Research Article

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Addendum: Local Elliptic Regularity for the Dirichlet Fractional Laplacian

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Abstract: In [1], for $1 < p < \infty$, we proved the $W^{2s,p}_{\mathrm{loc}}$ local elliptic regularity of weak solutions to the Dirichlet problem associated with the fractional Laplacian $(-\Delta)^s$ on an arbitrary bounded open set of $\mathbb{R}^N$. Here we make a more precise and rigorous statement. In fact, for $1 < p < 2$ and $s \neq \frac{1}{2}$, local regularity does not hold in the Sobolev space $W^{2s,p}_{\mathrm{loc}}$, but rather in the larger Besov space $(B^{2s}_{p,2})_{\mathrm{loc}}$.

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MSC 2010: 35B65, 35R11, 35S05

In [1], we have analyzed the local regularity for the solutions to the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an arbitrary bounded open set and the fractional Laplace operator $(-\Delta)^s$ is defined for $s \in (0, 1)$ as the singular integral

$$(-\Delta)^s u(x) := C_{N,s}\, \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy,$$

where $C_{N,s}$ is a normalization constant.

In [1, Theorem 1.4], we stated and proved the following maximal local regularity result for the weak solutions to (1): if $f \in L^p(\Omega)$, $1 < p < \infty$, the corresponding weak solution to system (1) satisfies $u \in W^{2s,p}_{\mathrm{loc}}(\Omega)$.

However, although this is true for $p \geq 2$, when $1 < p < 2$ the result is correct only for $s = \frac{1}{2}$. When $1 < p < 2$ and $s \neq \frac{1}{2}$, instead, u belongs to the Besov space $(B^{2s}_{p,2})_{\mathrm{loc}}(\Omega)$, which is strictly larger than $W^{2s,p}_{\mathrm{loc}}(\Omega)$.

This is due to the fact that in the proof [1, Theorem 1.4], the $W^{2s,p}_{\mathrm{loc}}$ combines a cut-off argument and global results for the fractional Poisson-type equation

$$(-\Delta)^s u = F \quad \text{in } \mathbb{R}^N. \tag{2}$$

When $F \in L^p(\mathbb{R}^N)$, the solution u belongs to $W^{2s,p}(\mathbb{R}^N)$ for $p \geq 2$ and for $1 < p < 2$, $s = \frac{1}{2}$. But not when $1 < p < 2$ and $s \neq \frac{1}{2}$. However [1, Theorem 2.7] does not make this distinction stating $W^{2s,p}(\mathbb{R}^N)$ regularity for

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all $1 < p < \infty$ and all $0 < s < 1$. When $1 < p < 2$ and $s \neq \frac{1}{2}$, we have regularity in the Besov space $B^{2s}_{p,2}(\mathbb{R}^N)$, which is strictly larger than $W^{2s,p}(\mathbb{R}^N)$. This has been also mentioned in [4, Remark 7.1].

The correct statement of [1, Theorem 2.7] should be the following.

Theorem 1. *Let $1 < p < \infty$. Given $F \in L^p(\mathbb{R}^N)$, let u be the unique weak solution to the fractional Poisson-type equation*

$$(-\Delta)^s u = F \quad \text{in } \mathbb{R}^N.$$

Then $u \in \mathscr{L}^p_{2s}(\mathbb{R}^N)$, where

$$\mathscr{L}^p_{2s}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : (-\Delta)^s u \in L^p(\mathbb{R}^N)\} \tag{3}$$

has been introduced for example in [5, Chapter V, Section 3.3, formula (38)].

As a consequence, we have the following:

(i) *If $1 < p < 2$ and $s \neq \frac{1}{2}$, then $u \in B^{2s}_{p,2}(\mathbb{R}^N)$.*
(ii) *If $1 < p < 2$ and $s = \frac{1}{2}$, then $u \in W^{2s,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$.*
(iii) *If $2 \leq p < \infty$, then $u \in W^{2s,p}(\mathbb{R}^N)$.*

Remark 2. The space $\mathscr{L}^p_{2s}(\mathbb{R}^N)$, which in the literature is called potential space, is sometimes denoted as $H^s_p(\mathbb{R}^N)$ (see, e.g., [7, Section 1.3.2]).

Accordingly, [1, Theorem 1.4] should be rephrased as follows.

