

# Controllability under constraints

**Dario Pighin**-Universidad Autónoma de Madrid  
joint work with: professor **Enrique Zuazua**

Funded ERC Advanced Grant DYCON-Dynamic Control

Benasque, August, 30<sup>th</sup>, 2017



European Research Council



## Motivations

- Controllability of PDE has been widely investigated in the past decades;
- On the other hand, on many PDE models describing biological or physical phenomena some constraints are imposed.

Our goal: obtain some **Controllability** results under **state** and/or **control** constraints.

## Existing results for the heat equation

This constrained Controllability issue for the heat equation has already been investigated in:

J. Lohéac, E. Trélat and E. Zuazua

Minimal controllability time for the heat equation under unilateral state or control constraints

*Mathematical Models and Methods in Applied Sciences*, Vol. 27  
no. 09 (2017), pp. 1587 – 1644.

## Existing results for the heat equation

### Theorem (Lohéac, Trélat and Zuazua)

Let  $y_0 \in L^2(\Omega)$  be an initial datum and  $y_1$  be a steady state.

Assume  $\text{Tr}(y_1) \upharpoonright_{\partial\Omega} \geq \nu > 0$ . Then, in **time large**, we can drive the system:

$$\begin{cases} y_t - \Delta y = 0 & \text{in } (0, T) \times \Omega \\ y = u & \text{on } (0, T) \times \partial\Omega. \end{cases}$$

from  $y_0$  to  $y_1$  by means of control  $u$  satisfying the **control constraint**:

$$u \geq 0 \quad \text{a.e. } (0, T) \times \partial\Omega.$$

If  $y_0 \geq 0$  a.e. in  $\Omega$ ,  $y$  fulfills the **state constraint**:

$$y \geq 0 \quad \text{a.e. } (0, T) \times \Omega.$$

# Outline of the talk

Intro

Finite dimensional case

The infinite dimensional linear case

Dissipative case

The infinite dimensional semilinear case

General case

Dissipative case

Simulations

## Finite dimensional systems-General case

Let  $A$  an  $N \times N$  real matrix and  $B$  an  $N \times M$  real matrix satisfying Kalman rank condition. Let  $y_1$  and  $y_2$  be solutions to the steady problem:

$$Ay_i = B\bar{u}_i$$

with  $\bar{u}_i \geq \nu > 0$ . We aim to drive the system

$$\frac{d}{dt}y + Ay = Bu$$

from  $y_0$  to  $y_1$  by means of a control  $u \in C^0([0, T]; \mathbb{R}^M)$  fulfilling the **control constraint**:

$$u \geq 0.$$

## Idea of the proof

By the Kalman's Theorem, we know that for any  $\varepsilon > 0$  we can find a corresponding  $\delta_\varepsilon > 0$  such that for any pair of initial and final data  $y_0$  and  $y_1$  such that:

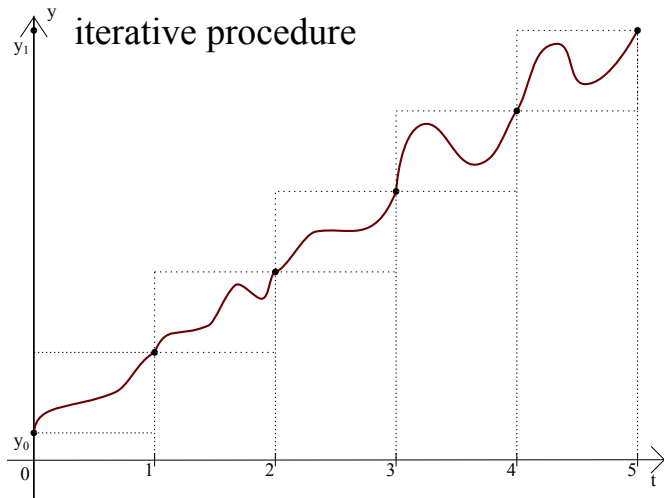
$$\|e^A y_1 - y_0\| < \delta_\varepsilon,$$

there exists a control  $u \in C^0([0, 1]; \mathbb{R}^M)$  steering in time 1 our system from  $y_0$  to  $y_1$  such that:

$$\|u\|_{C^0} < \varepsilon.$$

## The stair-case method: from local to global

We connect  $y_0$  and  $y_1$  stepwise joining steady states at distance  $< \delta_\nu$  along the arc  $\gamma(s) := (1-s)y_0 + sy_1$ .





