WKB expansion for a fractional Schrödinger equation with applications to controllability

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Abstract

This paper is devoted to the construction of localized quasi-solutions in general optics for a one-dimensional nonlocal Schrödinger equation involving the fractional Laplace operator. A suitable ansatz for the solutions to the problem is obtained adapting a classical WKB approach. As an application, the controllability problem for the fractional Schrödinger equation is analyzed, finding confirmations of previously known results.

Keywords. WKB, fractional Laplacian, Schrödinger equation.

1 Introduction

In this paper, we are interested in the construction of ray-like solutions in geometric optics for the following one-dimensional non-local equation

$$\mathcal{P}_s u := \left[i\partial_t + (-d_x^2)^s\right] u = 0, \quad (x,t) \in \mathbb{R} \times (0,+\infty), \tag{1.1}$$

with highly oscillatory initial datum

$$u(x,0) = u_{\text{in}}(x)e^{i\frac{\xi_0}{\varepsilon}x} := u_0(x), \quad \xi_0 \in \mathbb{R}. \tag{1.2}$$

In eq. (1.1), $(-d_x^2)^s$ is the fractional Laplacian, defined for all $s \in (0,1)$ and for any function f sufficiently smooth as the following singular integral

$$(-d_x^2)^s f(x) := c_{1,s}\; P.V. \int_{\mathbb{R}} \frac{f(x)-f(y)}{|x-y|^{1+2s}}\, dy,$$

with $c_{1,s}$ a normalization constant given by (see [9, Section 3] and [24, Appendix A])

$$c_{1,s} := \left(\int_{\mathbb{R}} \frac{1-\cos(z)}{|z|^{1+2s}}\, dz\right)^{-1} = \frac{s2^{2s}\Gamma\left(s+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(1-s)}, \tag{1.3}$$

where Γ is the usual Gamma function. The parameter ε in eq. (1.1) and in eq. (1.2) represents the fast space and time scale introduced in the equation, as well as the typical wavelength of oscillations

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of the initial data. Moreover, we will assume the initial phase u_{in} to be an $L^2(\mathbb{R})$ function, so that we have $u_0 \in L^2(\mathbb{R})$.

Space-fractional Schrödinger equations have been introduced by Laskin in quantum mechanics ([14, 15, 16]), since they provide a natural extension of the standard local model when the Brownian trajectories in Feynman path integrals are replaced by Levy flights, which are generated by the fractional Laplacian. Applications of eq. (1.1) may be found in the study of a condensed-matter realization of Lévy crystals ([26]). More recently, the fractional Schrödinger equation was introduced into optics by Longhi in [21], with applications to laser implementation.

The present paper deals with the construction of asymptotic approximations in geometric optics for the solutions to eq. (1.1), and with their application to the study of controllability problems.

Asymptotic analysis for wave-like equations through geometric optics (also known as the Wentzel-Kramers-Brillouin (WKB) method or ray-tracing, [6, 13, 25, 31]) is nowadays a classical tool that has been developed in several directions. An incomplete biography on the topic includes [18, 19, 20]. It is by now well-known that wave-type equations, in a local framework, have solutions that are localized near curves $(t, x(t))$ in space-time, also called rays. With this observation in mind, asymptotic methods allow to study the behavior of several wave-type phenomena, with applications, e.g., in geophysics ([8, 10]), acoustic wave equations ([17, 27]) or gravity waves ([28]).

To the best of our knowledge, a WKB approach has not yet been fully developed in a nonlocal setting. On the other hand, this is certainly an interesting issue, not only from a purely mathematical perspective, but also due to the the several applications that we mentioned above. Our work represents a first step in this direction, providing a complete procedure for obtaining a WKB expansion of equation eq. (1.1).