Theorem 3. *Let $1 < p < \infty$. Given $f \in L^p(\Omega)$, let u be the unique weak solution to the Dirichlet problem* (1). *Then $u \in (\mathscr{L}^p_{2s})_{\text{loc}}(\Omega)$. As a consequence, we have the following result:*

(i) *If $1 < p < 2$ and $s \neq \frac{1}{2}$, then*
$$u \in (B^{2s}_{p,2})_{\text{loc}}(\Omega).$$
(ii) *If $1 < p < 2$ and $s = \frac{1}{2}$, then*
$$u \in W^{2s,p}_{\text{loc}}(\Omega) = W^{1,p}_{\text{loc}}(\Omega).$$
(iii) *If $2 \leq p < \infty$, then*
$$u \in W^{2s,p}_{\text{loc}}(\Omega).$$

We provide below the explanation of these facts:

- According to [5, Chapter V, Section 5.3, Theorem 5 (B)], when $1 < p < 2$, the space $\mathscr{L}^p_{2s}(\mathbb{R}^N)$ introduced in (3) is included in the Besov space $B^{p,2}_{2s}(\mathbb{R}^N)$. Moreover, an explicit counterexample showing that sharper inclusions are not possible has been given in [5, Chapter V, Section 6.8]. But if $2s$ is an integer, that is, if $s = \frac{1}{2}$, then
$$\mathscr{L}^p_{2s}(\mathbb{R}^N) = \mathscr{L}^p_1(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$$
(see [5, Chapter V, Section 3.3, Theorem 3]). In view of these observations, we have that if $F \in L^p(\mathbb{R}^N)$, $1 < p < 2$, then the solution u to (2) belongs to the Besov space $B^{p,2}_{2s}(\mathbb{R}^N)$ if $s \neq \frac{1}{2}$, and belongs to $W^{1,p}(\mathbb{R}^N)$ if $s = \frac{1}{2}$.
- Recall also that the Sobolev space $W^{2s,p}(\mathbb{R}^N)$ is, by definition, the space $B^{p,p}_{2s}(\mathbb{R}^N)$ (see, e.g., [5, Chapter V, Section 5.1, formula (60)]) and that the latter is strictly included in
$$B^{p,2}_{2s}(\mathbb{R}^N) \quad \text{for } 1 < p < 2.$$
Hence, the counterexample mentioned above implies that the space of functions whose fractional Laplacian is in $L^p(\mathbb{R}^N)$ is strictly larger than $B^{p,p}_{2s}(\mathbb{R}^N)$ (which is by definition $W^{2s,p}(\mathbb{R}^N)$) when $1 < p < 2$ and $s \neq \frac{1}{2}$. In other words, when $1 < p < 2$ and $s \neq \frac{1}{2}$, there are functions whose fractional Laplacian belongs to $L^p(\mathbb{R}^N)$, but they do not belong to $W^{2s,p}(\mathbb{R}^N)$.

These arguments provide a proof of Theorem 1. Instead, the proof of Theorem 3 presented in [1] is correct. Indeed, it is based on a cut-off argument which is not affected by the discussion above.

During the revision process of our original manuscript, we became aware that similar results were obtained using pseudo-differential calculus (see, e.g., [3, Section 7] or [6, Chapter XI, Proposition 2.4, Theorem 2.5 and Exercise 2.1]). We already pointed out this fact in [1], saying that, in [3, Section 7], Grubb proved that, under the restriction $s > \frac{N}{p}$, the assumption $f \in W^{\tau,p}(\Omega)$ for some real number $\tau \geq 0$ implies that the corresponding solution u of (1) belongs to

$$W^{\tau+2s,p}_{\mathrm{loc}}(\Omega).$$

A more careful reading of Grubb's work and a discussion with the author made us realize that, for $1 < p < 2$, this result does not hold in the classical Sobolev setting, but rather in $(\mathscr{L}^p_{\tau+2s})_{\mathrm{loc}}(\Omega)$. Also, the restriction $s > \frac{N}{p}$ mentioned above is not necessary and this regularity is true for all $1 < p < \infty$.

Our approach complements the pseudo-differential one, using merely classical PDE techniques in the context of linear and nonlinear elliptic and parabolic equations.

In fact, our techniques and results extend to the following parabolic problem:

$$\begin{cases} u_t + (-\Delta)^s u = f & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u(\cdot\,, 0) = 0 & \text{in } \Omega. \end{cases} \tag{4}$$

In particular, we have the following theorem.

Theorem 4. *Let $1 < p < \infty$. Given $f \in L^p(\Omega \times (0, T))$, let u be the unique weak solution to the parabolic problem* (4)*. Then*

$$u \in L^p\big((0, T); (\mathscr{L}^p_{2s})_{\mathrm{loc}}(\Omega)\big).$$

As a consequence, we have the following result:

(i) *If $1 < p < 2$ and $s \neq \frac{1}{2}$, then*

$$u \in L^p\big((0, T); (B^{2s}_{p,2})_{\mathrm{loc}}(\Omega)\big).$$

(ii) *If $1 < p < 2$ and $s = \frac{1}{2}$, then*

$$u \in L^p\big((0, T); W^{2s,p}_{\mathrm{loc}}(\Omega)\big) = L^p\big((0, T); W^{1,p}_{\mathrm{loc}}(\Omega)\big).$$

(iii) *If $2 \leq p < \infty$, then*

$$u \in L^p\big((0, T); W^{2s,p}_{\mathrm{loc}}(\Omega)\big).$$

We refer to [2] for more details on this topic.

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