## Dissipative case

We now assume further that  $A$  is symmetric and strictly positive definite. Then, in time large, we can find a **nonnegative** control  $u$  driving a **generic**  $y_0 \in \mathbb{R}^N$  to a steady state  $y_1$  defined by a control  $\bar{u}_1 \geq \nu$ .

## Idea of the proof-Dissipative Case

We introduce the state variable  $z = y - y_1$  reducing ourselves to prove that, in time large, we can drive the system from  $z_0 = y_0 - y_1$  to 0 by a control:

$$v \geq -Tr(y_1).$$

Then, the control  $u = v + Tr(y_1)$  will drive the system from  $y_0$  to  $y_1$  and

$$u = v + Tr(y_1) \geq -Tr(y_1) + Tr(y_1) = 0.$$

## Idea of the proof-Dissipative Case

We fix  $\tau > 0$  and we take a generic  $T > \tau$ . Then, we determine the control  $v$  as follows:

1. we let the system evolve for a long time interval  $[0, T - \tau]$ . Since the system **dissipative**, we have:

$$\|y(T - \tau)\| \leq e^{-\lambda(T-\tau)} \|y_0 - y_1\|,$$

where  $\lambda$  is the first eigenvalue of  $A$ .

2. we steer the system from  $z(T - \tau)$  to 0 in the small time interval  $[T - \tau, T]$  by a control  $w \in C^0$  such that:

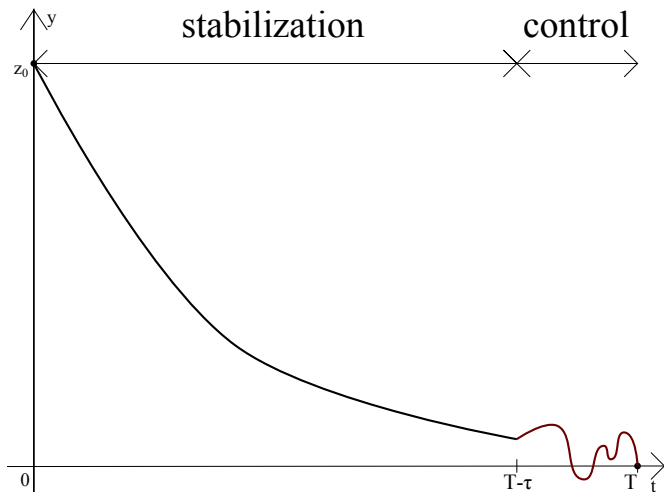
$$\|w\|_{C^0} \leq C(\tau) \|z(T - \tau)\| \leq C(\tau) e^{-\lambda(T-\tau)} \|y_0 - y_1\|.$$

Then,  $v := w\chi_{[T-\tau, T]}$  drives our control system from  $y_0 - y_1$  to 0 and, if  $T$  is large enough,

$$\|v\|_{C^0} \leq C(\tau) e^{-\lambda(T-\tau)} \|y_0 - y_1\| < \nu.$$

This implies that  $v \geq -\nu \geq -Tr(y_1)$  as required.

## Idea of the proof-Dissipative Case



## Infinite dimensional linear systems-General case

Take  $c \in L^\infty(\Omega)$ . Let us consider the control problem:

$$\begin{cases} y_t - \Delta y + c(x)y = 0 & \text{in } (0, T) \times \Omega \\ y = u. & \text{on } (0, T) \times \partial\Omega. \end{cases}$$

Assume that the first eigenvalue of the operator  $A(y) = -\Delta y + cy$  is  $\lambda < 0$ . For simplicity, we assume 0 is not an eigenvalue of  $A$ .

Then, we take  $y_0 := \phi_1$  the **nonnegative** first eigenfunction of  $A$  and  $y_1 := y_1$  steady state with  $Tr(y_1) \geq \nu$ .

