we have

$$\begin{aligned} \Delta\varphi_n(x) &= -\lambda_n \varphi_n(x) && (x \in D), \\ \varphi_n(x) &= 0, \quad \nabla\varphi_n(x)\cdot \mathrm{n}(x) = \alpha_n && (x \in \partial D). \end{aligned}$$

Let $T > 0$ and $u^T \in L^1((0,T)\times\partial D)$ be a non-negative control such that the solution y of (3.1) with initial condition y^0 satisfies $y(T) = \mathrm{y}^1$. For every $n \in \mathbb{N}^*$, we set $y_n(t) = \int_D y(t,x)\varphi_n(x)\mathrm{d}x$. Then, we have

$$\begin{aligned} \dot{y}_n(t) &= \int_D \Delta y(t,x)\varphi_n(x)\mathrm{d}x \\ &= \int_{\partial D} \nabla y(t,x)\cdot \mathrm{n}(x)\varphi_n(x)\mathrm{d}\Gamma_x - \int_D \nabla y(t,x)\cdot\nabla\varphi_n(x)\mathrm{d}x \\ &= -\int_{\partial D} y(t,x)\nabla\varphi_n(x)\cdot\mathrm{n}(x)\mathrm{d}\Gamma_x + \int_D y(t,x)\Delta\varphi_n(x)\mathrm{d}x \\ &= -\alpha_n \int_{\partial D} u^T(t,x)\mathrm{d}\Gamma_x - \lambda_n \int_D y(t,x)\varphi_n(x)\mathrm{d}x \\ &= -\lambda_n y_n(t) - \alpha_n \int_{\partial D} u^T(t,x)\mathrm{d}\Gamma_x \end{aligned}$$

and hence,

$$y_n(T) = e^{-\lambda_n T} y_n(0) - \alpha_n \int_0^T e^{-\lambda_n (T-t)} \int_{\partial D} u^T(t,x)\mathrm{d}\Gamma_x\, \mathrm{d}t.$$

Setting $\mathrm{y}_n^i = \int_D \mathrm{y}^i(x)\varphi_n(x)\mathrm{d}x$ for $i = 0, 1$, we obtain (since u^T is a control such that $y(T) = \mathrm{y}^1$),

$$\mathrm{y}_n^1 - e^{-\lambda_n T}\mathrm{y}_n^0 = -\alpha_n \int_0^T e^{-\lambda_n(T-t)} \int_{\partial D} u^T(t,x)\mathrm{d}\Gamma_x \,\mathrm{d}t. \tag{3.4}$$

Since $u^T \geq 0$ and $\lambda_n > 0$, we obtain

$$e^{-\lambda_n T} \int_0^T \int_{\partial D} u^T(t,x)\mathrm{d}\Gamma_x \,\mathrm{d}t \leq \frac{\mathrm{y}_n^1 - e^{-\lambda_n T}\mathrm{y}_n^0}{-\alpha_n} \leq \int_0^T \int_{\partial D} u^T(t,x)\mathrm{d}\Gamma_x \,\mathrm{d}t,$$

that is,

$$\frac{\mathrm{y}_n^1 - e^{-\lambda_n T}\mathrm{y}_n^0}{-\alpha_n} \leq \int_0^T \int_{\partial D} u^T(t,x)\mathrm{d}\Gamma_x \,\mathrm{d}t \leq \frac{e^{\lambda_n T}\mathrm{y}_n^1 - \mathrm{y}_n^0}{-\alpha_n}. \tag{3.5}$$

Hence, if for every $T > 0$ such a non-negative control u^T exists, then

$$\lim_{T\to 0} \int_0^T \int_{\partial D} u^T(t,x)\mathrm{d}\Gamma_x \,\mathrm{d}t = \frac{\mathrm{y}_n^1 - \mathrm{y}_n^0}{-\alpha_n}, \quad \forall\, n \in \mathbb{N}^*.$$

But this limit (denoted by γ) must be independent of n. Consequently, y_n^0 must satisfy

$$\mathrm{y}_n^0 = \mathrm{y}_n^1 + \alpha_n\gamma \quad (n \in \mathbb{N}^*). \tag{3.6}$$

Since $\mathrm{y}^0 \in L^2(D)$, we also have,

$$\sum_{n=1}^{+\infty} \frac{|\mathrm{y}_n^0|^2}{\|\varphi_n\|_{L^2(D)}^2} < +\infty. \tag{3.7}$$

But since

$$\int_D |\varphi_n(x)|^2 \,\mathrm{d}x = \omega_{d-1} \int_0^1 |p_n(r)|^2 r^{d-1} \,\mathrm{d}r \tag{3.8a}$$

$$= \frac{-\omega_{d-1}}{\lambda_n} \int_0^1 p_n(r) \frac{\mathrm{d}}{\mathrm{d}r}(r^{d-1}p_n'(r))\mathrm{d}r$$

$$= \frac{\omega_{d-1}}{\lambda_n} \int_0^1 |p_n'(r)|^2 r^{d-1} \,\mathrm{d}r \tag{3.8b}$$

$$= \frac{-\omega_{d-1}}{(d-1)\lambda_n} \int_0^1 p_n'(r)(\lambda_n p_n(r) + p_n''(r)) r^d \,\mathrm{d}r$$

$$= \frac{-\omega_{d-1}}{(d-1)} \int_0^1 p_n'(r) p_n(r) r^d \,\mathrm{d}r$$

$$- \frac{\omega_{d-1}}{(d-1)\lambda_n} \int_0^1 p_n'(r) p_n''(r) r^d \,\mathrm{d}r, \tag{3.8c}$$

where $\omega_{d-1} = \int_{\partial D} \mathrm{d}\Gamma_x$, and since

$$\int_0^1 p_n'(r)p_n(r)r^d\,\mathrm{d}r = \frac{1}{2}\int_0^1 p_n'(r)p_n(r)r^d\,\mathrm{d}r - \frac{1}{2}\int_0^1 p_n(r)\frac{\mathrm{d}}{\mathrm{d}r}(r^d p_n(r))\mathrm{d}r$$
$$= -\frac{d}{2}\int_0^1 |p_n(r)|^2 r^{d-1}\,\mathrm{d}r$$

and

$$\int_0^1 p_n'(r)p_n''(r)r^d\,\mathrm{d}r = \frac{1}{2}\int_0^1 p_n'(r)p_n''(r)r^d\,\mathrm{d}r - \frac{1}{2}\int_0^1 p_n'(r)\frac{\mathrm{d}}{\mathrm{d}r}(r^d p_n'(r))\mathrm{d}r + \frac{\alpha_n^2}{2}$$
$$= -\frac{d}{2}\int_0^1 |p_n'(r)|^2 r^{d-1}\,\mathrm{d}r + \frac{\alpha_n^2}{2},$$

(3.8c) together with the above equalities and (3.8a)–(3.8b) leads to

$$\int_D |\varphi_n(x)|^2\,\mathrm{d}x = \frac{d}{d-1}\int_D |\varphi_n(x)|^2\,\mathrm{d}x - \frac{\omega_{d-1}}{(d-1)\lambda_n}\frac{\alpha_n^2}{2}.$$

Consequently, $\|\varphi_n(x)\|_{L^2(D)}^2 = \frac{\omega_{d-1}}{2\lambda_n}\alpha_n^2$. Now combining (3.6) and (3.7) together with this equality, we find

$$\sum_{n=1}^{+\infty}\frac{|\mathrm{y}_n^1|^2}{\|\varphi_n\|_{L^2(D)}^2} + \frac{2\gamma}{\omega_{d-1}}\sum_{n=1}^{+\infty}\left(2\frac{\mathrm{y}_n^1}{\alpha_n} + \gamma\right)\lambda_n < +\infty.$$

The first sum is finite since $\mathrm{y}^1 \in L^2(D)$ and if $\gamma \neq 0$ the second sum can be finite only if $\lim_{n\to\infty}\lambda_n(2\mathrm{y}_n^1/\alpha_n + \gamma) = 0$. But, we have

$$\lambda_n \mathrm{y}_n^1 = \lambda_n\int_\Omega \mathrm{y}^1(x)\varphi_n(x)\mathrm{d}x = -\int_\Omega \mathrm{y}^1(x)\Delta\varphi_n(x)\mathrm{d}x = -\alpha_n\int_{\partial\Omega}\mathrm{y}^1(x)\mathrm{d}\Gamma_x$$

and hence,

$$\lambda_n\left(\frac{2\mathrm{y}_n^1}{\alpha_n} + \gamma\right)\lambda_n = -2\int_{\partial\Omega}\mathrm{y}^1(x)\mathrm{d}\Gamma(x) + \lambda_n\gamma \quad (n \in \mathbb{N}^*).$$

Consequently, unless $\gamma = 0$, we have $\lim_{n\to\infty}\lambda_n|2\mathrm{y}_n^1/\alpha_n + \gamma| = +\infty$. This ensures that we necessarily have $\gamma = 0$.

All in all, we have proved that if for every $T > 0$, there exists a non-negative control $u^T \in L^1((0,T)\times\partial D)$ steering y^0 to y^1 in time T, then we have

$$\lim_{T\to 0}\int_0^T\int_{\partial D} u(t,x)\mathrm{d}\Gamma_x\,\mathrm{d}t = 0.$$

Let us now prove that, if for every $T > 0$ there exists a non-negative control $u^T \in L^1((0,T)\times\partial D)$ steering the solution y of (3.1) from y^0 to y^1, then we must have $\mathrm{y}^0 \equiv \mathrm{y}^1$.

Let φ be an eigenfunction of the Dirichlet–Laplacian operator, with eigenvalue $\lambda > 0$, i.e. $\Delta\varphi = -\lambda\varphi$ on D and $\varphi = 0$ on ∂D. It is well known that any such eigenfunction is smooth and that there exists an orthonormal basis of $L^2(D)$ consisting of such eigenfunctions. As for (3.4), we obtain

$$\begin{aligned}\int_D \mathrm{y}^1(x)\varphi(x)\mathrm{d}x - e^{-\lambda T}\int_D \mathrm{y}^0(x)\varphi(x)\mathrm{d}x \\ = -\int_0^T e^{-\lambda(T-t)}\int_D \nabla\varphi(x)\cdot \mathrm{n}(x)u^T(t,x)\mathrm{d}\Gamma_x\,\mathrm{d}t.\end{aligned}$$

But since φ is smooth and since $\int_0^T\int_D u^T(t,x)\mathrm{d}\Gamma_x\,\mathrm{d}t = \int_0^T\int_D |u^T(t,x)|\mathrm{d}\Gamma_x\,\mathrm{d}t \to 0$ as $T \to 0$, we conclude that

$$\begin{aligned}0 &= -\lim_{T\to 0}\int_0^T e^{-\lambda(T-t)}\int_D \nabla\varphi(x)\cdot \mathrm{n}(x)u^T(t,x)\mathrm{d}\Gamma_x\,\mathrm{d}t \\ &= \lim_{T\to 0}\left(\mathrm{y}^1\int_D \varphi(x)\mathrm{d}x - e^{-\lambda T}\int_D \mathrm{y}^0(x)\varphi(x)\mathrm{d}x\right) \\ &= \int_D \mathrm{y}^1(x)\varphi(x)\mathrm{d}x - \int_D \mathrm{y}^0(x)\varphi(x)\mathrm{d}x.\end{aligned}$$

This means that y^1 and y^0 have the same L^2-projections on any eigenfunction of the Dirichlet–Laplacian operator, and thus they do coincide. This shows that $\underline{T}(\mathrm{y}^0,\mathrm{y}^1)$ is necessarily positive.

Set $\underline{T} = \underline{T}(\mathrm{y}^0,\mathrm{y}^1)$. Let us now prove the existence of a control in $\mathcal{M}((0,\underline{T})\partial D)$. First of all, for every $n \in \mathbb{N}^*$, there exists a control $u^n \in L^1((0,\underline{T}+\frac{1}{n})\times\partial D)$, $u^n \geq 0$, steering y, solution of (3.1), from y^0 to y^1 in time $\underline{T}+\frac{1}{n}$. According to (3.5), this control satisfies

$$\|u^n\|_{L^1((0,\underline{T}+\frac{1}{n})\times\partial D)} \leq \frac{e^{\lambda_1(\underline{T}+\frac{1}{n})}\mathrm{y}_1^1 - \mathrm{y}_1^0}{-\alpha_1} \leq \frac{e^{\lambda_1(\underline{T}+1)}|\mathrm{y}_1^1| + |\mathrm{y}_1^0|}{|\alpha_1|}.$$

Then, the sequence $(u^n)_n$ is bounded in $L^1(0,\underline{T}+1)$ and therefore, by weak compactness of $\mathcal{M}(0,\underline{T})$, up to a subsequence it converges in the weak sense of Radon measures to some control $\underline{u}$ in $\mathcal{M}(0,\underline{T})$.

Clearly, the limit controls satisfy the non-negativity constraint and the limit solution solves the limit non-homogeneous Dirichlet problem (3.1) in the sense of transposition and the limit solution reaches the target y^1 in time $\underline{T}$. □

Remark 3.3. In the proof of Theorem 3.1, proving that $\underline{T}(\mathrm{y}^0,\mathrm{y}^1) > 0$ (unless $\mathrm{y}^0 = \mathrm{y}^1$) did not require y^1 to be a steady state. In fact, we always have $\underline{T}(\mathrm{y}^0,\mathrm{y}^1) > 0$ if $\mathrm{y}^0 \neq \mathrm{y}^1$, with the convention that $\underline{T}(\mathrm{y}^0,\mathrm{y}^1) = +\infty$ if y^1 is not reachable from y^0.

As a consequence of the proof, and in particular of (3.5), we get the following lower bound for $\underline{T}(\mathrm{y}^0,\mathrm{y}^1)$, in the case where y^0 is constant.

Corollary 3.1. *Let* $y^0 \in \mathbb{R}$ *and* $y^1 \in (0, +\infty)$ *be arbitrary. Then the minimal time* $\underline{T} = \underline{T}(y^0, y^1)$ *is such that*

$$\sup_{n\in\mathbb{N}^*} \left(\frac{y^1}{\lambda_n} - \frac{e^{-\lambda_n \underline{T}} y^0}{\lambda_n} \right) \leq \inf_{n\in\mathbb{N}^*} \left(\frac{e^{\lambda_n \underline{T}} y^1}{\lambda_n} - \frac{y^0}{\lambda_n} \right)$$

and

$$0 \leq \inf_{n\in\mathbb{N}^*} \left(\frac{e^{\lambda_n \underline{T}} y^1}{\lambda_n} - \frac{y^0}{\lambda_n} \right),$$

where $(\lambda_n)_{n\in\mathbb{N}^*}$ *is defined by the Sturm–Liouville problem* (3.3).

3.2. *Non-negativity state constraints in general domains*

Let Ω be a bounded domain $\mathbb{R}^d$ of class C^2. We consider the control problem

$$\partial_t y(t,x) = \Delta y(t,x) \quad (t > 0, x \in \Omega), \tag{3.9a}$$

$$y(t,x) = u(t,x) \qquad (t > 0, x \in \partial\Omega), \tag{3.9b}$$

with non-negative initial condition $y^0 \in L^2(\Omega)$,

$$y(0,x) = y^0(x) \geq 0 \quad (x \in \Omega). \tag{3.9c}$$

The system is well known to be null controllable in any time $T > 0$ with controls $u \in L^2((0,T) \times \partial\Omega)$ (see Refs. 7, 12 and 28). The question we analyze is whether controllability is true as well under the additional non-negativity requirement on the state

$$y(t,x) \geq 0, \quad t > 0, \; x \in \Omega, \tag{3.10}$$

and whether this state constraint may cause a positive minimal time (or waiting time).