A motivation and a natural application for our construction will then be the study of controllability properties for the one dimensional fractional Schrödinger equation

$$\begin{cases} iu_t + (-d_x^2)^s u = g\chi_{\omega\times(0,T)}, & (x,t) \in (-1,1) \times (0,T) \\ u \equiv 0, & (x,t) \in (-1,1)^c \times (0,T) \\ u(x,0) = u_0(x), & x \in (-1,1), \end{cases} \tag{1.4}$$

where ω is a neighborhood of the boundary of the space domain $(-1, 1)$. As we already proved in [3] by means of Ingham's techniques, for eq. (1.4) we have the following control properties:

- For $s > 1/2$, null controllability holds in any finite time $T > 0$. In other words, given any $u_0 \in L^2(-1, 1)$ there exists a control function $g \in L^2(\omega \times (0, T))$ such that the solution to eq. (1.4) satisfies $u(x, T) = 0$.
- For $s = 1/2$, the same result holds if we assume the controllability time T to be large enough, i.e $T \geq T_0 > 0$.
- For $s < 1/2$, the equation eq. (1.4) is not null controllable.

Through the construction of localized solution that we are going to present in this work, it will be possible to give a further confirmation to the above facts.

The approach that we are going to use for building localized solutions is quite standard. In particular, given a plane wave solution u^ε, we will look for quasi-solutions to eq. (1.1) with an ansatz of the type

$$z^\varepsilon(x,t) = u^\varepsilon(x,t)a^\varepsilon(x,t), \quad a^\varepsilon(x,t) = \sum_{j\geq 0} \varepsilon^{pj} a_j(x,t),$$

with $p \in \mathbb{R}$, and where the functions a_j have to be determined. The identification of the a_j-s will then be carried out imposing

$$\mathcal{P}_s z^\varepsilon = O(\varepsilon^\infty),$$

thus obtaining a series of PDEs in which it will be possible to clearly separate the leading order terms, with respect to ε, from several remainders which will vanish as $\varepsilon \to 0$. This will generate a cascade system for the functions a_j, which can then be determined as the solution of certain given Partial Differential Equations.

This paper is organized as follows. In Section 3, we will present the construction of the ansatz for the asymptotic expansion of the solutions to our fractional Schrödinger equation 1.1. In Section 4, we will show that the quasi-solution z^ε obtained through our procedure are a good approximation of the real solutions u to our original equation. Finally, Section 5 will be devoted to the discussion on the application of our methodology to the study of control properties.

2 Preliminaries

Before presenting the construction of our ansatz, we introduce some preliminary facts that we are going to need for our further analysis.

First of all, a classical result on the Schrödinger equation tells us that the $H^s(\mathbb{R})$-norm of the solution u to eq. (1.1) is conserved. This is an easy consequence of the skew-adjointness of the operator $i(-d_x^2)^s$, which allows to readily check that

$$0 = \left\langle u_t - i(-d_x^2)^s u, u + (-d_x^2)^s u\right\rangle_{L^2(\mathbb{R})} = \frac{1}{2}\frac{d}{dt}\left(\|u\|^2_{L^2(\mathbb{R})} + [u]^2_{H^s(\mathbb{R})}\right) = \frac{1}{2}\frac{d}{dt}\|u\|^2_{H^s(\mathbb{R})}.$$

In particular, $\|u\|_{H^s(\mathbb{R})}$ represents an energy for our equation.

As a second thing, we recall here the definition of *null bicharacteristics*, which will have a fundamental role in our later construction.

Given a general pseudo-differential operator Ψ with principal symbol $\psi = \psi(x,t,\xi,\tau)$, a null bicharacteristic of Ψ is defined to be a solution of the following system of ordinary differential equations

$$\begin{cases} \dot{x}(\sigma) = \psi_\xi(x(\sigma),t(\sigma),\xi(\sigma),\tau(\sigma)) \\ \dot{t}(\sigma) = \psi_\tau(x(\sigma),t(\sigma),\xi(\sigma),\tau(\sigma)) \\ \dot{\xi}(\sigma) = -\psi_x(x(\sigma),t(\sigma),\xi(\sigma),\tau(\sigma)) \\ \dot{\tau}(\sigma) = -\psi_t(x(\sigma),t(\sigma),\xi(\sigma),\tau(\sigma)) \end{cases}$$

with initial data $(x(0),t(0),\xi(0)) = (x_0,t_0,\xi_0) \in \mathbb{R}^3$ and $\tau(0) \in \mathbb{R}$ chosen so that $\psi(x_0,t_0,\xi_0,\tau(0)) = 0$. Then, the projection of a null bicharacteristic to the physical time-space, $(t,x(t))$, traces a curve in $(0,+\infty)\times\mathbb{R}$ which is called a *ray* of Ψ.