## General case

Then, by the Maximum Principle for parabolic problems, for any **nonnegative** control  $u$ , the unique solution  $y$  to:

$$\begin{cases} y_t - \Delta y + c(x)y = 0 & \text{in } (0, T) \times \Omega \\ y = u & \text{on } (0, T) \times \partial\Omega \\ y(0) = \phi_1 & \text{in } \Omega \end{cases}$$

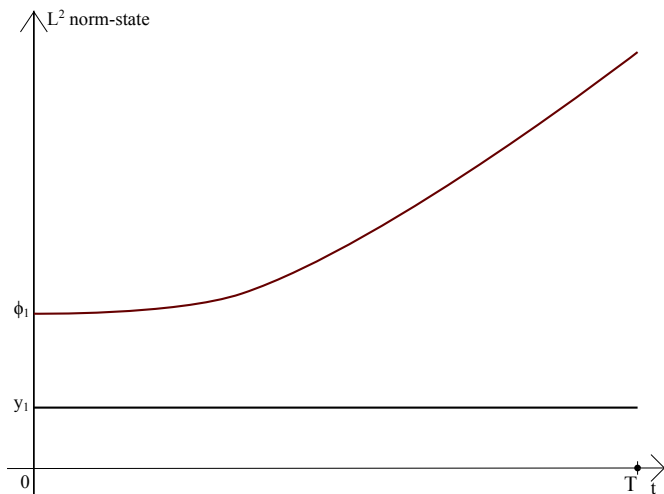
is such that:

$$y \geq e^{-\lambda t} \phi_1.$$

Then, for  $T$  large enough,  $y(T, \cdot) \neq y_1$ .

Thus, the **constrained controllability fails to be realized in the nondissipative case for general initial data**. On the other hand, it holds provided that both the initial datum and the final target are steady states.

## General case



## $\varepsilon$ -Controllability

We know that for any  $z_0 \in L^2$  we can find a control  $w \in L^\infty((0, T) \times \partial\Omega)$  steering the control system from  $z_0$  to 0 in time 1 and

$$\|w\|_{L^\infty} \leq C(1)\|z_0\|_{L^2}.$$

Then, for any  $\varepsilon > 0$ , there exists  $\delta_\varepsilon > 0$  such that:

$$\|z_0\|_{L^2} < \delta_\varepsilon \quad \Rightarrow \quad \|w\|_{L^\infty} < \varepsilon.$$

Take  $\varepsilon = \nu$ . If  $\|y_1 - y_0\|_{L^2} < \delta_\nu$ , then, we are able to find a control  $v \in L^\infty$  of size  $\|v\|_{L^\infty} < \nu$  such that  $u := v + Tr(y_1)$  drives the system from  $y_0$  to  $y_1$  in time 1. Moreover,  $u \geq -\nu + \nu = 0$ .



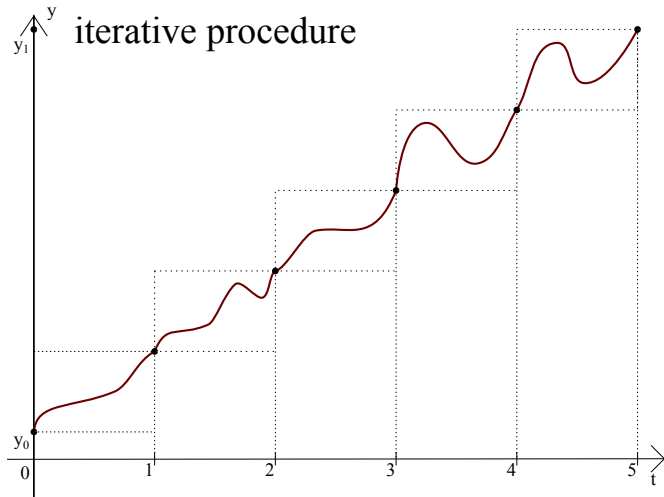
## The stair-case method: from local to global

We introduce the following continuous arc joining  $y_0$  and  $y_1$  in the set of steady states.

$$\gamma(s) := (1 - s)y_0 + sy_1.$$

## The stair case method: from local to global

Then, we link  $y_0$  by  $y_1$  by a step by step procedure joining the steady states along  $\gamma$  at distance less then  $\delta_\nu$ .