Remark 3.4. Here again, as in Remark 3.1, by the comparison principle, the non-negativity state constraint $y \geq 0$ is equivalent to the non-negativity control constraint $u \geq 0$.

Theorem 3.2. *Let* $y^0 \in L^2(\Omega)$ *be such that* $y^0 \geq 0$, *and let* $y^1 \in L^2(\Omega)$ *be a steady state of* (3.9). *We assume that* $y^0 \neq y^1$ *and that there exists* $\varepsilon > 0$ *such that* $y^1(x) \geq \varepsilon$ *for every* $x \in \Omega$. *Then*:

- *There exist* $T > 0$ (*large enough*) *and a control* $u \in L^2((0,T) \times \partial\Omega)$ *such that the corresponding solution of* (3.9) *satisfies the state constraint* (3.10) *and reaches* $y(T) = y^1$.
- *Defining*

$$\underline{T}(y^0, y^1) = \inf\{T > 0, \exists\, u \in L^1((0,T) \times \Gamma_0) \text{ s.t. } y(t,x) \geq 0 \text{ and } y(T,\cdot) = y^1\},$$

we have $\underline{T}(y^0, y^1) > 0$.

- *For* $T = \underline{T}(\mathrm{y}^0, \mathrm{y}^1)$, *there exists a control* $\underline{u} \in \mathcal{M}((0,T) \times \partial\Omega)$ (*the space of Radon measures*) *steering the heat equation* (3.9) *from* y^0 *to* y^1 *in time* T *under the non-negativity state constraint* (3.10).

Remark 3.5. Proceeding as in the proof of Proposition 2.1, one can show that for every $\tau > 0$, there exists a control in time $\underline{T}(\mathrm{y}^0, \mathrm{y}^1) + \tau$ with arbitrary regularity.

Proof of Theorem 3.2. The proof of constrained controllability in large time is the same as in 1D. We give a sketch. Working on the shifted state $z = y - \mathrm{y}^1$, it is sufficient to address the problem of controlling the system to the zero final state. Then, using the fact that the cost of controlling the system decreases exponentially as $T \to +\infty$, and that, by regularity considerations, the controls can be taken in L^∞, we conclude that, for T large enough, the L^∞-norm of the control v driving z to 0 in time T is smaller than y^1. The control u for the original state y is then $u = v + \mathrm{y}^1|_{\partial\Omega}$, which can then be guaranteed to be non-negative.

The positivity of the minimal time is established by comparison, based on the result of the previous subsection in the case where Ω is a ball. Indeed, let D be the largest ball contained in Ω and such that $\mathrm{y}^0|_D \neq \mathrm{y}^1|_D$. Assume that the heat equation (3.9) is controllable in time T under the positivity constraint. Let z be equal to the restriction of y to D, and let v be the restriction of y to ∂D. Then, obviously, v is a control for z in the ball D, preserving the control constraint. This immediately implies that the control time T has to satisfy the lower bounds of the previous section. Of course, this argument applies to any ball D included in Ω and, as indicated in the remark hereafter, is also valid for other boundary conditions.

The proof of the existence of a non-negative control $\mathcal{M}((0, \underline{T}(\mathrm{y}^0, \mathrm{y}^1)) \times \partial\Omega)$ is similar to the one of Theorem 3.1. More precisely, denoting by λ_0 the first eigenvalue of the Dirichlet–Laplacian operator and by φ_0 the corresponding normalized eigenvector, and defining $y_0(t) = \int_\Omega y(t,x)\varphi_0(x)\mathrm{d}x$, we have $\dot{y}_0(t) = -\int_{\partial\Omega} u(t,x)\nabla\varphi_0(x)\cdot \mathrm{n}(x)\mathrm{d}\Gamma_x - \lambda_0 y_0(t)$, and hence,

$$-\int_0^T e^{-\lambda_0(T-t)} \int_{\partial\Omega} u(t,x)\nabla\varphi_0(x)\cdot \mathrm{n}(x)\mathrm{d}\Gamma_x\, \mathrm{d}t = y_0(T) - e^{-\lambda_0 T} y_0(0).$$

It is well known that φ_0 keeps a constant sign, which can be chosen to be positive, and that its normal derivative does not vanish. Therefore, $\inf_{\partial\Omega}(-\nabla\varphi_0\cdot \mathrm{n}) = \alpha_0 > 0$ and hence,

$$\int_0^T \int_{\partial\Omega} u(t,x)\mathrm{d}\Gamma_x\, \mathrm{d}t \le \frac{1}{\alpha}(e^{\lambda_0 T}\mathrm{y}_0^1 - \mathrm{y}_0^0),$$

with $\mathrm{y}_0^i = \int_\Omega \mathrm{y}^i(x)\varphi_0(x)\mathrm{d}x$ for $i = 0, 1$. This ensures that any non-negative control is bounded in L^1-norm. We conclude as in the proof of Theorem 3.1. □

Several remarks are in order.

Remark 3.6. We consider the case where the control acts only on a proper subset $\Gamma_0 \subset \partial\Omega$,

$$\partial_t y(t,x) = \Delta y(t,x) \quad (t>0, x\in\Omega), \tag{3.11a}$$

$$y(t,x) = u(t,x) \quad (t>0, x\in\Gamma_0), \tag{3.11b}$$

$$y(t,x) = 0 \quad (t>0, x\in\partial\Omega\backslash\Gamma_0), \tag{3.11c}$$

$$y(0,x) = \mathrm{y}^0(x) \geq 0 \quad (x\in\Omega). \tag{3.11d}$$

In this situation, we want to control the state trajectory to a non-negative steady state of (3.11), i.e. to some $\mathrm{y}^1 \in L^2(\Omega)$ solution of $\Delta \mathrm{y}^1(x) = 0$ in Ω, $\mathrm{y}^1 = \mathrm{u}^1$ on Γ_0 and $\mathrm{y}^1 = 0$ on $\partial\Omega\backslash\Gamma_0$, for some given non-negative $\mathrm{u}^1 \in L^2(\Gamma_0)$. Given $\mathrm{y}^0 \in L^2(\Omega)$, if y^1 is reachable from y^0 (in sufficiently large time) with non-negative controls, then $\underline{T}(\mathrm{y}^0,\mathrm{y}^1) > 0$. This follows, similarly, from a localization argument in a ball contained in Ω, combined with Theorem 3.1 and Remark 3.3.

The main difficulty here is to establish reachability in large enough time. Following the proof of Theorem 2.1, it is possible to prove that, if $\inf_{\Gamma_0} \mathrm{u}^1$ is positive, then there exist $T>0$ and a non-negative control $u \in L^2((0,T)\times\Gamma_0)$ steering the solution of (3.11) from y^0 to y^1 in time T. However, when $\inf_{\Gamma_0} \mathrm{u}^1 = 0$, this is not possible anymore. In particular when $\mathrm{u}^1 = 0$, i.e. when $\mathrm{y}^1 = 0$, due to the comparison principle, it is not possible to steer to 0 any $\mathrm{y}^0 \geq 0$, with $\mathrm{y}^0 \in L^2(\Omega)\backslash\{0\}$.

Remark 3.7. The result can be extended to more general elliptic constant coefficient operators, not necessarily coinciding with the Laplacian. They hold in particular for more general parabolic problems of the form

$$\partial_t y = \mathrm{div}(A\nabla y),$$

where $A \in \mathbb{R}^{d\times d}$ is a constant coefficient positive matrix.

Indeed, there exists an orthogonal matrix $P \in \mathbb{R}^{d\times d}$ such that $A = PP^\top$. Setting $\tilde{x} = Px$ and $\tilde{y}(t,\tilde{x}) = y(t,x)$, the parabolic problem above reduces to $\partial_t \tilde{y}(t,\tilde{x}) = 1/|P|\Delta\tilde{y}(t,\tilde{x})$.

Remark 3.8. The positivity of the minimal time follows from the comparison with the Dirichlet control problem with non-negativity control constraints over the largest ball included in Ω. Similarly to Lemma 2.2, this leads to lower bounds on the minimal time.

Remark 3.9. The minimal time is the same both for non-negativity constraints on the state and on the control. This can be easily seen by the comparison principle since solutions with non-negative initial data and Dirichlet controls are non-negative everywhere. Similarly, if the solution itself is non-negative, of course, the Dirichlet controls, which are simply the restriction to the boundary of the state, are non-negative as well.

Remark 3.10. The same result is valid for other boundary conditions, for instance of Neumann type, provided that we deal with non-negativity state constraints. This is the aim of Sec. 4.2. But, as we shall see in Sec. 4.4, dealing with constrained Neumann controls leads to different results.

Remark 3.11. Similar results hold when the control is acting in some interior subdomain ω of Ω, i.e. for the model

$$\partial_t y(t,x) = \Delta y(t,x) + u(t,x)\mathbf{1}_\omega(x),$$

under any boundary conditions ensuring that constant states are steady states. Here $u = u(t,x)$ is the control and $\mathbf{1}_\omega$ is the characteristic function of the subset ω where the control is applied. The proof relies again on comparison arguments. Here, it suffices to consider a ball $D \subset \Omega\backslash\omega$ and to apply the arguments above, since the action of the external force u applied in ω is not seen anymore. This is developed with more details in Sec. 4.3.

3.3. *Relationship with the viscous Hamilton–Jacobi equation*

As observed by Tucsnak in Ref. 24, the waiting time phenomenon for the control under non-negativity state constraints of the linear heat equation is related to that on the control of the viscous Hamilton–Jacobi equations developed in Ref. 17.

Indeed, let us consider the Dirichlet boundary control problem for the heat equation under the state constraint $y(t,x) \geq 0$. The logarithmic change of variable $z(t,x) = -\ln y(t,x)$ transforms the linear heat equation into the viscous Hamilton–Jacobi equation

$$\dot{z} - \Delta z + |\nabla z|^2 = 0 \quad (t > 0, x \in \Omega), \tag{3.12}$$

with the initial condition $z(0,x) = -\ln \mathrm{y}^0(x)$ and constant target state $\mathrm{z}^1 = -\ln \mathrm{y}^1$.

The null controllability of (3.12), with interior control localized in a subset ω of the domain Ω and Dirichlet boundary conditions, has been studied in Ref. 17. More precisely, in Theorem 1.1 of Ref. 17, the authors prove that any initial condition z^0 for (3.12) can be steered to 0 in any time T larger than some time T_* depending only on $\|\mathrm{z}^0\|_{L^\infty(\Omega)}$. This result does not ensure that there always exists a positive minimal time for every initial condition z^0, but rather it establishes the existence of a waiting time when controlling all initial data of a given size. It would be interesting to compare the lower bounds on the waiting time obtained in this paper directly for the linear heat equation with those in Ref. 17. In any case, in some sense, both results are of the same nature since when applying the logarithmic change of variables to the first eigenfunction of the Dirichlet Laplacian one gets the classical barrier functions for elliptic viscous Hamilton–Jacobi equations employed in Ref. 17.

Our results in Sec. 4 show the existence of a waiting time whatever the initial non-negative initial condition $\mathrm{y}^0 \neq \mathrm{y}^1$ is. This leads to the existence of a positive minimal time for the controllability of z, taking into account that $y(t,x) = \exp(-z(t,x))$, thus complementing the results in Ref. 17. Of course this argument, which allows one to compare the control of the linear heat equation and that of the viscous Hamilton–Jacobi equation with the logarithmic change of variables, can only be applied under the condition $y \geq 0$.

4. Generalizations

4.1. *Mixed Dirichlet–Neumann boundary conditions*

In 1D, the symmetry properties of the controlled solutions allow to obtain some particular results for mixed boundary conditions of Dirichlet–Neumann type.

Going back to Theorem 2.1 and proceeding with symmetry considerations, when y^0 is symmetric with respect to $x = 1/2$ and, in particular, when it is a constant state, we can take $u_0 = u_1$. Then u_0 can also be viewed as the only boundary control for the mixed Dirichlet–Neumann system formulated in the half-interval $(0, 1/2)$:

$$\partial_t y(t,x) = \partial_x^2 y(t,x) \quad \text{in } (0,T) \times \left(0, \frac{1}{2}\right), \tag{4.1a}$$

$$\partial_x y\left(t, \frac{1}{2}\right) = 0 \qquad (t \in (0,T)), \tag{4.1b}$$

$$y(t,0) = u_0(t) \qquad (t \in (0,T)), \tag{4.1c}$$

$$y(0,x) = \mathrm{y}^0 \qquad (x \in (0,1)), \tag{4.1d}$$

steering the solution from y^0 to y^1 in time T.

Note that in this system the control enters through the Dirichlet boundary condition at the left boundary $x = 0$, while the homogeneous Neumann boundary condition is satisfied at the right boundary $x = 1/2$.

The converse result holds as well. Indeed, the even (with respect to $x = 1/2$) extension of the solution of (4.1) solves the Dirichlet control problem in $(0, 1)$ with equal controls at both extremities $x = 0$ and $x = 1$. Consequently, the time required to steer the solution of (1.1), with control constraints (2.1), from y^0 to y^1, coincides with the one required to steer (4.1), with the control constraint $u_0(t) \geq 0$, from y^0 to y^1.

Similar symmetry considerations can also be developed in the multi-dimensional case when the domain Ω under consideration enjoys adequate symmetry properties. This allows one to relate the Dirichlet control problem in the full domain Ω to the mixed Dirichlet–Neumann problem in some subdomains. This can be applied, for instance, when the domain Ω is a square, linking the Dirichlet control problem in Ω to the mixed control problem in the rectangle corresponding to half of Ω.

4.2. *Neumann control under non-negativity state constraints*

Let Ω be a bounded domain of $\mathbb{R}^d$ with C^2-boundary.

We first consider the heat equation with Neumann boundary conditions

$$\partial_t y(t,x) = \Delta y(t,x) \quad (t > 0, x \in \Omega), \tag{4.2a}$$

$$\partial_\nu y(t,x) = v(t,x) \qquad (t > 0, x \in \partial\Omega), \tag{4.2b}$$

with initial condition $y(0,x) = \mathrm{y}^0 \in L^2(\Omega)$, with $\mathrm{y}^0 \geq 0$. Note that, here, the Neumann control acts along the whole boundary. We consider the problem of reaching the constant target state $\mathrm{y}^1 > 0$ (with $\mathrm{y}^0 \neq \mathrm{y}^1$) under the non-negativity state constraint (3.10).

Controllability, without taking into account any constraint, has been established in Ref. 19. We claim that controllability under the non-negativity state constraint can be achieved in sufficiently large time, and that the minimal time for this constrained control problem is positive. Actually, the minimal times under state constraints both for Dirichlet and Neumann control problems coincide with the minimal time under non-negativity constraints on the Dirichlet control. Indeed, the Neumann control v can be taken to be the trace on $\partial\Omega$ of the normal derivative of the controlled trajectory by means of Dirichlet controls, under state constraints.