In the case of our fractional Schrödinger equation, notice that $\mathcal{P}_s = i\partial_t + (-d_x^2)^s$ is a pseudo-differential operator with symbol $p_s(x,t,\xi,\tau) = \tau - |\xi|^{2s}$. Therefore, the bicharacteristic system is given by

$$\begin{cases} \dot{x}(\sigma) = \pm 2s|\xi(\sigma)|^{2s-1}, & x(0) = x_0 \\ \dot{t}(\sigma) = 1, & t(0) = t_0 \\ \dot{\xi}(\sigma) = 0, & \xi(0) = \xi_0 \\ \dot{\tau}(\sigma) = 0, & \tau(0) = |\xi_0|^{2s}. \end{cases} \tag{2.1}$$

Moreover, without losing generality we may assume $t_0 = 0$. Then, eq. (2.1) can be solved explicitly, and we obtain the following expressions for the bicharacteristics

$$\begin{cases} x(\sigma) = x_0 \pm 2s|\xi_0|^{2s-1}\sigma \\ t(\sigma) = \sigma \\ \xi(\sigma) = \xi_0 \\ \tau(\sigma) = |\xi_0|^{2s}. \end{cases}$$

In particular, the rays of $\mathcal{P}_s$ are given by the curves $(t, x_0 \pm 2s|\xi_0|^{2s-1}t) \in (0, +\infty) \times \mathbb{R}$. Notice that, as one expects since the operator has constant coefficients, these rays are straight lines.

3 Construction of the ansatz

This section is devoted to a heuristic exposition of the key ideas leading to the construction of ray-like solutions for our equation. We begin by seeking approximate solutions with an ansatz of WKB type with linear phase:

$$z^{\varepsilon}(x,t) = c(\varepsilon)u^{\varepsilon}(x,t)a^{\varepsilon}(x,t), \quad a^{\varepsilon}(x,t) = \sum_{j\geq 0} \varepsilon^{pj} a_j(x,t), \tag{3.1}$$

where $p \in \mathbb{R}$ and the functions a_j have to be determined. The constant $c(\varepsilon)$, instead, will be chosen asking that the function z^{ε} has $H^s(\mathbb{R})$-norm of the order $\mathcal{O}(1)$. We start by observing that, since for any $\xi_0 \in \mathbb{R}$ we have

$$(-d_x^2)^s e^{i\xi_0\varepsilon^{-1}x} = |\xi_0|^{2s}\varepsilon^{-2s}e^{i\xi_0\varepsilon^{-1}x}, \tag{3.2}$$

the plain wave

$$u^{\varepsilon}(x,t) := e^{i\left[\xi_0\varepsilon^{-1}x + |\xi_0|^{2s}\varepsilon^{-2s}t\right]}$$

satisfies $\mathcal{P}_s u^{\varepsilon} = 0$. Notice that

$$e^{i\left[\xi_0\varepsilon^{-1}x + |\xi_0|^{2s}\varepsilon^{-2s}t\right]} = e^{i\xi_0\varepsilon^{-1}\left(x - \frac{\varepsilon^{1-2s}}{2s}x(t)\right)}.$$

At this point, for identifying the correct value of the parameter p in eq. (3.1), and for determining the functions a_j, we need to compute $\mathcal{P}_s z^{\varepsilon}$ and gather the terms that we obtain according to their order with respect to ε. Moreover, in what follows we shall employ the following expression for the fractional Laplacian of the product of two functions

$$(-d_x^2)^s(fg) = f(-d_x^2)^s g + g(-d_x^2)^s f - I_s(f,g), \tag{3.3}$$

where the term I_s is given by (see, e.g., [5, 24])

$$I_s(f,g)(x) := c_{1,s}\, P.V. \int_{\mathbb{R}^N} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{1+2s}}\, dy.$$