## Idea of the proof-Dissipative case

As in the pure heat equation, in the dissipative case the **constrained controllability** holds for **general initial data** in  $L^2$ . Indeed, we introduce the state variable  $z = y - y_1$  reducing ourselves to prove that, in time large, we can drive the system from  $y_0 - y_1$  to 0 by a control:

$$v \geq -Tr(y_1).$$

Then, the control  $u = v + Tr(y_1)$  will drive the system from  $y_0$  to  $y_1$  and

$$u = v + Tr(y_1) \geq -Tr(y_1) + Tr(y_1) = 0.$$



## Idea of the proof-Dissipative Case

We fix  $\tau > 0$  and we take an arbitrary  $T > \tau$ . Then, we determine the control  $v$  as follows:

1. we let the system evolve for a long time interval  $[0, T - \tau]$ .  
Since the system **dissipative**, we have:

$$\|z(T - \tau)\|_{L^2} \leq e^{-\lambda(T-\tau)} \|y_0 - y_1\|_{L^2},$$

where  $\lambda$  is the first eigenvalue of  $-\Delta y + cy$ .

2. we steer the system from  $z(T - \tau)$  to 0 in the small time interval  $[T - \tau, T]$  by a control  $w \in L^\infty$  such that:

$$\|w\|_{L^\infty} \leq C(\tau) \|z(T - \tau)\|_{L^2} \leq C(\tau) e^{-\lambda(T-\tau)} \|y_0 - y_1\|_{L^2}.$$

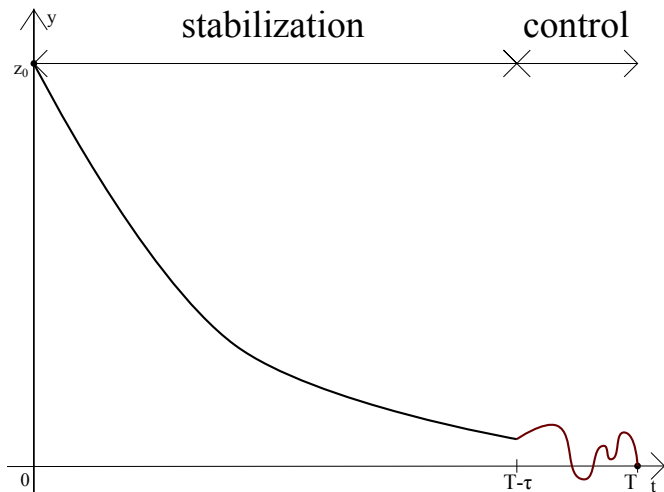
Then,  $v := w\chi_{[T-\tau, T]}$  drives our control system from  $y_0 - y_1$  to 0 and, if  $T$  is large enough,

$$\|v\|_{L^\infty} \leq C(\tau) e^{-\lambda(T-\tau)} \|y_0 - y_1\|_{L^2} < \nu.$$

This implies that  $v \geq -\nu \geq -Tr(y_1)$  as required.



## Idea of the proof-Dissipative Case



## Infinite dimensional SemiLinear systems-General case

We consider the control system:

$$\begin{cases} y_t - \Delta y + f(y) = 0 & \text{in } (0, T) \times \Omega \\ y = u & \text{on } (0, T) \times \partial\Omega \\ y(0) = y_0. & \text{in } \Omega \end{cases}$$

where  $f$  is  $C^1$  function such that  $f(0) = 0$ . We have **NO monotonicity assumptions** on  $f$ . Then, for an initial datum  $y_0 \in L^\infty(\Omega)$  and a boundary control  $u \in L^\infty((0, T) \times \partial\Omega)$ , the above system admits solution locally in time. **Blow up** phenomena in finite time may occur.

## General case

Let  $\mathcal{S}$  be the set of **bounded steady states** endowed with the uniform topology. Then we take two steady states  $y_0$  and  $y_1$  connected in  $\mathcal{S}$  by a continuous arc  $\gamma$  such that:

$$\text{Tr}(\gamma(s)) \upharpoonright_{\partial\Omega} \geq \nu > 0 \quad \forall s \in [0, 1].$$

Then, in time large, we can steer the system from  $y_0$  to  $y_1$  in time  $T$  by a **nonnegative** control  $u$ .

## Idea of the proof-General Case

First of all, we have then to check the local steady states controllability of the above system. We go by step. The first step is to prove a controllability Theorem for the case  $f$  globally Lipschitz and distributed control, employing the approach of:

E. Fernández-Cara and E. Zuazua

*Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, Vol. 17 no. 5 (2000), pp. 583 – 616.