Let us now consider the more general situation where the Neumann control acts only on a proper subset $\Gamma_0 \subset \partial\Omega$ of the boundary:

$$\partial_t y(t,x) = \Delta y(t,x) \quad (t > 0, x \in \Omega), \tag{4.3a}$$

$$\partial_\nu y(t,x) = v(t,x) \quad (t > 0, x \in \Gamma_0), \tag{4.3b}$$

$$\partial_\nu y(t,x) = 0 \quad (t > 0, x \in \partial\Omega \backslash \Gamma_0), \tag{4.3c}$$

$$y(0,x) = \mathrm{y}^0(x) \quad (x \in \Omega). \tag{4.3d}$$

Theorem 4.1. *Let* $\mathrm{y}^0 \in L^2(\Omega)\backslash\{0\}$ *be such that* $\mathrm{y}^0(x) \geq 0$ *for every* $x \in \Omega$, *and let* $\mathrm{y}^1 \in L^2(\Omega)$ *be a steady state of* (4.3). *We assume that there exists* $\varepsilon > 0$ *such that* $\mathrm{y}^1(x) \geq \varepsilon$ *for every* $x \in \Omega$. *Then*:

- *There exist* $T > 0$ *and a control* $v \in L^2((0,T) \times \Gamma_0)$ *such that the corresponding solution* y *of* (4.3) *with initial condition* $y(0) = \mathrm{y}^0$ *satisfies* $y(T) = \mathrm{y}^1$ *and* $y(t,x) \geq 0$ *for all* $(t,x) \in (0,T) \times \Omega$.
- *Defining*

$$\underline{T}(\mathrm{y}^0, \mathrm{y}^1) = \inf\{T > 0, \exists\, v \in L^1((0,T) \times \Gamma_0) \text{ s.t. } y(t,x) \geq 0 \text{ and } y(T) = \mathrm{y}^1\}, \tag{4.4}$$

we have $\underline{T}(\mathrm{y}^0, \mathrm{y}^1) > 0$ *whenever* $\mathrm{y}^0 \neq \mathrm{y}^1$.

Remark 4.1. In the above definition of $\underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ we take controls in L^1, in order to have a large class of controls (see also Remark 4.2 further). It would be interesting to prove the existence of a Radon measure control u realizing the controllability exactly in time $\underline{T}(\mathrm{y}^0, \mathrm{y}^1)$.

Proof of Theorem 4.1. The previous argument does not apply directly. First, the positivity of the minimal control time is established similarly: indeed, if the solution of the control problem (4.3) satisfies the non-negativity state constraint, then it can also be viewed as a controlled trajectory with non-negative Dirichlet controls. But the fact that the system can be controlled while preserving the non-negativity of the state in large time requires further analysis. This goes as follows. Recall that, for the Dirichlet control problem, the property of constrained controllability was established by using the fact that the cost of controlling the system tends to 0 exponentially as $T \to +\infty$. This argument does not apply directly in the context of Neumann boundary controls. In fact the existence of nontrivial steady states for the adjoint Neumann problem:

$$-\partial_t \varphi(t,x) = \Delta\varphi(t,x) \quad (t \in (0,T), x \in \Omega), \tag{4.5a}$$

$$\partial_\nu \varphi(t,x) = 0 \quad (t \in (0,T), x \in \partial\Omega), \tag{4.5b}$$

$$\varphi(T,x) = \varphi^T \quad (x \in \Omega) \tag{4.5c}$$

is an impediment for the observability constant to decay exponentially. The relevant observability constant in the present setting, which is

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C(T) \int_0^T \int_{\Gamma_0} \varphi^2 \, \mathrm{d}\sigma \, \mathrm{d}t$$

is well known to be satisfied for any $T > 0$ (see Ref. 19). But the existence of trivial constant solutions $\varphi \equiv 1$ prevents the observability constant $C(T)$ from decaying exponentially. It is however easy to see that $C(T)$ decays as $O(1/T)$ when $T \to +\infty$. This is so because, once the existence of an observability constant C^* is established for some specific value of $T = T^*$, the observability constant $C(T)$ can be guaranteed to be of the order of C^*/k for $T = kT^*$.

This also ensures that one can steer the system (4.3) to 0 in time T with a control $u \in L^2((0,T) \times \Gamma_0)$ such that

$$\|u\|^2_{L^2((0,T)\times\Gamma_0)} \leq C(T)\|\mathrm{y}^0\|^2_{L^2(\Omega)}.$$

Similar results hold when the observation is taken in a weaker space. This leads in particular to the existence of a control $u \in L^2((0,T); H^r(\Gamma_0)) \cap H^s((0,T); L^2(\Gamma_0))$, for any given $r, s \geq 0$, satisfying

$$\|u\|^2_{L^2((0,T);H^r(\Gamma_0))} + \|u\|^2_{H^s((0,T);L^2(\Gamma_0))} \leq C_{r,s}(T)\|\mathrm{y}^0\|^2_{L^2(\Omega)} \quad (T > 0),$$

steering (4.3) to 0 in time T.

Furthermore, in view of Sec. 13.1 and Theorem 2.1 of Ref. 14, for r and s large enough and for y^0 regular enough, the solution y of (4.3) with initial condition $y(0) = \mathrm{y}^0$ satisfies $y(t,\cdot) \in L^\infty(\Omega)$, and due to well-posedness of the heat equation, there also exists $K(T) > 0$, not depending on the control u, such that

$$\|y(t) - y_0(t)\|_{L^\infty(\Omega)} \leq K(T)(\|u\|_{L^2((0,T);H^r(\Gamma_0))} + \|u\|_{H^s((0,T);L^2(\Gamma_0))}) \quad (t \in (0,T)),$$

where y_0 is the solution of (4.3) with control $u = 0$ and initial condition y^0. All in all, we have obtained the following local controllability result.

Lemma 4.1. *Let* $\mathrm{y}^0 \in H^1(\Omega)$ *and let* $\tau > 0$ *be arbitrary. There exist* $\kappa(T) > 0$ *and a control* $u \in L^2((0,T) \times \Gamma_0)$ *such that the solution* y *of* (4.3) *with initial condition*

$y(0) = \mathrm{y}^0$ *and control* u *satisfies* $y(T) = 0$ *and* $\|y(t) - y_0(t)\|_{L^\infty(\Omega)} \leq \kappa(T)\|\mathrm{y}^0\|_{L^2(\Omega)}$, *where* y_0 *is solution of* (4.3) *with control* $u = 0$ *and initial condition* y^0.

We are now in a position to prove the theorem. In order to avoid additional notations, we only present the case where y^1 is a constant steady state. The general case can be addressed in the same way. Let $\tau > 0$ and let $\varepsilon > 0$ be arbitrary. We will proceed in several steps:

Step 1. We take a null control during a long time $T_0(\varepsilon)$ so that the solution y of (4.3) with null control and initial condition $y(0) = \mathrm{y}^0$ satisfies

$$\left\| y(T_0(\varepsilon)) - \frac{1}{|\Omega|}\int_\Omega \mathrm{y}^0 \right\|_{L^\infty(\Omega)} < \varepsilon \quad \text{and} \quad \left\| y(T_0(\varepsilon)) - \frac{1}{|\Omega|}\int_\Omega \mathrm{y}^0 \right\|_{L^2(\Omega)} < \varepsilon.$$

We set $\mathrm{y}_0 = y(T_0(\varepsilon))$ and $\bar{\mathrm{y}}_0 = 1/|\Omega| \int_\Omega \mathrm{y}^0$.

Step 2. We build a control u_0 steering (4.3) from the initial condition $y(0) = \mathrm{y}_0$ to the target $\bar{\mathrm{y}}_0$ in time τ.

Step 3. We define a finite sequence $(\bar{\mathrm{y}}_k)_{k=0,\dots,K}$ of positive values such that $\bar{\mathrm{y}}_K = \mathrm{y}^1$ and $|\bar{\mathrm{y}}_{k+1} - \bar{\mathrm{y}}_k| < \varepsilon$ for every $k \in \{0, \dots, K-1\}$.

Step 4. For $k \in \{1, \dots, K\}$, we build a control u_k steering (4.3) from the initial condition $y(0) = \bar{\mathrm{y}}_{k-1}$ to the target $\bar{\mathrm{y}}_k$ in time τ.

Once all this process is done, we check that there exists $\varepsilon = \varepsilon(\tau) > 0$ small enough such that the solution y with initial condition $y(0) = \mathrm{y}^0$ and with control

$$u(t) = \begin{cases} 0 & \text{if } t \in (0, T_0(\varepsilon)), \\ u_0(t - T_0(\varepsilon)) & \text{if } t \in (T_0(\varepsilon), T_0(\varepsilon) + \tau), \\ \vdots & \\ u_K(t - T_0(\varepsilon) - K\tau) & \text{if } t \in (T_0(\varepsilon) + K\tau, T_0(\varepsilon) + (K+1)\tau), \end{cases}$$

for $t \in (0, T_0(\varepsilon) + (K+1)\tau)$, satisfies $y(t, x) \geq 0$ for all $t \in (0, T_0(\varepsilon) + (K+1)\tau)$ and all $x \in \Omega$.

Step 1. Using a spectral expansion, it is easy to see that the solution y of (4.3) with null control and initial condition $y(0) = \mathrm{y}^0$ converges in any Sobolev norm to $\bar{\mathrm{y}}_0$ as $t \to +\infty$. Therefore, there exists $T_0(\varepsilon) > 0$ such that $\|y(T_0(\varepsilon)) - \bar{\mathrm{y}}_0\|_{L^\infty(\Omega)} < \varepsilon$ and $\|y(T_0(\varepsilon)) - \bar{\mathrm{y}}_0\|_{L^2(\Omega)} < \varepsilon$. Furthermore, since y^0 is non-negative and nontrivial, we have $\bar{\mathrm{y}}_0 > 0$ and $y(t, x) \geq 0$ on $(0, T_0(\varepsilon)) \times \Omega$. We set $\mathrm{y}_0 = y(T_0(\varepsilon)) \in H^1(\Omega)$.

Step 2. According to Lemma 4.1, there exist $\kappa(\tau) > 0$, only depending on τ, and a control $u_0 \in L^2((0, \tau) \times \Gamma_0)$ such that the solution y of (4.3) with initial condition $y(0) = \mathrm{y}_0$ and control u_0 satisfies $y(\tau) = \bar{\mathrm{y}}_0$ and

$$\|y(t) - y_0(t)\|_{L^\infty(\Omega)} \leq \kappa(\tau)\|\mathrm{y}_0 - \bar{\mathrm{y}}_0\|_{L^2(\Omega)} \leq \varepsilon\kappa(\tau) \quad (t \in (0, \tau)),$$

where y_0 is the solution of (4.3) with initial condition y_0 and null control. Since $\|y_0(t) - \bar{\mathrm{y}}_0\|_{L^\infty(\Omega)} \leq \varepsilon$, we easily obtain that $\|y(t) - \bar{\mathrm{y}}_0\|_{L^\infty(\Omega)} \leq \varepsilon(1 + \kappa(\tau))$ for $t \in (0, \tau)$, i.e. $y(t, x) \geq \bar{\mathrm{y}}_0 - \varepsilon(1 + \kappa(\tau))$ on $(0, \tau) \times \Omega$.

Step 3. Setting $K = K(\varepsilon) \geq |\mathrm{y}^1 - \bar{\mathrm{y}}_0|/\varepsilon$ and $\bar{\mathrm{y}}_k = \frac{k}{K}\mathrm{y}^1 + (1 - \frac{k}{K})\bar{\mathrm{y}}_0$ for $k \in \{0, \ldots, K\}$, we have $|\bar{\mathrm{y}}_{k+1} - \bar{\mathrm{y}}_k| \leq \varepsilon$ and $\bar{\mathrm{y}}_K = \mathrm{y}^1$.

Step 4. Let $k \in \{1, \ldots, K(\varepsilon)\}$. According to Lemma 4.1, there exist $\kappa(\tau) > 0$, only depending on τ (and independent of k, this constant is the same as in Step 2), and a control $u_k \in L^2((0, \tau) \times \Gamma)$ such that the solution y of (4.3) with initial condition $y(0) = \mathrm{y}_{k-1}$ and control u_k satisfies $y(\tau) = \bar{\mathrm{y}}_k$ and

$$\|y(t) - \bar{\mathrm{y}}_{k-1}\|_{L^\infty(\Omega)} \leq \kappa(\tau)\|\bar{\mathrm{y}}_{k-1} - \bar{\mathrm{y}}_k\|_{L^2(\Omega)} \leq \varepsilon\kappa(\tau)\sqrt{|\Omega|} \quad (t \in (0, \tau)),$$

and thus $y(t, x) \geq \bar{\mathrm{y}}_{k-1} - \varepsilon\kappa(\tau)\sqrt{|\Omega|}$ on $(0, \tau) \times \Omega$.

Conclusion of the proof. Setting

$$\varepsilon = \varepsilon(\tau) = \min\left(\frac{\min(\bar{\mathrm{y}}_0, \mathrm{y}^1)}{\kappa(\tau)\sqrt{|\Omega|}}, \ \frac{\bar{\mathrm{y}}_0}{1 + \kappa(\tau)}\right),$$

we have obtained a control $u \in L^2((0, T) \times \Gamma_0)$, with $T = T_0(\varepsilon(\tau)) + (K(\varepsilon(\tau)) + 1)\tau$, such that the solution y of (4.3) with initial condition y^0 and control u satisfies $y(t, x) \geq 0$ and $y(T, x) = \mathrm{y}^1$. □

Remark 4.2. Assuming moreover that $\inf_\Omega \mathrm{y}^0 > 0$, it can be also proved that, for every $\tau > 0$, there exists a control v of arbitrary regularity steering the solution of (4.3) from $\mathrm{y}^0 \geq 0$ in $L^2(\Omega)$ to $\mathrm{y}^1 \in (0, +\infty)$ in time $\underline{T}(\mathrm{y}^0, \mathrm{y}^1) + \tau$. Let us sketch the proof hereafter. We set $\underline{T} = \underline{T}(\mathrm{y}^0, \mathrm{y}^1)$.

Step 1. Taking $\varepsilon \in (0, \min(\mathrm{y}^1, \inf_\Omega \mathrm{y}^0))$, there exists a minimal time $\underline{T}^\varepsilon$ such that for every $T > \underline{T}^\varepsilon$, there exists a control $v^\varepsilon \in L^1((0, T) \times \Gamma_0)$ such that the solution y^ε of (4.3) with control v^ε satisfies $y^\varepsilon(T) = \mathrm{y}^1$ and $y^\varepsilon \geq \varepsilon$ on $(0, T) \times \Omega$.

It can be checked that $\underline{T}^\varepsilon \geq \underline{T}$ and that $\underline{T}^\varepsilon \to \underline{T}$ as $\varepsilon \to 0$. Hence, taking $\varepsilon > 0$ small enough, there exists a control $v^\varepsilon \in L^1((0, \underline{T} + \tau/3) \times \Gamma_0)$ such that the corresponding solution y^ε of (4.3) satisfies $y^\varepsilon(\underline{T} + \tau/3) = \mathrm{y}^1$ and $y^\varepsilon \geq \varepsilon$ on $(0, \underline{T} + \tau/3) \times \Gamma_0$.

Step 2. We regularize the control v^ε of the previous step: we design a control $v_0 \in C^\infty((0, \underline{T} + \tau/3) \times \Gamma_0)$ such that all derivatives of v_0 at time $\underline{T} + \tau/3$ vanish and such that v_0 is close to v^ε in $L^1((0, \underline{T} + \tau/3) \times \Gamma_0)$ norm. Let y_0 be the corresponding solution of (4.3). If v_0 is close enough to v^ε in $L^1((0, \underline{T} + \tau/3) \times \Gamma_0)$-norm, then $y_0 \geq 0$ on $(0, \underline{T} + \tau/3) \times \Gamma_0$ and $y_0(\underline{T} + \tau/3)$ is close to y^1 in the $H^{-1}(\Omega)$-norm. Set $\mathrm{y}^1_0 = y_0(\underline{T} + \tau/3)$.

Step 3. The conclusion is then similar to the one in Proposition 2.1:

(a) We take the zero control over $(\underline{T} + \tau/3, \underline{T} + 2\tau/3)$, in order to regularize y^1_0. Then the corresponding solution y^1_1 at time $\underline{T} + 2\tau/3$ is close to y^1 in any Sobolev norm.