By means of eq. (3.2) and eq. (3.3), we can immediately see that

$$\mathcal{P}_s z^{\varepsilon} = c(\varepsilon)u^{\varepsilon}\Big[ia^{\varepsilon} + (-d_x^2)^s a^{\varepsilon} - K_s(a^{\varepsilon})\Big] \tag{3.4}$$

with

$$
\begin{aligned}
K_s(a^\varepsilon) &= c_{1,s}\, P.V. \int_{\mathbb{R}} \frac{1-e^{i\frac{\xi_0}{\varepsilon}(y-x)}}{|x-y|^{1+2s}} \Big[a^\varepsilon(x,t) - a^\varepsilon(y,t)\Big]\, dy \\
&= c_{1,s} \frac{\xi_0^{2s}}{\varepsilon^{2s}}\, P.V. \int_{\mathbb{R}} \frac{1-e^{iq}}{|q|^{1+2s}} \left[a^\varepsilon(x,t) - a^\varepsilon\left(x + \frac{\varepsilon}{\xi_0} q, t\right)\right] dq.
\end{aligned}
$$

Now, employing a fractional Taylor expansion ([12, 29, 30]), for any $\beta \in (0,1)$ and $x \le \theta \le x + \frac{\varepsilon}{\xi_0} q$ we can write

$$
a^\varepsilon\left(x + \frac{\varepsilon}{\xi_0} q, t\right) = a^\varepsilon(x,t) + \frac{\mathcal{D}^\beta a^\varepsilon(x,t)}{\Gamma(1+\beta)} \left(\frac{\varepsilon}{\xi_0} q\right)^\beta + \frac{\mathcal{D}^{2\beta} a^\varepsilon(x,t)}{\Gamma(1+2\beta)} \left(\frac{\varepsilon}{\xi_0} q\right)^{2\beta} + \frac{\mathcal{D}^{3\beta} a^\varepsilon(\theta,t)}{\Gamma(1+3\beta)} \left(\frac{\varepsilon}{\xi_0} q\right)^{3\beta}, \tag{3.5}
$$

where with $\mathcal{D}^\beta$ we indicate the following fractional derivative of order β

$$
\mathcal{D}^\beta f(x) := \frac{1}{\Gamma(1-\beta)} \int_{-\infty}^{x} \frac{f'(y)}{(x-y)^\beta}\, dy, \quad \beta \in (0,1).
$$

Thanks to eq. (3.5), we get

$$
\begin{aligned}
K_s(a^\varepsilon) = \quad & -c_{1,s} \frac{\xi_0^{2s-\beta}}{\Gamma(1+\beta)} \varepsilon^{\beta-2s} \mathcal{D}^\beta a^\varepsilon(x,t)\, P.V. \int_{\mathbb{R}} \frac{1-e^{iq}}{|q|^{1+2s}} q^\beta\, dq \\
& -c_{1,s} \frac{\xi_0^{2s-2\beta}}{\Gamma(1+2\beta)} \varepsilon^{2\beta-2s} \mathcal{D}^{2\beta} a^\varepsilon(x,t)\, P.V. \int_{\mathbb{R}} \frac{1-e^{iq}}{|q|^{1+2s}} q^{2\beta}\, dq \\
& -c_{1,s} \frac{\xi_0^{2s-3\beta}}{\Gamma(1+3\beta)} \varepsilon^{2\beta-3s} \mathcal{D}^{3\beta} a^\varepsilon(\theta,t)\, P.V. \int_{\mathbb{R}} \frac{1-e^{iq}}{|q|^{1+2s}} q^{3\beta}\, dq.
\end{aligned}
$$

Moreover, we observe that since $|1-e^{iq}| = 2 - 2\cos(q)$, for all $q \in \mathbb{R}$ and $\beta < 2s/3$ the integrals in the above expression are finite. In particular, we have

- $\displaystyle \left| P.V. \int_{\mathbb{R}} \frac{1-e^{iq}}{|q|^{1+2s}} q^\beta\, dq \right| \le 4\Gamma(\beta - 2s - 1) \cos\left[\frac{(2s-\beta)\pi}{2}\right],$
- $\displaystyle \left| P.V. \int_{\mathbb{R}} \frac{1-e^{iq}}{|q|^{1+2s}} q^{2\beta}\, dq \right| \le 4\Gamma(2\beta - 2s - 1) \cos\big[(s-\beta)\pi\big],$
- $\displaystyle \left| P.V. \int_{\mathbb{R}} \frac{1-e^{iq}}{|q|^{1+2s}} q^{3\beta}\, dq \right| \le 4\Gamma(3\beta - 2s - 1) \cos\left[\frac{(2s-3\beta)\pi}{2}\right].$