Basically, for any  $\eta \in L^\infty((0, T) \times \Omega)$ , we consider the linear system

$$\begin{cases} z_t - \Delta z + \frac{f(\eta+y_1)-f(y_1)}{\eta-y_1} z = u \chi_\omega & \text{in } (0, \tau) \times \Omega \\ z = 0 & \text{on } (0, \tau) \times \partial\Omega. \end{cases}$$

and we apply Kakutani's fixed point Theorem to prove the desired result.



## Idea of the proof-General Case

By extension-restriction arguments, we get the same result for the case of boundary control and  $f$  globally Lipschitz. Finally, this yields the following local null controllability result in case  $f$  is only locally Lipschitz.

### Proposition

*There exists  $\delta > 0$  such that, for any initial datum  $y_0 \in L^\infty$  and final target  $y_1$  bounded steady state such that:*

$$\|y_1 - y_0\|_{L^\infty} \leq \delta,$$

*there exists a control  $v \in L^\infty((0, T) \times \partial\Omega)$  such that:*

- *$v$  drives the system from  $y_0$  to  $y_1$ ;*
- 

$$\|v\|_{L^\infty} \leq C \|y_1 - y_0\|_{L^\infty}.$$

## Idea of the proof-General case

By the local controllability of the system, for any  $\varepsilon > 0$  there exists  $\delta_\varepsilon > 0$  such that for any pair of bounded steady states  $y_0$  and  $y_1$  such that

$$\|y_1 - y_0\|_{L^\infty} < \delta_\varepsilon,$$

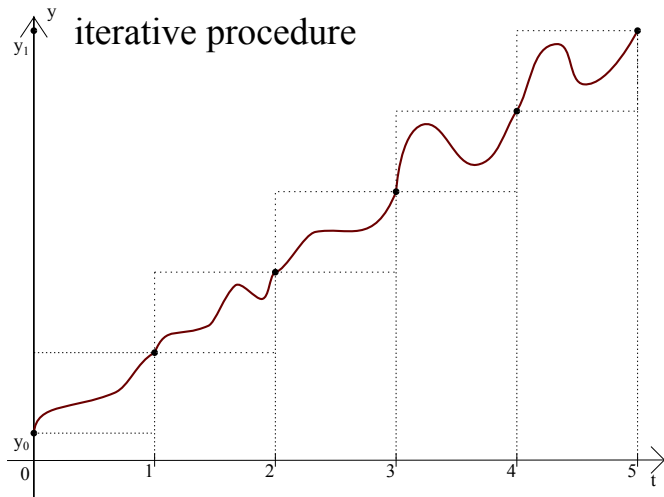
we can find a control  $u = v + Tr(y_1) \in L^\infty$  driving the system from  $y_0$  to  $y_1$  in time 1. Moreover, we have:

$$\|v\|_{L^\infty} < \varepsilon.$$

If  $\varepsilon = \nu$ . Then,  $u = v + Tr(y_1) \geq -\nu + \nu = 0$ .

## Idea of the proof-General case

We connect  $y_0$  and  $y_1$  stepwise joining steady states along  $\gamma$  at distance less than  $\delta_\nu$ .



## Dissipative case

We consider the control system:

$$\begin{cases} y_t - \Delta y + f(y) = 0 & \text{in } (0, T) \times \Omega \\ y = u & \text{on } (0, T) \times \partial\Omega \\ y(0) = y_0. & \text{in } \Omega \end{cases}$$

We assume  $f$  is a  $C^1$  **nondecreasing** function such that  $f(0) = 0$ . Then, thanks to the nondecreasing character of  $f$ , for any  $y_0 \in L^2(\Omega)$  and  $u \in L^2((0, T) \times \partial\Omega)$ , there exists a unique solution

$$y \in L^2((0, T) \times \Omega) \cap C^0([0, T], H^{-1}(\Omega)).$$

## Dissipative case

As in the dissipative heat equation, the constrained controllability holds for general initial data in  $L^2$ .