(b) We steer y^1_1 to y^1 with a regular control in time $\tau/3$ having its derivatives equal to 0 at time 0. If y^1_1 is close enough to y^1, then we can ensure that the corresponding state trajectory is non-negative. Note that y_1^1 can be taken arbitrarily close to y^1 (as a consequence of the regularization of v^ε in Step 2 above).

The more general case $\mathrm{y}^0 \geq 0$ is open. It is more difficult, due to the fact that the regularization process in Step 2 does not preserve non-negativity of the trajectory. A way to establish this result would be to take the zero control over some small time interval $(0, \tau)$ so that, at time τ, we have obtained a state $\tilde{\mathrm{y}}^0$ which is positive and close to y^0. Then we could proceed as above. But the main difficulty is to prove that $\underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ is close to $\underline{T}(\tilde{\mathrm{y}}^0, \mathrm{y}^1)$, i.e. that the mapping $\mathrm{y} \in L^2(\Omega) \mapsto \underline{T}(\mathrm{y}, \mathrm{y}^1)$ is continuous.

4.3. *Internal control with Neumann boundary conditions*

Let Ω be a bounded domain of $\mathbb{R}^d$ with C^2-boundary. We consider the internally-controlled heat equation with Neumann homogeneous boundary conditions:

$$\partial_t y(t,x) = \Delta y(t,x) + \mathbf{1}_\omega(x) u(t,x) \quad (t > 0, x \in \Omega), \tag{4.6a}$$

$$\partial_\nu y(t,x) = 0 \quad (t > 0, x \in \partial\Omega), \tag{4.6b}$$

$$y(0,x) = \mathrm{y}^0(x) \quad (x \in \Omega), \tag{4.6c}$$

with ω an open nonempty subset of Ω, support of the control.

Given $\mathrm{y}^0 \in L^2(\Omega)$ such that $\mathrm{y}^0 \geq 0$, the objective is to steer (4.6) to a given steady state y^1 (with $\mathrm{y}^1 \neq \mathrm{y}^0$ and $\inf_\Omega \mathrm{y}^1 > 0$), under the non-negativity state constraint (1.3), by means of a control $u = u(t,x)$ localized in ω.

Note that, if $\omega = \Omega$, then we have a trivial solution: the control $u(t,x) = (\mathrm{y}^1 - \mathrm{y}^0 - (T-t)\Delta \mathrm{y}^0 - t\Delta \mathrm{y}^1)/T$ for $(t,x) \in [0,T] \times (0,1)$, with $T > 0$, steers the solution y of (4.6) from y^0 to y^1 in time T with state $((T-t)\mathrm{y}^0 + t\mathrm{y}^1)/T$, which of course satisfies the state constraint (1.3).

This strategy holds when $\mathrm{y}^0 \in H^2(\Omega)$ and $\partial_\nu \mathrm{y}^0 = 0$ on $\partial\Omega$. When $\mathrm{y}^0 \in L^2(\Omega)$ only, we can use the regularizing property of the heat equation. More precisely, for any $\tau > 0$:

(1) We take the zero control over $(0, \tau)$, so that, then, we have to steer some regular y_0^0 to y^1 (with $\mathrm{y}_0^0 = y(\tau)$).
(2) Setting $\Omega_\varepsilon = \Omega \backslash (\partial\Omega + B(0, \varepsilon))$ for some $\varepsilon > 0$, the solution y_1 of (4.6) with initial condition $y_1(0) = \mathrm{y}_0^0$, with the control $u = \frac{1}{\tau}(\mathrm{y}^1 - \mathrm{y}_0^0 - (\tau - t)\Delta \mathrm{y}_0^0 - t\Delta \mathrm{y}^1)|_{\Omega_\varepsilon}$, satisfies $y_1(t)|_{\Omega_\varepsilon} = (\tau - t)/\tau \mathrm{y}_0^0 + t/\tau \mathrm{y}^1$, and in $\Omega \backslash \Omega_\varepsilon$ it is solution of

$$\partial_t y_1(t,x) = \Delta y_1(t,x) \quad (t > 0, x \in \Omega \backslash \Omega_\varepsilon),$$

$$\partial_\nu y_1(t,x) = 0 \quad (t > 0, x \in \partial\Omega),$$

$$y_1(t,x) = \frac{\tau - t}{\tau} \mathrm{y}_0^0 + \frac{t}{\tau} \mathrm{y}^1 \quad (t > 0, x \in \partial\Omega_\varepsilon),$$

$$y_1(0,x) = \mathrm{y}_0^0(x) \quad (x \in \Omega \backslash \Omega_\varepsilon).$$

Now, setting $y_1^0 = y_1(\tau)$, we have $\|y_1^0 - y^1\|_{L^2(\Omega)} \leq \sqrt{|\Omega\backslash\Omega_\varepsilon|}\|y_0^0 - y^1\|_{L^\infty(\Omega)}$, which converges to 0 as $\varepsilon \to 0$.

(3) For $\varepsilon > 0$ small enough, one can steer y_1^0 to y^1 in time τ while preserving non-negativity of the trajectory. This result is a consequence of Lemma 4.1 (adapted to internal control).

We now assume that $\omega \neq \Omega$. Then we have the following result, similarly as before: there exist $T > 0$ and a control $u \in L^2((0,T) \times \omega)$ such that the solution y of (4.6), with $y(0) = y^0$, satisfies $y(T) = y^1$ and the non-negativity state constraint (1.3) (same argument as in the proof of Theorem 4.1). Then, defining as before the minimal time, we have $\underline{T}(y^0, y^1) > 0$. Indeed, restricting the solution of (4.6) to a ball contained in $\Omega\backslash\omega$ and taking the trace on the boundary of this ball lead to a Dirichlet problem as the one studied in Sec. 3.1 and it has been shown in Theorem 3.1 that this control problem with non-negative control constraint cannot be solved within arbitrarily small time.

Consequently, we have proved that, if $y^0|_{\Omega\backslash\omega} \neq y^1|_{\Omega\backslash\omega}$, then $\underline{T}(y^0, y^1) > 0$.

Remark 4.3. If $y^0|_{\Omega\backslash\omega} = y^1|_{\Omega\backslash\omega}$ and $y^0|_\omega \neq y^1|_\omega$, then $\underline{T}(y^0, y^1) = 0$. In fact, if $y^0 \in H^2(\Omega)$, $y^0|_{\Omega\backslash\omega} = y^1|_{\Omega\backslash\omega}$ and $\partial_\nu y^0 = 0$ on $\partial\Omega$, then for any $T > 0$, the control $u = (y^1 - y^0 - (T-t)\Delta y^0 - t\Delta y^1)/T$ is supported in ω and steers y^0 to y^1 in time T (and the associated trajectory $((T-t)y^0 + ty^1)/T$ is non-negative).

The general case $y^0 \in L^2(\Omega)$ can be treated with the same strategy as the one used to prove that $\underline{T}(y^0, y^1) = 0$ when $\omega = \Omega$.

Remark 4.4. We have presented here the result with Neumann homogeneous boundary condition. But the boundary conditions do not affect the result on the positivity of $\underline{T}(y^0, y^1)$.

4.4. *Neumann boundary control with control constraints in* 1D

Let $y^0 \in \mathbb{R}$ and $y^1 \in \mathbb{R}$ be two real numbers with $y^0 \neq y^1$. We consider the 1D heat equation with Neumann boundary controls:

$$\partial_t y(t,x) = \partial_x^2 y(t,x) \quad (t > 0, x \in (0,1)), \tag{4.7a}$$

$$\partial_x y(t,0) = v_0(t) \qquad (t > 0), \tag{4.7b}$$

$$\partial_x y(t,1) = v_1(t) \qquad (t > 0), \tag{4.7c}$$

with *constant* initial condition $y(0,x) = y^0$ (steady state). We consider the question of steering (1.1) to y^1 under the non-negativity control constraints

$$v_0(t) \geq 0 \quad \text{and} \quad v_1(t) \geq 0 \quad (t > 0 \text{ a.e.}). \tag{4.8}$$

The following result shows that this is impossible.

Theorem 4.2. *Given any $T > 0$, there do not exist any non-negative controls v_0 and v_1 in $\mathcal{M}(0,T)$ such that the corresponding solution y of* (4.2) *satisfies $y(0) = y^0$ and $y(T) = y^1$.*

In Sec. 4.2, we considered Neumann boundary controls with constraints on the state. The results of the present section are of complementary interest since they show that different constraints can have various effects on the control property, making it impossible to be achieved in the Neumann case with constraints on the control.

Proof. We follow the arguments of Theorem 2.1. Assume by contradiction that there exist $T > 0$ and controls v_0 and $v_1 \in \mathcal{M}(0,T)$ satisfying (4.8) such that the solution y of (4.2) satisfies $y(0) = \mathrm{y}^0$ and $y(T) = \mathrm{y}^1$. Defining $y_n(t) = \int_0^1 y(t,x)\cos(n\pi x)\mathrm{d}x$ for $t \in [0,T]$ and $n \in \mathbb{N}$, we have, by integrations by parts, $\dot{y}_n(t) = -n^2\pi^2 y_n(t) + (-1)^n v_1(t) - v_0(t)$, and thus

$$y_n(T) = e^{-n^2\pi^2 T} y_n(0) + \int_0^T e^{-n^2\pi^2(T-t)}\mathrm{d}((-1)^n v_1 - v_0)(t).$$

Since $y(0) = \mathrm{y}^0$ and $y(T) = \mathrm{y}^1$, we must have $y_0(0) = \mathrm{y}^0$ and $y_n(0) = 0$ if $n > 0$, and $y_0(T) = \mathrm{y}^1$ and $y_n(T) = 0$ if $n > 0$. Hence,

$$\mathrm{y}^1 - \mathrm{y}^0 = \int_{[0,T]} \mathrm{d}(v_1 - v_0)(t), \tag{4.9a}$$

$$0 = \int_{[0,T]} e^{-(2p)^2\pi^2(T-t)}\mathrm{d}(v_1 - v_0)(t) \quad (p \in \mathbb{N}^*), \tag{4.9b}$$

$$0 = \int_{[0,T]} e^{-(2p+1)^2\pi^2(T-t)}\mathrm{d}(v_1 + v_0)(t) \quad (p \in \mathbb{N}^*). \tag{4.9c}$$

The conditions (4.9b) and (4.8) lead to $v_0 = v_1 = 0$, which is incompatible with the first condition (4.9a) (since $\mathrm{y}^0 \neq \mathrm{y}^1$). We get a contradiction. □

Remark 4.5. We have stated the failure of the controllability property with non-negative controls in $\mathcal{M}(0,T)$, which is the most general space to consider non-negativity constraints. As we see in the proof, the Fourier series expansion is well justified in that control setting since the duality between controls and the time real exponentials (which are continuous) is well justified.

Remark 4.6. The result of Theorem 4.2 is valid as well when both controls are of constant sign with respect to t but not necessarily the same.

Some remarks are in order on the negative result of Theorem 4.2.

Remark 4.7. An alternative argument of proof is the following. If there were to exist non-negative nontrivial controls v_0 and v_1, then, for $p \in \mathbb{N}$, the $(2p+1)$th Fourier coefficient of $y(T,\cdot)$, namely $\int_0^1 y(T,x)\cos((2p+1)\pi x)\mathrm{d}x$, would be positive. In other words, the action of the controls generates some Fourier modes which cannot be cancelled. Since the modes associated with the final target $y(T) = \mathrm{y}^1$ are zero, this would lead to a contradiction.

Remark 4.8. Still, one can wonder why, as it occurs in the context of Dirichlet control, the constrained controllability property cannot be guaranteed if the control time is large enough.

To prove controllability under constraints in large time, one could use a quasi-static strategy (see Ref. 6 and Chap. 7 of Ref. 5), which consists first of considering a path of equilibria joining the initial and final data. The path of equilibria in this case is $\bar{\mathrm{y}}(\tau) = (1-\tau)\mathrm{y}^0 + \tau\mathrm{y}^1$, the corresponding Neumann traces are the trivial ones $\bar{v}_0(\tau) = \bar{v}_1(\tau) = 0$, for $\tau \in [0,1]$, and they lie on the boundary of the constraints that we impose. In other words, the path of state-controls pairs $\tau \in [0,1] \mapsto (\bar{\mathrm{y}}(\tau), \bar{v}_0(\tau), \bar{v}_1(\tau)) \in L^2(0,1) \times \mathbb{R}^2$ does not belong to the interior of the set $L^2(0,1) \times (0,+\infty)^2$. This causes the failure of the quasi-static control strategy, which requires adding extra controls to correct the defect of the trajectory when satisfying the heat equation.

Remark 4.9. The argument based on using a translation of the state and the fact that the observability constant decays as T tends to infinity fails as well. Indeed, let us introduce the new state $z = y - \mathrm{y}^1$. The goal is to drive z from $\mathrm{y}^0 - \mathrm{y}^1$ to 0 in time T. This can be achieved by means of boundary controls $w_0(t)$ and $w_1(t)$, and they can be shown to be small in $L^\infty(0,T)$ if the control time T is large enough. But these controls are of oscillatory nature and they cannot be guaranteed to keep a constant sign. Going back to the original state y and due to the boundary conditions of Neumann type, we observe that the Neumann controls for y are precisely $w_0(t)$ and $w_1(t)$. This translation argument does not allow to conclude the Neumann control with non-negative controls even if T is large. But, in fact, as shown above, this is impossible whatever T is.

Remark 4.10. Whether the negative 1D result of Theorem 4.2 is valid also in multi-dimensional case is an open question.

5. Numerical Simulations

In this section, we run some numerical simulations to illustrate our results. We focus on the 1D heat equation with Neumann or Dirichlet boundary controls. To simplify the presentation, we will take positive and constant initial and final states y^0 and y^1.

According to Remark 3.9 and as a consequence of the comparison principle for the solutions of the heat equation, the minimal time for the control of the heat equation with state constraints coincides with the minimal time arising when the constraints are imposed only on the control. According to Lemma 2.2 in both situations, this minimal time satisfies lower estimates given by (2.7). Moreover, for the Dirichlet control problem we can take $u_0 = u_1$, and for the Neumann control problem we can take $v_0 = -v_1$.

5.1. *Dirichlet boundary controls with non-negativity control constraints*

Let us consider the minimal time control problem for the 1D heat equation on $(0, 1)$ with Dirichlet boundary controls submitted to a non-negativity constraint. In order to perform numerical simulations, we choose the simplest possible discretization scheme: finite differences in space, Euler explicit scheme in time, with a uniform space-time grid $t_i = i\frac{T}{N_t}$, $i = 0, \ldots, N_t$, $x_j = \frac{i}{N_x}$, $j = 0, \ldots, N_x$, where N_t and N_x are positive integers satisfying the Courant–Friedrich–Lewy condition $2\Delta t \leq (\Delta x)^2$, with $\Delta t = T/N_t$ and $\Delta x = 1/N_x$.

Taking into account that we are dealing with non-negativity constraints, it would have been more natural to consider an implicit Euler discretization for which the comparison principle of the solutions of the heat equation is preserved. However, we have observed no substantial change on the numerical results, while they were computationally less expensive when employing explicit Euler method.

In all the numerical examples presented hereafter, we will have $T < 0.25$, and we choose $N_x = 30$ and $N_t = 450$.