Therefore, we can rewrite

$$
K_s(a^\varepsilon) = -\mathcal{C}_\beta \varepsilon^{\beta-2s} \mathcal{D}^\beta a^\varepsilon(x,t) - \mathcal{C}_{2\beta} \varepsilon^{2\beta-2s} \mathcal{D}^{2\beta} a^\varepsilon(x,t) - \mathcal{C}_{3\beta} \varepsilon^{3\beta-2s} \mathcal{D}^{3\beta} a^\varepsilon(\theta,t),
$$

with

$$
\mathcal{C}_\gamma := c_{1,s} \frac{\xi_0^{2s-\gamma}}{\Gamma(1+\gamma)} P.V. \int_{\mathbb{R}} \frac{1-e^{iq}}{|q|^{1+2s}} q^\gamma\, dq,
$$

and we then obtain

$$\mathcal{P}_s z^{\varepsilon}(x,t) = c(\varepsilon) u^{\varepsilon}\Big[i a_t^{\varepsilon}(x,t) + (-d_x^2)^s a^{\varepsilon}(x,t) \quad + \mathcal{C}_\beta \varepsilon^{\beta-2s} \mathcal{D}^{\beta} a^{\varepsilon}(x,t) \\ + \mathcal{C}_{2\beta} \varepsilon^{2\beta-2s} \mathcal{D}^{2\beta} a^{\varepsilon}(x,t) + \mathcal{C}_{3\beta} \varepsilon^{3\beta-2s} \mathcal{D}^{3\beta} a^{\varepsilon}(\theta,t)\Big].$$

Furthermore, let us introduce the following rescaling of the time variable $t \mapsto \tau := \varepsilon^{2s-\beta} t$. In this way, eq. (3.4) finally becomes

$$\mathcal{P}_s z^{\varepsilon} = c(\varepsilon)\varepsilon^{\beta-2s} u^{\varepsilon} \sum_{j\geq 0} \varepsilon^{pj} \Big[i\partial_\tau a_j + \mathcal{C}_\beta \mathcal{D}^{\beta} a_j + \varepsilon^{2s-\beta}(-d_x^2)^s a_j + \mathcal{C}_{2\beta}\varepsilon^{\beta}\mathcal{D}^{2\beta} a_j + \mathcal{C}_{3\beta}\varepsilon^{2\beta}\mathcal{D}^{3\beta} a_j(\theta,\tau)\Big]. \tag{3.6}$$

For determining from eq. (3.6) the expression of the functions a_j, we will now impose, for any $j \geq 0$,

$$\mathcal{P}_s z^{\varepsilon} = O(\varepsilon^{\infty}),$$

thus obtaining a series of PDEs in which will be possible to clearly separate the leading order terms, with respect to ε, from several remainder which will vanish as $\varepsilon \to 0$. During this procedure, we will also identify the values of the parameters p and β that we shall employ.

Let us start firstly with $j = 0$. In this case, it is trivial to identify the leading equation, and we immediately have that the function $a_0(x,\tau)$ has to satisfy

$$i\partial_\tau a_0 + \mathcal{C}_\beta \mathcal{D}^\beta a_0 = 0. \tag{3.7}$$

For $j = 1$, instead, choosing $p = \beta$ we obtain from eq. (3.6) and eq. (3.7)

$$\varepsilon^{\beta}\Big[i\partial_\tau a_1 + \mathcal{C}_\beta \mathcal{D}^{\beta} a_1 + \mathcal{C}_{2\beta}\mathcal{D}^{2\beta} a_0 \quad + \varepsilon^{2s-2\beta}(-d_x^2)^s a_0 + \mathcal{C}_{3\beta}\varepsilon^{\beta}\mathcal{D}^{3\beta} a_0(\theta,\tau) \\ + \varepsilon^{2s-\beta}(-d_x^2)^s a_1 + \mathcal{C}_{2\beta}\varepsilon^{\beta}\mathcal{D}^{2\beta} a_1 + \mathcal{C}_{3\beta}\varepsilon^{2\beta}\mathcal{D}^{3\beta} a_1(\theta,\tau)\Big],$$

and we thus find that the function $a_1(x,\tau)$ has to satisfy

$$i\partial_\tau a_1 + \mathcal{C}_\beta \mathcal{D}^{\beta} a_1 + \mathcal{C}_{2\beta}\mathcal{D}^{2\beta} a_0 = 0. \tag{3.8}$$