Indeed, we observe that  $z = y - y_1$  satisfies:

$$\begin{cases} z_t - \Delta z + f(z + y_1) - f(y_1) = 0 & \text{in } (0, T) \times \Omega \\ z = u - Tr(y_1) & \text{on } (0, T) \times \partial\Omega \\ z(0) = y_0 - y_1. & \text{in } \Omega. \end{cases}$$

We have to prove that, in time large, we can drive the above system from  $y_0 - y_1$  to 0 by means of a control  $v \geq -Tr(y_1)$ . Then,  $u = v + Tr(y_1) \geq 0$  will be the desired control.

## Idea of the proof-Dissipative Case

Let  $\tau > 0$  be fixed and  $T > \tau$  time horizon.

1. First of all, we enjoy the regularizing effect and the **dissipative** nature of the system for a long time. Indeed, for any  $\delta > 0$ , taking the control to be zero in  $[0, T - \tau]$ , we have that the unique solution  $z$  to:

$$\begin{cases} z_t - \Delta z + f(z + y_1) - f(y_1) = 0 & \text{in } (0, T - \tau) \times \Omega \\ z = 0 & \text{on } (0, T - \tau) \times \partial\Omega \\ z(0) = y_0 - y_1 & \text{in } \Omega. \end{cases}$$

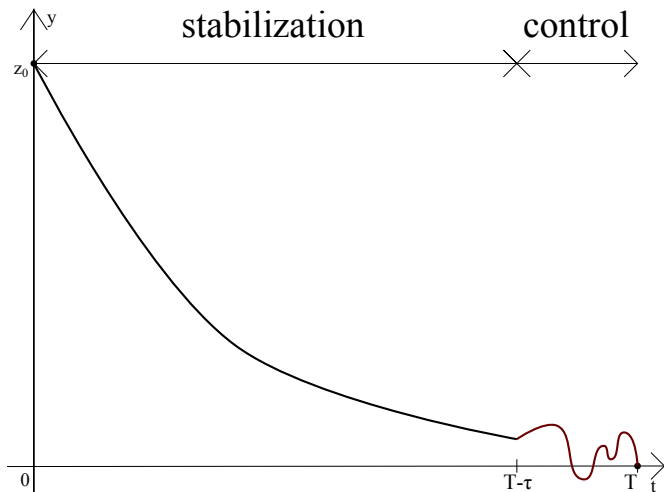
is such that  $z(T - \tau, \cdot) \in L^\infty$  and

$$\|z(T - \tau, \cdot)\|_{L^\infty} \leq \delta,$$

by taking  $T$  large enough;

2. to conclude, we drive  $z(T - \tau, \cdot)$  to 0 by a control  $w$  of size  $\|w\|_{L^\infty} < \nu$ .

## Idea of the proof-Dissipative Case



## Simulations

Let consider the control system:

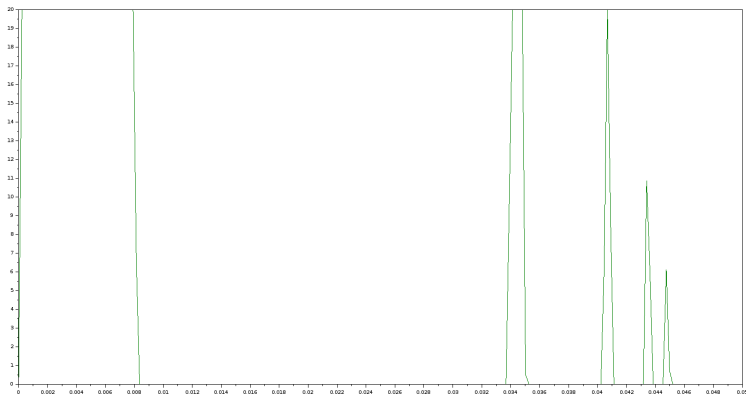
$$\begin{cases} y_t - y_{xx} + \sin(\pi y) = 0 & (t, x) \in (0, T) \times (0, 1) \\ y(t, 0) = u_1(t) & t \in (0, T) \\ y(t, 1) = u_2(t). & t \in (0, T) \end{cases}$$

Take as initial datum  $y_0 \equiv 1$  and as final target  $y_1 \equiv 2$ . Both are steady states for the system. Simulations performed by using AMPL-IPOPT agrees with our steady state constrained controllability result.





## Graph of the control



Thank you for the attention!!!