The discrete state is then a $((N_t+1)\times(N_x+1))$-component discrete matrix $(Y_{i,j})$ representing the approximation of $y(t, x)$ over the grid points, the two $(N_t + 1)$-component column vectors (U_i) and (V_i) representing the discretized controls both at $x = 0$ and $x = 1$, and the scalar $T \geq 0$ which is the final time. These discrete state-control pairs are linked by the discrete relations representing the dynamics, the boundary constraints and terminal condition at $t = T$. The resulting optimization problem, under discrete control constraints, is the following:

$$\text{minimize } T$$

under the constraints

$$\begin{aligned}
&T = N_t\Delta t,\\
&\frac{Y_{i+1,j} - Y_{i,j}}{\Delta t} = \frac{Y_{i,j+1} - 2Y_{i,j} + Y_{i,j-1}}{(\Delta x)^2}, && j = 1, \ldots, N_x - 1, \quad i = 0, \ldots, N_t - 1,\\
&Y_{i,0} = U_i^0, \quad Y_{i,N_x} = U_i^1, && i = 0, \ldots, N_t,\\
&U_i^0 \geq 0, \quad U_i^1 \geq 0, && i = 0, \ldots, N_t,\\
&Y_{0,j} = \mathrm{y}^0, \quad Y_{N_t,j} = \mathrm{y}^1, && j = 1, \ldots, N_x - 1.
\end{aligned}$$

This is a standard finite-dimensional constrained optimization problem, its dimension being larger as the discretization is finer. To solve it numerically, we use the expert interior-point optimization routine `IpOpt` (see Ref. 26) combined with automatic differentiation and the modeling language `AMPL` (see Ref. 9). We refer to Refs. 1, 20 and 21 for a survey on numerical methods in optimal control and how to implement them efficiently according to the context.

The interest of the simulations we perform hereafter is justified by the convergence result in Sec. 5.3 (see further) that guarantees that the discrete optimal time converges, as the mesh size tends to 0, to the minimal time for the continuous constrained control problem.

Case $\mathbf{y^0 \equiv 5}$ and $\mathbf{y^1 \equiv 1}$. According to Sec. 2 and Example 2.1, we must have $T \geq 0.165297$. Figures 3 and 4 display the numerical solution of the above problem, with $N_t = 450$ and $N_x = 30$.

The minimal time we obtain from our simulations is $T \simeq 0.1931$, which is compatible with the theoretical lower bound prediction, but larger, which is consistent with the fact that our analysis yields only lower bounds that, probably, are not sharp. In particular, we observe that, taking larger N_t and N_x, the value of the resulting minimal time does not change significantly. This is in accordance with the convergence result given in Sec. 5.3 of the minimal time for the discretized problem, to the minimal time of the continuous one, as N_t and N_x tend to $+\infty$ (preserving the CFL condition).

It is interesting to note in Fig. 4 that the two controls coincide and are identically equal to 0 over long time subintervals. We have shown that the controls can always be chosen to coincide; but, in general, they are not unique. And yet, the controls we obtain in our numerical experiments are always symmetric. This raises the question of whether, at the minimal time, the controls are unique (if so, they must be equal). The numerical simulations also raise the interesting question of whether the controls in the minimal control time necessarily present a "sparse" structure with long lags where they are identically zero.

The evolution of the state that we observe in Fig. 4 presents the following features:

- The solution starts from the constant initial state $y^0 \equiv 5$.

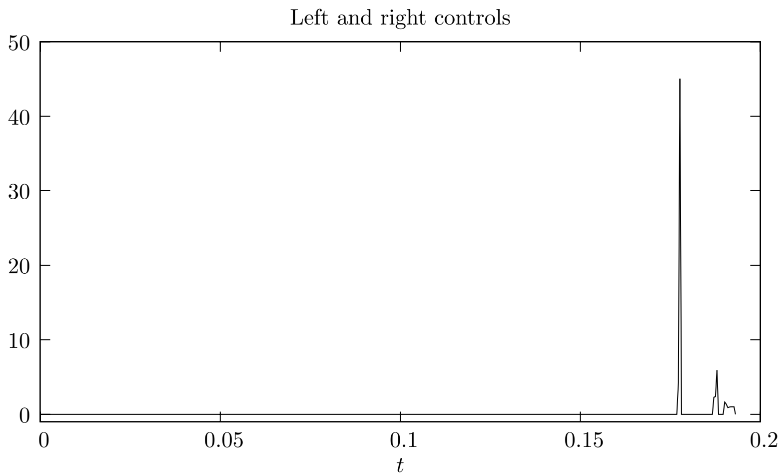


Fig. 2. Evolution of the controls while steering the system from the initial value (= 5) to the final one (= 1) (left and right Dirichlet controls are equal). The minimal computed time is $T \simeq 0.1931$. See Figs. 3 and 4 for the associated state trajectory.

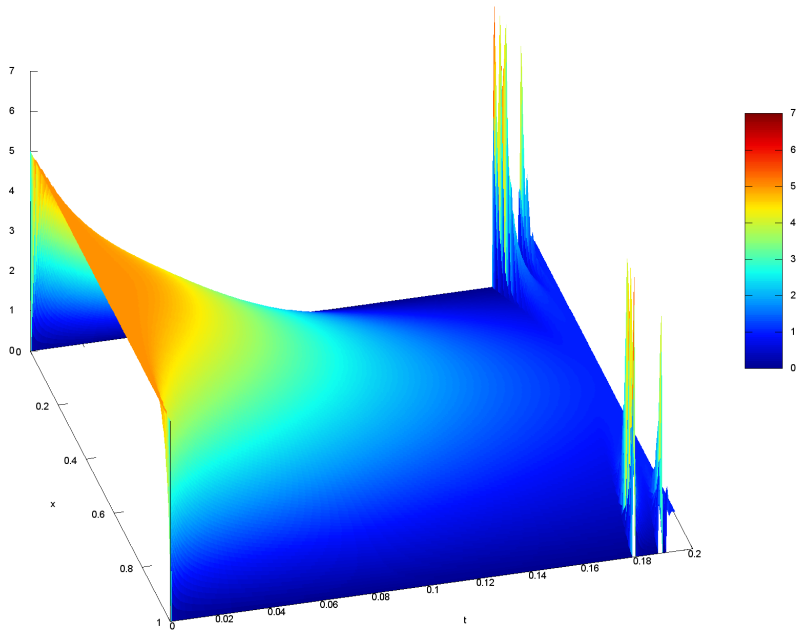


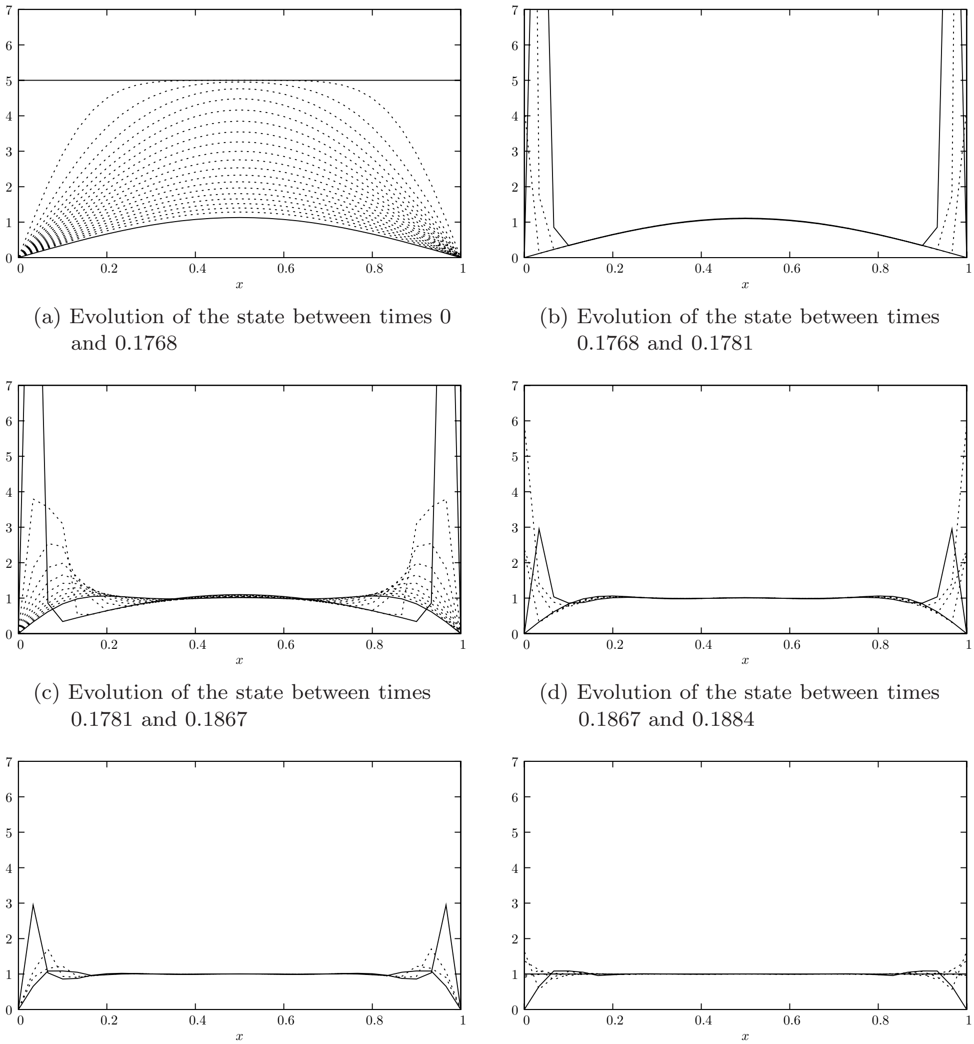
Fig. 3. (Color online) Evolution of the state from the initial value (= 5) to the final one (= 1) in the minimal computed time $T \simeq 0.1931$. See Fig. 2 for the associated controls.

- Since we first have $u_0(t) = u_1(t) = 0$ during a significantly long time subinterval, the solution follows the dynamics of the heat equation with null boundary conditions, in the absence of controls (see Fig. 4(a)).

  Using the Fourier expansion of the solutions, the state trajectory on this time interval is given by

$$y(t,x) = 4\mathrm{y}^0 \sum_{p=0}^{+\infty} \frac{e^{-(2p+1)^2\pi^2 t}}{(2p+1)\pi} \sin((2p+1)\pi x).$$

  We observe that, during this time interval, the state approximately coincides with the first Fourier mode, i.e. $y(t,x) \simeq \frac{20}{\pi} e^{-\pi^2 t} \sin(\pi x)$. In agreement with this analytical observation the state looks as a concave function, vanishing at the boundaries $x = 0$ and $x = 1$. The maximum of this function, which is symmetric with respect to at $x = 1/2$, is reached precisely at $x = 1/2$, and this maximum decreases as time increases.
- These dynamics of free heat equation with homogeneous Dirichlet boundary conditions remain so until $\max_x y(t_1,x) = y(t_1,1/2) \simeq 1$, that is, at $t_1 \simeq 0.1764$. Then, the control takes large positive values (over a short time interval $(0.1764, 0.1781)$), and, accordingly, so does the solution at the boundary points $x = 0$ and $x = 1$ (see Fig. 4(b)). After this short interval, the solution adjusts itself to reach exactly $\mathrm{y}^1 = 1$ (see Figs. 4(c)–4(f)).



(a) Evolution of the state between times 0 and 0.1768

(b) Evolution of the state between times 0.1768 and 0.1781

(c) Evolution of the state between times 0.1781 and 0.1867

(d) Evolution of the state between times 0.1867 and 0.1884

(e) Evolution of the state between times 0.1884 and 0.1897

(f) Evolution of the state between times 0.1897 and the final time, 0.1931

Fig. 4. Evolution of the state from the initial value (= 5) to the final one (= 1) in the minimal computed time $T \simeq 0.1931$. See Fig. 2 for the associated controls and Fig. 3 for the continuous time evolution of the state.

- Controls present an "off-bang-off-bang-off" structure. In other words, roughly, the optimal (minimal time) strategy consists of, first, letting the solution damp itself out, exponentially decrease to 0 while staying positive; when the solution becomes small enough (just below 1 but touching 1 at its maximum), the controls switch on taking positive values to increase the solution, on the left and on the right, and make it match the desired target solution.

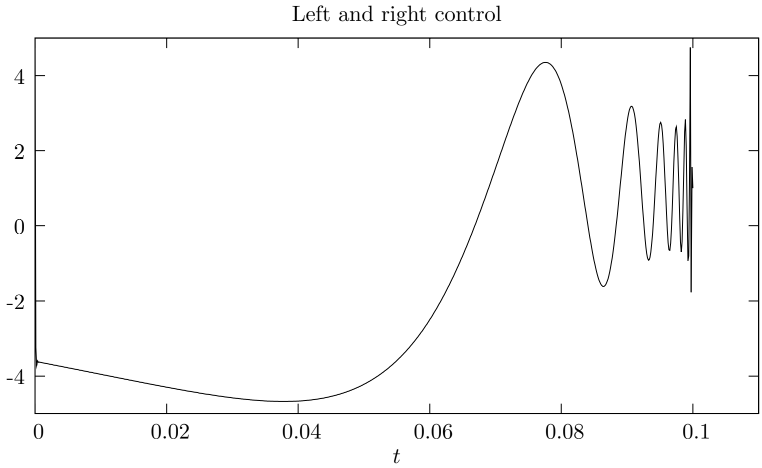


Fig. 5. Control of minimal L^2-norm steering $\mathrm{y}^0 = 5$ to $\mathrm{y}^1 = 1$ in time $T = 0.1$ without constraint. (Left and right Dirichlet controls are equal.)

This strategy corresponds to intuition, and follows from the non-negativity control constraints. Other numerical simulations confirm that, as expected, if we remove this non-negativity control constraint, then one can steer $\mathrm{y}^0 = 5$ to $\mathrm{y}^1 = 1$ in arbitrarily small time with oscillating (changing sign) controls. For instance, in Fig. 5 we plot the control of minimal L^2-norm steering $\mathrm{y}^0 = 5$ to $\mathrm{y}^1 = 1$ in time $T = 0.1$.

Case $\mathbf{y^0 \equiv 1}$ and $\mathbf{y^1 \equiv 5}$. According to Sec. 2 and Example 2.1, we must have $T > 0.023076$. We adopt the same discretization scheme, with $N_x = 20$. Moreover, we add temporarily an upper constraint on the controls, assuming that $0 \leq u_0(t) \leq M$ and $0 \leq u_1(t) \leq M$ for some $M > 0$ that, in practice, is chosen large enough. We do so because our theoretical results predict that controls may be Radon measures including some singular components (Dirac deltas). If that were the case, the possible need of Dirac masses should be visible in the numerical experiments by taking M to be larger and larger. This is precisely what is observed in our numerical simulations (see Figs. 7 and 8 and Figs. 10 and 11).

Consider first Fig. 8. The minimal time we obtain is $T \simeq 0.0498$, which is compatible with the theoretical prediction ($T > 0.02307$). In this numerical simulation, the optimal strategy develops the following features:

- We start with a bang arc along which $u_0(t) = u_1(t) = 50 = M$, the maximal authorized value (see the corresponding state trajectory in Fig. 8(a)), which is compatible with the possible presence of Dirac components on the controls at the initial time.

  Along this arc, at any time instance, the solution looks as a convex function, equal to 50 at both boundaries $x = 0$ and $x = 1$.

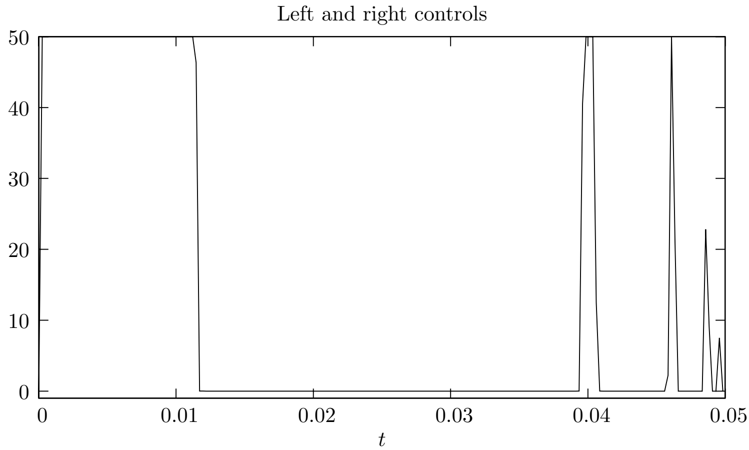


Fig. 6. $M = 50$: Time evolution of the controls while steering the system from the initial value (= 5) to the final one (= 1), with the constraint $|u_i(t)| \leq M$ (left and right Dirichlet controls are equal). The minimal computed time is $T \simeq 0.0498$. See Figs. 7 and 8 for the associated state trajectory.