Let us now continue with $j = 2$. In this case, we have

$$\varepsilon^{2\beta}\Big[\varepsilon^{-2\beta} \quad \underbrace{(i\partial_\tau a_0 + \mathcal{C}_\beta \mathcal{D}^{\beta} a_0)}_{=0} + \varepsilon^{2s-3\beta}(-d_x^2)^s a_0 + \mathcal{C}_{3\beta}\mathcal{D}^{3\beta} a_0(\theta,\tau) \\ + \varepsilon^{-\beta}\underbrace{(i\partial_\tau a_1 + \mathcal{C}_\beta \mathcal{D}^{\beta} a_1 + \mathcal{C}_{2\beta}\mathcal{D}^{2\beta} a_0)}_{=0} + \varepsilon^{2s-2\beta}(-d_x^2)^s a_1 + \mathcal{C}_{3\beta}\varepsilon^{\beta}\mathcal{D}^{3\beta} a_1(\theta,\tau) \\ + \mathcal{C}_{2\beta}\mathcal{D}^{2\beta} a_1 + i\partial_\tau a_2 + \mathcal{C}_\beta \mathcal{D}^{\beta} a_2 + \varepsilon^{2s-\beta}(-d_x^2)^s a_2 + \mathcal{C}_{2\beta}\varepsilon^{\beta}\mathcal{D}^{2\beta} a_2 + \mathcal{C}_{3\beta}\varepsilon^{2\beta}\mathcal{D}^{3\beta} a_2(\theta,\tau)\Big].$$

and we thus find that the function $a_2(x,\tau)$ has to satisfy

$$i\partial_\tau a_2 + \mathcal{C}_\beta \mathcal{D}^{\beta} a_2 + \mathcal{C}_{2\beta}\mathcal{D}^{2\beta} a_1 + \mathcal{C}_{3\beta}\mathcal{D}^{3\beta} a_0(\theta,\tau) = 0. \tag{3.9}$$

For $j=3$, choosing $\beta = s/2$ (observe that this is admissible since $s/2 < 2s/3$), and employing eq. (3.7), eq. (3.8) and eq. (3.9), we obtain

$$\varepsilon^{\frac{3s}{2}}\Big[i\partial_\tau a_3 + \mathcal{C}_{\frac{s}{2}}\mathcal{D}^{\frac{s}{2}}a_3 \quad +\mathcal{C}_s\mathcal{D}^s a_2 + \mathcal{C}_{\frac{3s}{2}}\mathcal{D}^{\frac{3s}{2}}a_1(\theta,\tau) + (-d_x^2)^s a_0 + \varepsilon^{\frac{s}{2}}(-d_x^2)^s a_1 + \varepsilon^s(-d_x^2)^s a_2$$
$$+\mathcal{C}_{\frac{3s}{2}}\varepsilon^{\frac{s}{2}}\mathcal{D}^{\frac{3s}{2}}a_2(\theta,\tau) + \varepsilon^{\frac{3s}{2}}(-d_x^2)^s a_3 + \mathcal{C}_s\varepsilon^{\frac{s}{2}}\mathcal{D}^s a_3 + \mathcal{C}_{\frac{3s}{2}}\varepsilon^s\mathcal{D}^{\frac{3s}{2}}a_3(\theta,\tau)\Big].$$

Therefore, the function $a_3(x,\tau)$ has to satisfy

$$i\partial_\tau a_3 + \mathcal{C}_{\frac{s}{2}}\mathcal{D}^{\frac{s}{2}}a_3 + \mathcal{C}_s\mathcal{D}^s a_2 + \mathcal{C}_{\frac{3s}{2}}\mathcal{D}^{\frac{3s}{2}}a_1(\theta,\tau) + (-d_x^2)^s a_0.$$