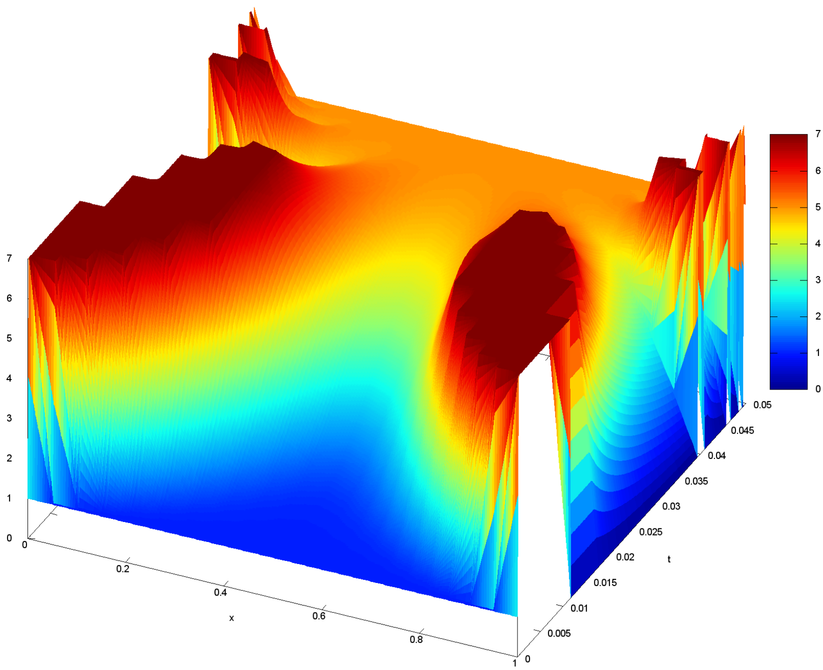
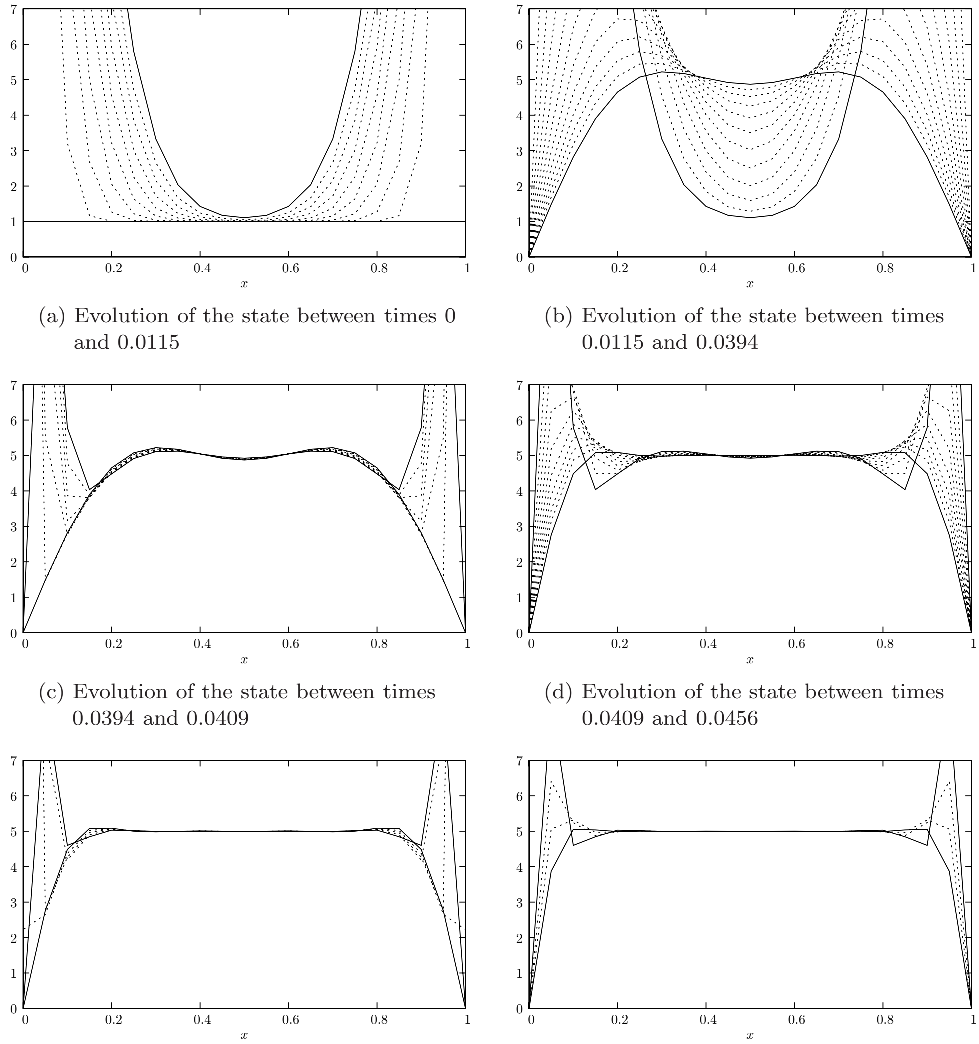


Fig. 7. (Color online) $M = 50$: Evolution of the state from the initial value (= 5) to the final one (= 1) in the minimal computed time $T \simeq 0.0498$. See Fig. 6 for the associated controls.

Using Fourier expansion, the state trajectory on this time interval can be computed and at the final time of this arc, $t_1 \simeq 0.0115$, the state is given by

$$y(t_1, x) = M + 4(\mathrm{y}^0 - M) \sum_{p=0}^{+\infty} \frac{e^{-(2p+1)^2\pi^2 t_1}}{(2p+1)\pi} \sin((2p+1)\pi x).$$

- Then the controls switch to $u_0(t) = u_1(t) = 0$ over a quite long subinterval (compared to the total length of the control horizon $[0, T]$) and then the



(a) Evolution of the state between times 0 and 0.0115

(b) Evolution of the state between times 0.0115 and 0.0394

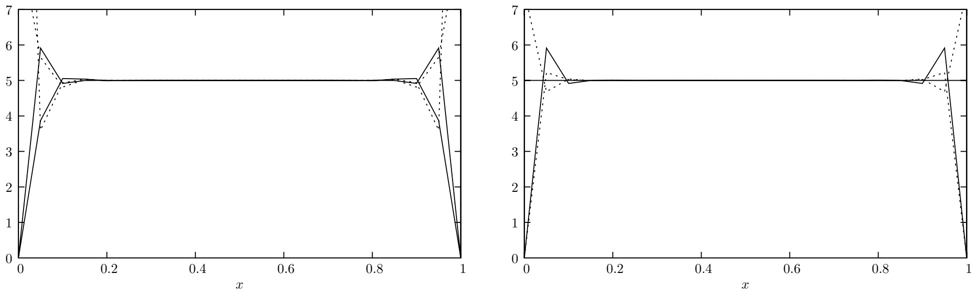
(c) Evolution of the state between times 0.0394 and 0.0409

(d) Evolution of the state between times 0.0409 and 0.0456

(e) Evolution of the state between times 0.0456 and 0.0466

(f) Evolution of the state between times 0.0466 and 0.0483

Fig. 8. $M = 50$: Evolution of the state from the initial value (= 5) to the final one (= 1) in the minimal computed time $T \simeq 0.0498$. See Fig. 6 for the associated controls and Fig. 7 for the continuous time evolution of the state.



(g) Evolution of the state between times 0.0483 and 0.0491

(h) Evolution of the state between times 0.0491 and the final time, 0.0498

Fig. 8. (*Continued*)

solution vanishes at the two boundaries, looking as an inverted double potential (see the corresponding state trajectory in Fig. 8(b)). During this time interval, the solution can also be computed, using Fourier expansion and is given by

$$y(t,x) = 4M \sum_{p=0}^{+\infty} \frac{e^{-(2p+1)^2\pi^2(t-t_1)}}{(2p+1)\pi} \sin((2p+1)\pi x)$$

$$+ 4(\mathrm{y}^0 - M) \sum_{p=0}^{+\infty} \frac{e^{-(2p+1)^2\pi^2 t}}{(2p+1)\pi} \sin((2p+1)\pi x).$$

Approximating this solution with the first three Fourier modes,

$$y(t,x) \simeq \left(M + (\mathrm{y}^0 - M)e^{-\pi^2 t_1}\right) \frac{4e^{-\pi^2(t-t_1)} \sin(\pi x)}{\pi}$$

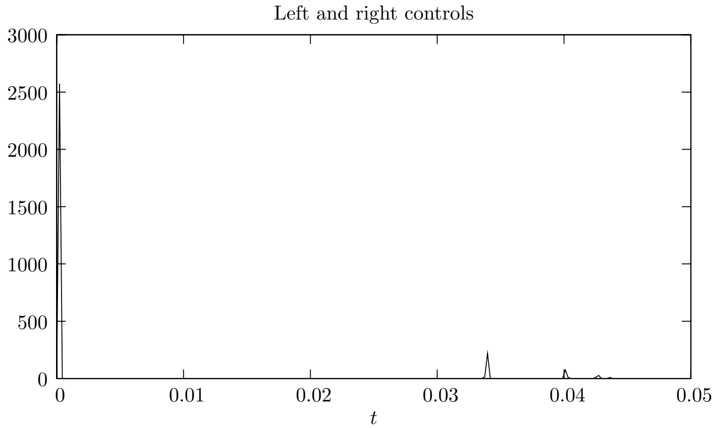
$$+ \left(M + (\mathrm{y}^0 - M)e^{-9\pi^2 t_1}\right) \frac{4e^{-9\pi^2(t-t_1)} \sin(3\pi x)}{3\pi},$$

we recover the inverted double potential shape of the solution.

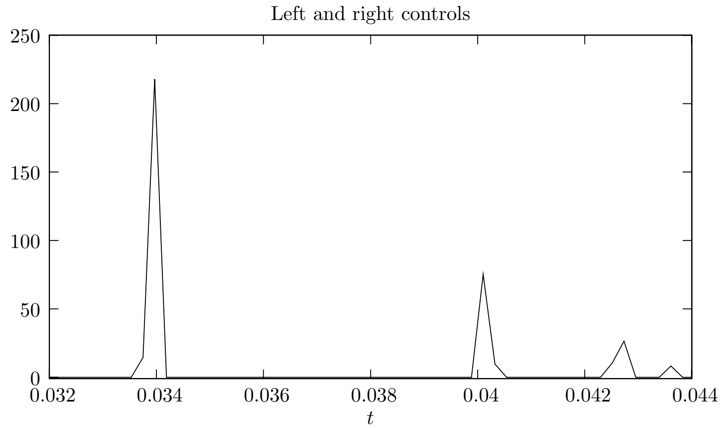
- At a later time t_1, that can be easily observed in the discrete dynamics, the solution becomes concave, less than 5 and touching 5 at $x = 1/2$. Then the controls switch and take positive values again, to adjust the solution to the final target $\mathrm{y}^1 = 5$ (see Figs. 8(c)–8(h)).

Consider now Fig. 10. It corresponds to the limit situation $M = +\infty$. The previous first bang arc $u_0 = u_1 = 50$ now becomes a very short arc along which the controls take a very large value, which is compatible with an approximation of a Dirac impulse. This is in accordance with our theoretical result, predicting that controls could belong to the set of Radon measures, and develop some singularities.

It is likely that the seemingly short impulse that one can see for the controls at $t \simeq 0.037$ is actually a Dirac mass, but the discretization is not fine enough to reproduce this fact faithfully.



(a) Global view

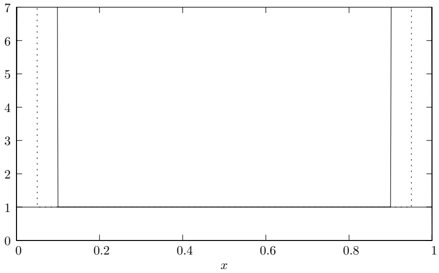


(b) Zoomed view on the tail

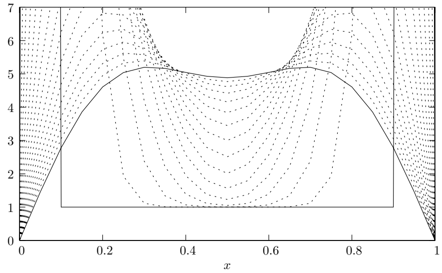
Fig. 9. $M = +\infty$: Time evolution of the controls while steering the system from the initial value (= 5) to the final one (= 1) in the minimal computed time $T \simeq 0.0438$ (left and right Dirichlet controls are equal). See Figs. 10 and 11 for the associated state trajectory.

Turnpike and/or sparsity structure? The structure of the controls observed in these simulations reveal a *turnpike* and *sparse* structure, meaning that the optimal trajectory, defined on $[0, T]$ approximately consists of three parts:

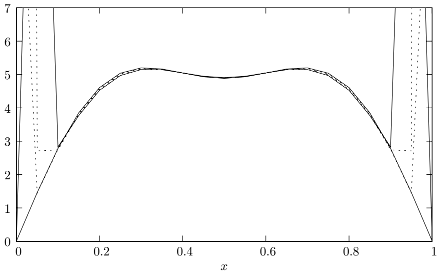
- A first short-time arc, on $[0, \varepsilon]$ with $\varepsilon \ll T$, along which the control possibly takes maximal values.
- A middle arc, on $[\varepsilon, T - \varepsilon]$, where the control remains in a steady configuration (the null one).
- A final short-time arc, on $[T - \varepsilon, T]$, along which the control possibly takes maximal values.



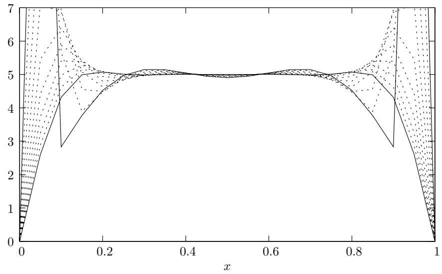
(a) Evolution of the state between times 0 and 0.0004



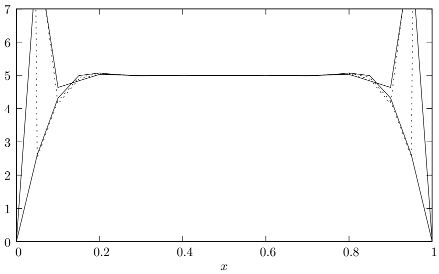
(b) Evolution of the state between times 0.0004 and 0.0335



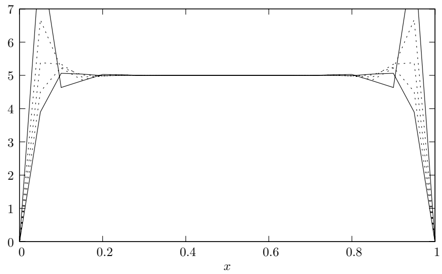
(c) Evolution of the state between times 0.0335 and 0.0342



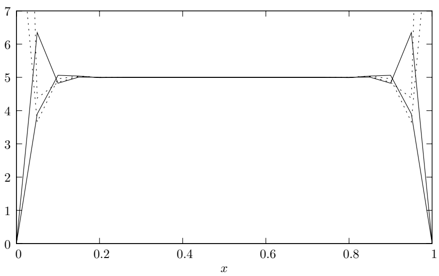
(d) Evolution of the state between times 0.0342 and 0.0399



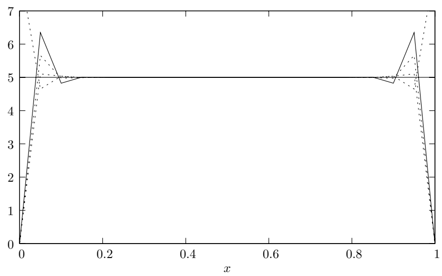
(e) Evolution of the state between times 0.0399 and 0.0405



(f) Evolution of the state between times 0.0405 and 0.0423



(g) Evolution of the state between times 0.0423 and 0.0430



(h) Evolution of the state between times 0.0430 and the final time, 0.0438

Fig. 10. $M = +\infty$: Evolution of the controls and of the state from the initial value (= 5) to the final one (= 1) in the minimal computed time $T \simeq 0.0438$. See Fig. 9 for the associated controls and Fig. 11 for the continuous time evolution of the state.

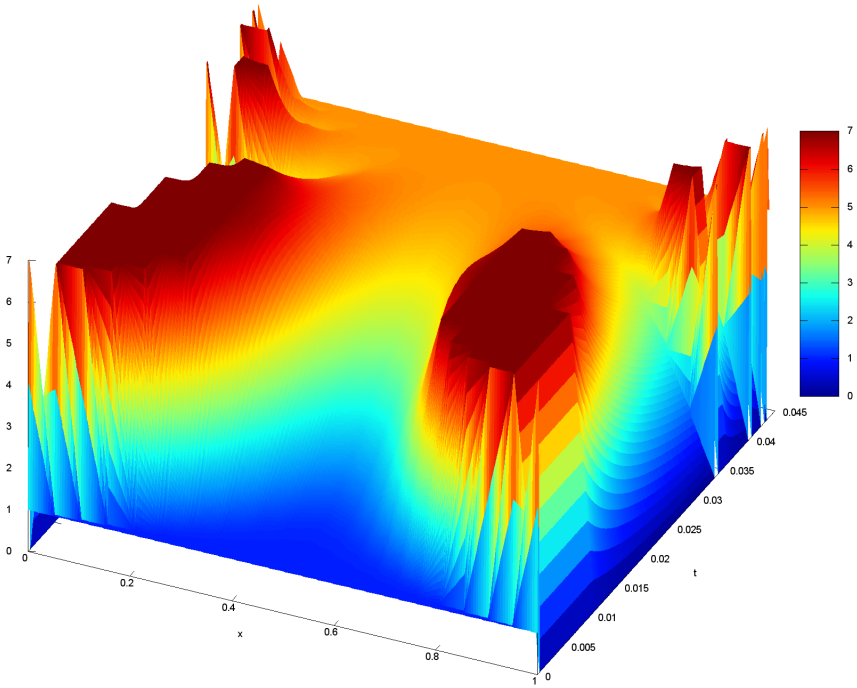


Fig. 11. (Color online) $M = +\infty$: Evolution of the controls and of the state from the initial value $(= 5)$ to the final one $(= 1)$ in the minimal computed time $T \simeq 0.0438$. See Fig. 9 for the associated controls.

The turnpike phenomenon was analyzed in Ref. 23 for general nonlinear optimal control problems in finite dimension, showing that, when one steers a control system from one initial configuration to a final configuration in fixed time T, then, for T large, the optimal trajectory enjoys the above qualitative behavior, and more precisely, the long middle arc corresponds to an *optimal steady state*, i.e. the optimal solution of an associated static optimal control problem. It has even been proved in this reference that, except at the beginning and at the end of the interval, if T is large enough, the solution, the control, and the adjoint vector (coming from the application of the Pontryagin maximum principle) are exponentially close to some constant values, corresponding to the solution of the static version of the optimal control problem. We have recently extended this analysis in Ref. 22 to infinite dimension, involving in particular the case of heat equations (see also Ref. 10 for a specific analysis on the wave equation).

Here, however, the context is slightly different because we minimize the time, and therefore the final time T is not expected to be large as in the above-mentioned references. Still, optimal solutions and control seem to develop a similar behavior.

The optimal controls present long lags where they take null values, saturating the constraint. Thus, their structure is of sparse nature, which is due to the fact that the control time is minimized. In recent years the existence of sparse controls has been derived for a variety of parabolic control problems but, normally, under the constraint that they are of minimal norm in the space of measures, a fact that enhances their concentration into Dirac deltas (see Ref. 4). Here, however, the sparsity seems to be due to the fact that controls are constrained by the non-negativity condition and the time is minimal. This issue requires further analysis and understanding.

5.2. *State constraints for the Neumann control problem*

Let us now consider the minimal time control problem for the 1D heat equation on $(0,1)$ with Neumann boundary controls, under a non-negativity constraint on the state $y(t,x) \geq 0$. Note that in this case we expect the states to be those that we obtain by imposing non-negativity constraints on the Dirichlet controls. Accordingly, the Neumann controls in the minimal time are expected to be the normal traces of the Dirichlet-controlled trajectories. Our numerical experiments hereafter confirm this fact.

We choose $\mathrm{y}^0 \equiv 5$ and $\mathrm{y}^1 \equiv 1$ and the following discretization:

$$\frac{Y_{i+1,j} - Y_{i,j}}{\Delta t} = \frac{Y_{i,j+1} - 2Y_{i,j} + Y_{i,j-1}}{(\Delta x)^2}, \qquad j = 1, \ldots, N_x - 1, \ i = 0, \ldots, N_t - 1,$$

$$\frac{Y_{i,1} - Y_{i,0}}{\Delta x} = V_i^0, \quad \frac{Y_{i,N_x} - Y_{i,N_x-1}}{\Delta x} = V_i^1, \quad i = 0, \ldots, N_t,$$

$$Y_{i,j} \geq 0, \qquad j = 0, \ldots, N_x, \ i = 0, \ldots, N_t,$$

$$Y_{0,j} = 5, \quad Y_{N_t,j} = 1, \qquad j = 0, \ldots, N_x.$$

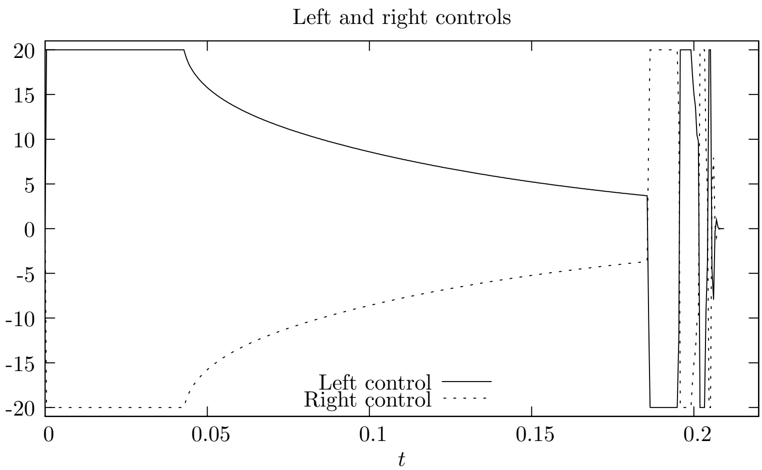


Fig. 12. $M = 20$: Time evolution of the Neumann boundary controls while steering the system from the initial value (= 5) to the final one (= 1) with the control constraint $|v_i(t)| \leq M$. The minimal computed time is $T \simeq 0.2093$. See Fig. 14 for the associated state trajectory.

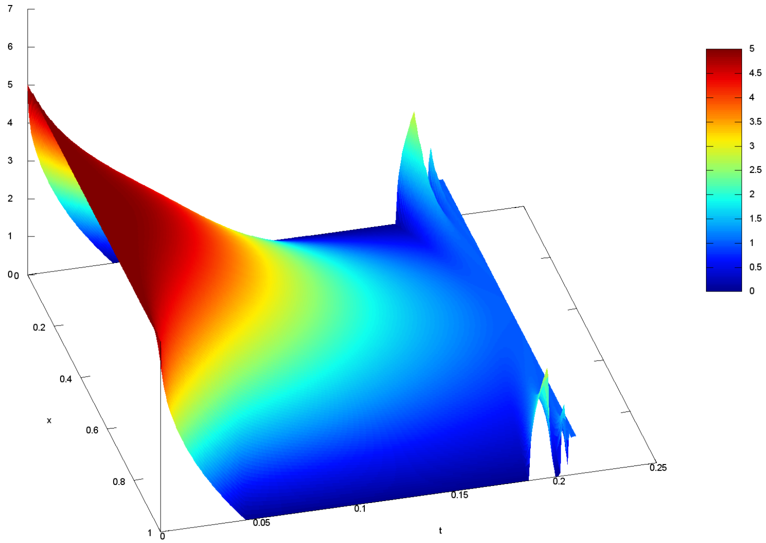


Fig. 13. (Color online) $M = 20$: Evolution of the state from the initial value (= 5) to the final one (= 1) in the minimal computed time $T \simeq 0.2093$. See Fig. 12 for the associated controls.

As before we add an upper bound on the controls, $|v_0(t)| \leq M$ and $|v_1(t)| \leq M$, with $M > 0$ to be chosen large.

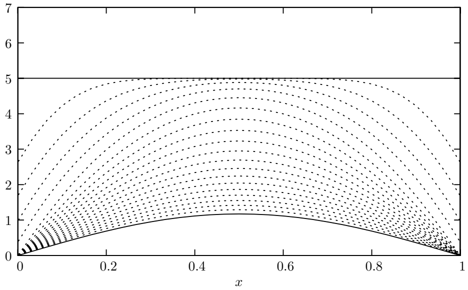
In Figs. 13 and 14, we report the numerical results obtained for $M = 20$. The minimal time that is obtained is then $T \simeq 0.2087$.

The optimal strategy starts with a bang arc with $v_0(t) = 20$ and $v_1(t) = -20$. Once more, we observe the odd symmetry $v_0 = -v_1$ of the controls (see Fig. 12). Along this arc, starting from $\mathrm{y}^0 \equiv 5$, the solution is concave, symmetric with respect to $x = 1/2$, and takes boundary values at $x = 0$ and $x = 1$, that are positive and decrease in time (see Fig. 14(a)).

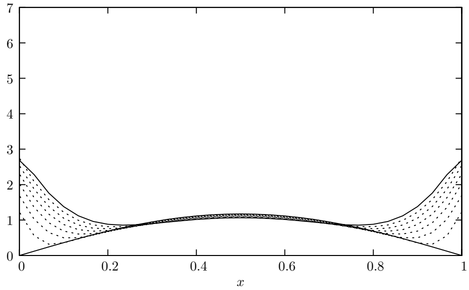
When these values reach 0, the controls switch and then evolve continuously, not saturating the control constraints, while the boundary values of the solution remain at 0, i.e. $y(t, 0) = y(t, 1) = 0$. This behavior is similar to the one of *singular arcs* or of *boundary arcs*. Along this arc, the solution remains concave, vanishes at both boundaries $x = 0$ and $x = 1$, and its maximum decreases (see Fig. 14(a)). This is so, until its maximum reaches the target value 1.

Then the controls switch again, and oscillate much (they switch between +20 and −20) before the solution finally reaches the target $\mathrm{y}^1 \equiv 1$ (see Figs. 14(b)–14(d)). A chattering phenomenon seems to occur just before the final time, and the controls switch very rapidly over a compact time interval. This could be a manifestation of the fact that for the continuous problem controls switch infinitely many times.

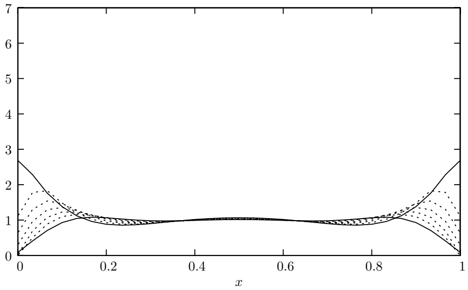
In Figs. 16 and 17, we report what happens for large values of M. The minimal time that is obtained is $T \simeq 0.1938$. This situation corresponds to the limit of the previous one as $M \to +\infty$. We observe that the first arc converges to a Dirac mass,



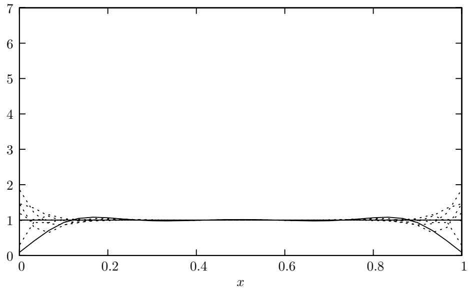
(a) Evolution of the state between times 0 and 0.1856



(b) Evolution of the state between times 0.1856 and 0.1954



(c) Evolution of the state between times 0.1954 and 0.2014



(d) Evolution of the state between times 0.2014 and the final time, 0.2093

Fig. 14. $M = 20$: Evolution of the state from the initial value (= 5) to the final one (= 1) in the minimal computed time $T \simeq 0.2093$. See Fig. 12 for the associated controls and Fig. 13 for the continuous time evolution of the state.

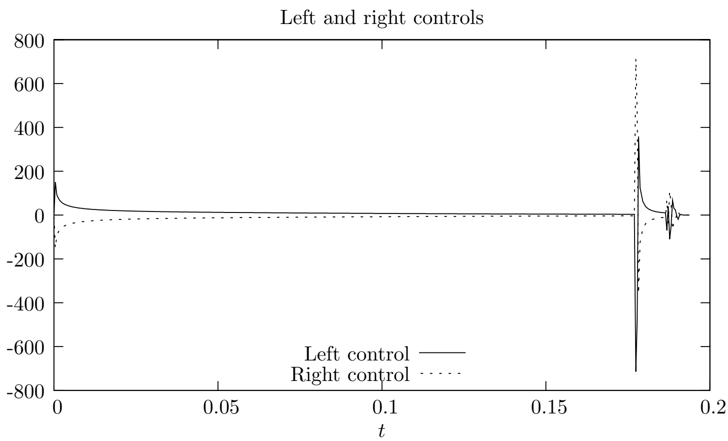


Fig. 15. Large value of M: Time evolution of the controls while steering the system from the initial value (= 5) to the final one (= 1). The minimal computed time is $T \simeq 0.1938$. See Fig. 17 for the associated state trajectory.

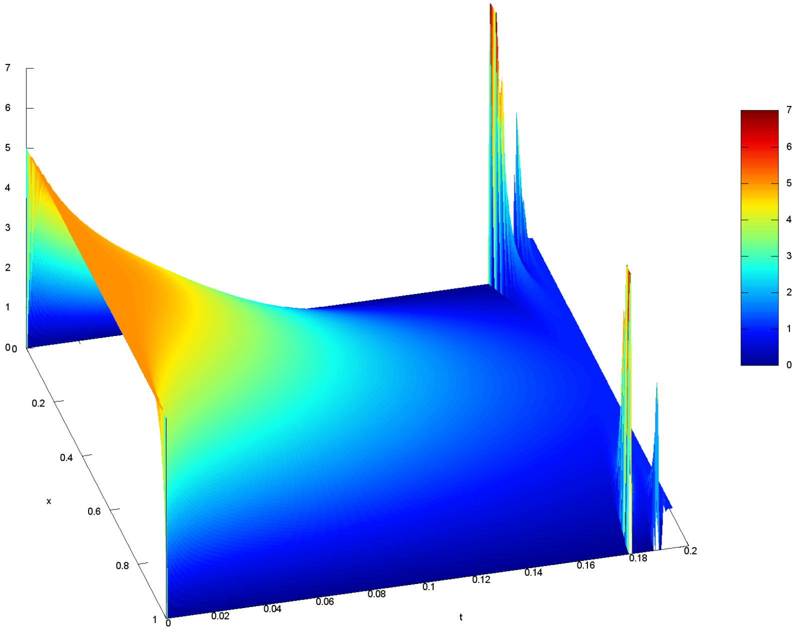


Fig. 16. (Color online) Large value of M: Evolution of the controls and of the state from the initial value (= 5) to the final one (= 1) in the minimal computed time $T \simeq 0.1938$. See Fig. 15 for the associated controls.

and oscillations between -20 and 20 at the end of the interval are replaced by oscillations of negative and positive Dirac impulses.