Furthermore, we have identified both the parameters p and β. Thus, we can iterate the procedure described above and we find the following expression for the quasi-solution z^ε

$$z^\varepsilon(x,t) = c(\varepsilon)e^{i\left[\xi_0\varepsilon^{-1}x + |\xi_0|^{2s}\varepsilon^{-2s}t\right]}\sum_{j\geq 0}\varepsilon^{\frac{s}{2}j}a_j\left(x,\varepsilon^{\frac{3}{2}s}t\right), \tag{3.10}$$

where the functions a_j, $j\geq 0$, are the solutions of the following cascade system

$$\begin{cases} i\partial_\tau a_0 + \mathcal{C}_{\frac{s}{2}}\mathcal{D}^{\frac{s}{2}}a_0 = 0 & \\ i\partial_\tau a_1 + \mathcal{C}_{\frac{s}{2}}\mathcal{D}^{\frac{s}{2}}a_1 + \mathcal{C}_s\mathcal{D}^s a_0 = 0 & \\ i\partial_\tau a_2 + \mathcal{C}_{\frac{s}{2}}\mathcal{D}^{\frac{s}{2}}a_2 + \mathcal{C}_s\mathcal{D}^s a_1 + \mathcal{C}_{\frac{3s}{2}}\mathcal{D}^{\frac{3s}{2}}a_0(\theta,\tau) = 0, & x\leq\theta\leq x+\dfrac{\varepsilon}{\xi_0}q \\ i\partial_\tau a_j + \mathcal{C}_{\frac{s}{2}}\mathcal{D}^{\frac{s}{2}}a_j + \mathcal{C}_s\mathcal{D}^s a_{j-1} + \mathcal{C}_{\frac{3s}{2}}\mathcal{D}^{\frac{3s}{2}}a_{j-2}(\theta,\tau) + (-d_x^2)^s a_{j-3}, & j\geq 3 \\ & x\leq\theta\leq x+\dfrac{\varepsilon}{\xi_0}q. \end{cases} \tag{3.11}$$

Notice that the classical Borel's theorem (see, e.g., [11, Chapter I, Theorem 1.2.6]) allows one to choose a C^∞-smooth function $a^\varepsilon(x,\tau)$ which has the expansion at $\varepsilon=0$

$$a^\varepsilon(x,\tau) = \sum_{j\geq 0}\varepsilon^{\frac{s}{2}j}a_j(x,\tau).$$

This, in particular, justifies the formal computations presented above. Consequently, we conclude that the function $z^\varepsilon(x,t)$ constructed in eq. (3.10) is C^∞-smooth and it is an infinitely accurate solution of the fractional Schrödinger equation in the sense that $\mathcal{P}_s z^\varepsilon = O(\varepsilon^\infty)$ in $\mathbb{R}\times(0,+\infty)$.

For concluding our construction, let us now compute the value of the normalization constant $c(\varepsilon)$. This is done asking that $\|z^\varepsilon\|_{H^s(\mathbb{R})} = \mathcal{O}(1)$ as $\varepsilon\to 0^+$. First of all, from the construction that we just presented we get

$$z^\varepsilon = c(\varepsilon)e^{i\left[\xi_0\varepsilon^{-1}x + |\xi_0|^{2s}\varepsilon^{-2s}t\right]}\left(a_0 + \mathcal{O}(\varepsilon^{\frac{s}{2}})\right).$$

Hence, in what follows we can consider only the term for $j=0$ in our expansion. Moreover, since we are working on the whole $\mathbb{R}$, we have

$$\begin{aligned}\|z^\varepsilon\|_{H^s(\mathbb{R})} &= \left(\|z^\varepsilon\|^2_{L^2(\mathbb{R})} + \left\|(-d_x^2)^{\frac{s}{2}}z^\varepsilon\right\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}} \\ &= \left(\|c(\varepsilon)u^\varepsilon a_0\|^2_{L^2(\mathbb{R})} + \left\|c(\varepsilon)u^\varepsilon\left(\frac{|\xi_0|^s}{\varepsilon^s}a_0 + (-d_x^2)^{\frac{s}{2}}a_0 + R\right)\right\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}},\end{aligned}$$

where, as we did before, the reminder term R can be written in the form

$$R = \frac{1}{\varepsilon^s}\frac{|\xi_0|^s}{\Gamma(1+s)}\left(P.V. \int_{\mathbb{R}} \frac{1-e^{iq}}{|q|^{1+2s}} q^s \, dq\right) \mathcal{D}^s a_0 = \frac{c(s)}{\varepsilon^s}\mathcal{D}^s a_0.$$