We observe that the minimal time obtained and the state trajectory plot in Fig. 17 are in accordance with the one obtained for the Dirichlet control case (see Fig. 4). As explained in Sec. 4.2, this fact was expected.

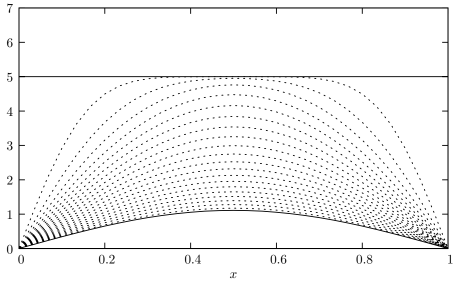
We can still observe the middle "singular" arc. As before, this is also a kind of turnpike and sparsity phenomenon, but this time, of a different nature, because the long middle arc is a kind of singular arc.

5.3. *Convergence result for the minimal time*

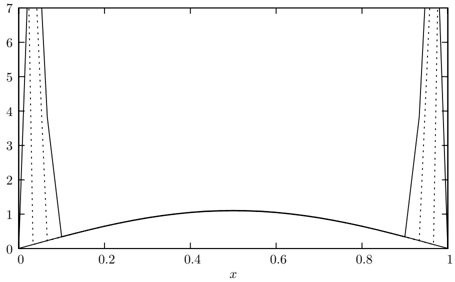
The numerical simulations that we have developed here exhibit the expected phenomena for the constrained control problems under consideration. The validity of these computational results can be confirmed by a convergence result showing that the minimal time of control for the discrete problem converges toward the discrete one as the mesh sizes tend to 0.

Here we briefly sketch the main ideas of a possible proof. Completing the details would require further analytical work combining the ingredients developed in the existing literature.

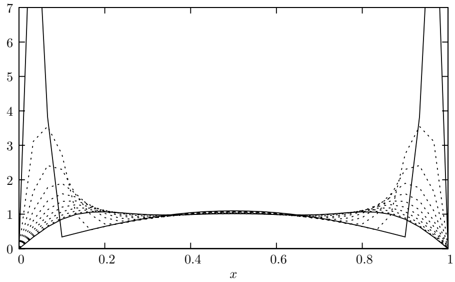
We denote by T the minimal time of control under constraint for fixed y^0 and y^1, as above, for the continuous heat equation. We denote similarly by T_N the



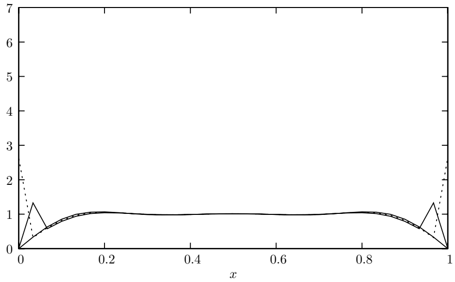
(a) Evolution of the state between times 0 and 0.1764



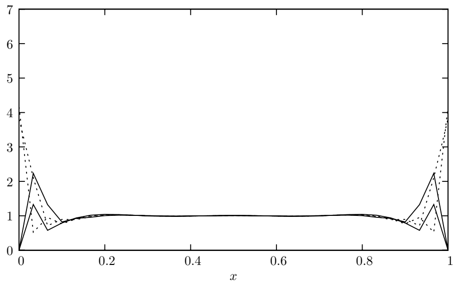
(b) Evolution of the state between times 0.1770 and 0.1783



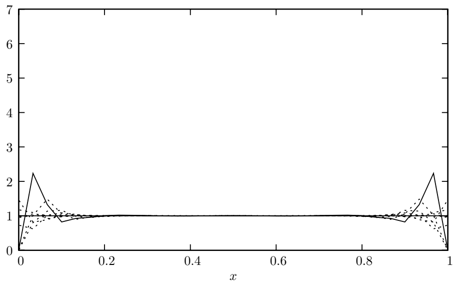
(c) Evolution of the state between times 0.1783 and 0.1864



(d) Evolution of the state between times 0.1864 and 0.1873



(e) Evolution of the state between times 0.1873 and 0.1886



(f) Evolution of the state between times 0.1886 and the final time, 0.1938

Fig. 17. Large value of M: Evolution of the controls and of the state from the initial value ($= 5$) to the final one ($= 1$) in the minimal computed time $T \simeq 0.1938$. See Fig. 15 for the associated controls and Fig. 16 for the continuous time evolution of the state.

minimal time of control for the discrete problem corresponding to the mesh-size parameters N_x and N_t satisfying the CFL condition for stability. We are interested in the convergence of T_N toward T, as N_x and N_t tend to infinity, which we write "$N \to +\infty$" in short. We proceed in two steps.

Step 1: $T \geq \limsup_{N\to\infty} T_N$. We argue by contradiction. We denote by u_0 and u_1 the optimal controls for the continuous heat equation (under state constraints) in minimal time T. We then plug a suitable time discretization of these controls into the discrete system. Using convergence results of the solutions of the discrete equation toward the continuous one (using the weak solutions in the sense of transposition as described above), we deduce that the corresponding discrete solutions, other than satisfying the state constraints, satisfy the convergence property $y_N(T) \to y(T)$ in $H^{-s}(0,1)$ for $s > 3/2$ as $N \to +\infty$. This means that the discretizations of the continuous controls are quasi-controls for the discrete models. Now, since these discrete systems are uniformly controllable (see Refs. 3, 11 and 27 for the analysis of uniform controllability of discrete versions of the 1D heat equation) and that the gap of the discrete final states $y_N(T)$ that we reach to the target tends to 0, in any time interval of length δ with $\delta > 0$ arbitrarily small, the discrete systems can be driven from $t = T$ to the target in time $t = T + \delta$ by means of controls whose size tends to 0 and, therefore, so that the state preserves the non-negativity constraint. This means that $T_N \leq T + \delta$ for all $\delta > 0$ and N large enough.

Step 2: $T \leq \liminf_{N\to\infty} T_N$. Let us denote by $T_- = \liminf_{N\to\infty} T_N$ and by $u_{0,N}$ and $u_{1,N}$ the corresponding discrete controls in the optimal times T_N. In view of the uniform controllability properties of these discrete models the controls should be bounded in the space of measures as $N \to +\infty$ (this specific aspect would require a finer study). Extracting subsequences, and passing weakly to the limit in the sense of measures, this would lead to limit controls u_0 and u_1 so that the corresponding solution of the continuous heat equation fulfills the non-negativity constraint and reaches the target in time T_-. In case $T_- < T$, this would contradict the definition of T being the minimal control time for the continuous model.

Remark 5.1. We have developed the numerical experiments in the 1D setting. Similar questions could be formulated in the multi-dimensional one. In that case the theory of uniform (with respect to the mesh size) controllability of numerical approximation schemes of the heat equation is much more complex. This is mainly due to the fact that Carleman inequalities are the main tool to prove observability inequalities for the continuous heat equation. The extension of these inequalities to the discrete setting yields added high-frequency remainder terms that require high-frequency filtering to be used to ensure the convergence of numerical controls. In other words, the controllability property of the discrete dynamics has to be relaxed to deal only with the control of a low-dimensional projection (see Ref. 2). A systematic numerical investigation of the constrained control problem in the multi-dimensional case remains to be done.

6. Conclusion, Open Problems and Perspectives

In this paper, we have addressed the problem of investigating controllability to steady states for heat equations under non-negativity state or control constraints. We have shown that, although controllability can be realized in arbitrarily small time when one does not take into account any constraint, such unilateral state or control constraints create a positive minimal time for realizing controllability.

Given an initial datum y^0 and a final steady state target y^1, we have proved that $\underline{T}(\mathrm{y}^0, \mathrm{y}^1) > 0$ (positive minimal time) in the following cases:

- 1D heat equation with Dirichlet boundary controls, under non-negativity state or control constraints (both are equivalent); controllability with controls that are Radon measures at time $T = \underline{T}(\mathrm{y}^0, \mathrm{y}^1)$, lower estimates for $\underline{T}(\mathrm{y}^0, \mathrm{y}^1)$.
- 1D heat equation with Neumann controls, under non-negativity state constraint; lower estimate for $\underline{T}(\mathrm{y}^0, \mathrm{y}^1)$.
- Multi-dimensional heat equation with Dirichlet boundary controls along the whole boundary, under non-negativity state or control constraints (both are equivalent); controllability exactly in time $\underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ with Radon measure controls; lower estimate for $\underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ when the domain is a ball.
- Multi-dimensional heat equation with Neumann boundary controls along the whole boundary, under non-negativity state constraints; in contrast, under non-negativity control constraints we have $\underline{T}(\mathrm{y}^0, \mathrm{y}^1) = +\infty$ (i.e. controllability fails: we have established it in 1D).
- Multi-dimensional heat equation with internal control and/or Neumann boundary controls, under non-negativity state constraints (existence of a Radon measure control in time $\underline{T}(\mathrm{y}^0, \mathrm{y}^1)$ is open).

The techniques presented in this paper can certainly be extended to some other related problems, among which:

- 1D parabolic equations with variable coefficients of the form $\partial_t y = \partial_x(a(x)\partial_x y) - p(x)\partial_x y$, with internal and/or boundary control (provided that the internal control is not acting everywhere). The results that we have established for the constant coefficient 1D heat equation can very likely be extended to this more general situation, by using spectral expansions and asymptotic properties of the spectrum of Sturm–Liouville problems, under suitable regularity assumptions on the coefficients.
- Finite-dimensional systems allowing for similar comparison properties and spectral expansions. More generally, in Ref. 15 we develop some specific finite-dimensional methods, exploiting the Brunovsky canonical form of controllable systems, allowing us to deal with various constraints on the controlled state.

We present hereafter a non-exhaustive list of open problems and perspectives.

Uniqueness of controls at the minimal time. As observed above, if $\mathrm{y}^0 \in L^2(0,1)$ is symmetric with respect to $1/2$, then we can take $u_0 = u_1$ (at least, if T is large enough for them to exist). Besides, we have observed that the controls that we get with numerical simulations always coincide. It is thus natural to address the question of uniqueness of controls at the minimal time.

Let $\underline{T}$ be the minimal control time and let $\underline{u}_0$ and $\underline{u}_1 \in \mathcal{M}(0,\underline{T})$ be non-negative controls. Setting $\underline{u} = (\underline{u}_0 + \underline{u}_1)/2$ and $\underline{h} = (\underline{u}_0 - \underline{u}_1)/2$, we have, using (2.5),

$$\frac{\mathrm{y}^1}{(2p+1)^2\pi^2} - \frac{e^{-(2p+1)^2\pi^2\underline{T}}\mathrm{y}^0_{2p+1}}{2(2p+1)\pi} = \int_{[0,\underline{T}]} e^{-(2p+1)^2\pi^2(\underline{T}-t)}\mathrm{d}\underline{u}(t)$$

$$\text{and} \quad \int_{[0,\underline{T}]} e^{(2p)^2\pi^2 t}\mathrm{d}\underline{h}(t) = 0 \quad (p \in \mathbb{N}). \tag{6.1}$$

These two equations characterize all possible controls $\underline{u}_0 = \underline{u} \pm \underline{h}$ and $\underline{u}_1 = \underline{u} \mp \underline{h} \in \mathcal{M}(0,\underline{T})$ in time $\underline{T}$.

In order to prove that there is a unique pair of controls in minimal time, both components being the same, one needs to show that if $\underline{u}$ and $\underline{h}$ solve (6.1) and $\underline{u} + \underline{h}$ and $\underline{u} - \underline{h}$ are non-negative, then $\underline{h} = 0$.

Obviously, if $\underline{h} \neq 0$, then it has to change sign. On the other hand, if $\underline{u}$ and $\underline{h}$ are solutions of (6.1) and $\underline{u} + \underline{h}$ and $\underline{u} - \underline{h}$ are non-negative, then we must have $\operatorname{supp}\underline{h} \subset \operatorname{supp}\underline{u}$.

With these observations in mind, nonuniqueness of the pair of controls might be expected to be true. Indeed, let $\underline{u}$ be non-negative and satisfy the moment equations above. Let $[\tau_1, \tau_2]$ be a subinterval along which $\underline{u} \geq \delta > 0$ for some $\delta > 0$. Now, any function $\underline{h}$, supported on $[\tau_1, \tau_2]$ and satisfying the orthogonality conditions in (6.1), must change its sign as mentioned above. Such a function exists because the set of real exponentials involved in these moment equations is not complete in $L^2(\tau_1, \tau_2)$. Furthermore, $\underline{h}$ can be taken arbitrarily small in $L^\infty(\tau_1, \tau_2)$-norm. In these conditions, the new pair $(\underline{u} + \underline{h}, \underline{u} - \underline{h})$ satisfies (6.1) and is non-negative. This would complete the argument for proving nonuniqueness, but to make it completely rigorous, we would need to show the existence of some subinterval $[\tau_1, \tau_2]$ along which $\underline{u} \geq \delta > 0$. The argument above would fail if the interior of the support of the control $\underline{u}$ was empty. But we did not find computational evidence of this possible lack of uniqueness.

This issue would require a finer study of the optimality system characterizing the optimal controls at the minimal time.

Regularity of controls at the minimal time. Our analysis shows that the controls in minimal time can be guaranteed to be non-negative Radon measures. Whether they actually belong to some smaller space (such as L^1) or really contain some nontrivial singular components is an open problem. This issue is related with the problem above on uniqueness of controls and their symmetry. Indeed, if controls were equal to Dirac deltas concentrated on some strategic time instances, then the argument above, which uses the fact that controls remain bounded away from zero on some open subintervals, would fail.

Turnpike structure. As mentioned above, the optimal pair that we obtain in the numerical experiments seems to present a turnpike and sparse structure with long lags along which the controls are identically equal to 0. A complete understanding of these properties for the continuous problem requires a finer analysis of the optimality system (Pontryagin maximum principle).

Convergence result for the minimal time. A complete analysis of the convergence of the minimal time for the discrete models toward the continuous one requires significant further work.

Sharpness of lower bounds. The numerical experiments indicate that the lower bounds that we have given for the minimal time are not sharp. Improving analytically these bounds is an open problem.

Numerical approximation in multi-dimensional setting. A systematic analysis of the numerical approximation issues of constrained control in the multi-dimensional setting is to be developed.

Multi-dimensional heat equations with variable coefficients. The arguments we have used to deal with the multi-dimensional heat equation rely on the use of the radially symmetric eigenfunctions in the case where Ω is a ball, and on the use of comparison arguments. This method cannot be extended to general multi-dimensional heat equations with variable coefficients which would require a separate treatment, certainly with different methods.

Nonlinear heat equations. Finally, similar questions arise naturally for nonlinear heat equations, and in particular semilinear ones, $\partial_t y = \Delta y + f(y)$. The arguments developed in this paper are based on spectral expansions and therefore are of a linear nature. Addressing nonlinearities is therefore a completely open challenge.

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