Therefore, we obtain

$$\|z^{\varepsilon}\|_{H^s(\mathbb{R})} = \left(\|c(\varepsilon)u^{\varepsilon}a_0\|^2_{L^2(\mathbb{R})} + \left\|c(\varepsilon)u^{\varepsilon}\left[\varepsilon^{-s}\left(|\xi_0|^s a_0 + c(s)\mathcal{D}^s a_0\right) + (-d_x^2)^{\frac{s}{2}}a_0\right]\right\|_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}.$$

Thus, choosing the normalization constant as $c(\varepsilon) = \varepsilon^s$, we immediately obtain

$$\begin{aligned}
\|z^{\varepsilon}\|_{H^s(\mathbb{R})} &= \left(\varepsilon^s\|u^{\varepsilon}a_0\|^2_{L^2(\mathbb{R})} + \left\|u^{\varepsilon}\left(|\xi_0|^s a_0 + c(s)\mathcal{D}^s a_0 + \varepsilon^s(-d_x^2)^{\frac{s}{2}}a_0\right)\right\|_{L^2(\mathbb{R})}\right)^{\frac{1}{2}} \\
&= \left(\varepsilon^s\|a_0\|^2_{L^2(\mathbb{R})} + \left\||\xi_0|^s a_0 + c(s)\mathcal{D}^s a_0 + \varepsilon^s(-d_x^2)^{\frac{s}{2}}a_0\right\|_{L^2(\mathbb{R})}\right)^{\frac{1}{2}} = \mathcal{O}\big(\|a_0\|_{H^s(\mathbb{R})}\big).
\end{aligned}$$

In this way, we find the final expression for our quasi-solution, which reads as follows

$$z^{\varepsilon}(x,t) = \varepsilon^s e^{i\left[\xi_0\varepsilon^{-1}x + |\xi_0|^{2s}\varepsilon^{-2s}t\right]}\sum_{j\geq 0}\varepsilon^{\frac{s}{2}j}a_j\left(x, \varepsilon^{\frac{3}{2}s}t\right). \tag{3.12}$$

Moreover, eq. (3.11) is uniquely solvable with initial conditions imposed at $t = 0$ and this, of course, allows to identify the expressions of the functions a_j. In more detail, it is possible to compute quasi-solutions to the initial value problem

$$\begin{cases} iu_t + (-d_x^2)^s u = 0, & (x,t) \in \mathbb{R}\times(0,+\infty) \\ u(x,0) = u_0(x), \end{cases} \tag{3.13}$$

by means of the following procedure.

Step 1. $j = 0$

Given any $g_0 \in L^2(\mathbb{R})$, we start by considering the equation

$$\begin{cases} i\partial_\tau a_0 + \mathcal{C}_{\frac{s}{2}}\mathcal{D}^{\frac{s}{2}}a_0 = 0, & (x,\tau) \in \mathbb{R}\times(0,+\infty) \\ a_0(x,0) = g_0(x). \end{cases} \tag{3.14}$$

Recall that $\tau = \varepsilon^{\frac{3}{2}s}t$. Moreover, it is classical that the solution to eq. (3.14) can be computed explicitly and it is given by

$$a_0(x,\tau) = \int_{\mathbb{R}} \mathcal{G}(y,\tau)g_0(x-y)\, dy, \tag{3.15}$$

where with $\mathcal{G}$ we indicated the Green function defined as the solution to

$$\begin{cases} i\partial_\tau \mathcal{G} + \mathcal{C}_{\frac{s}{2}}\mathcal{D}^{\frac{s}{2}}\mathcal{G} = 0, & (x,\tau) \in \mathbb{R}\times(0,+\infty) \\ \mathcal{G}(x,0) = \delta(x). \end{cases} \tag{3.16}$$

Equation eq. (3.16) can be easily solved with the help of the Fourier transform. With this purpose, let us recall that we have (see, e.g., [22, Page 59, Equation A.13])

$$\mathcal{F}\left[\mathcal{D}^{\frac{s}{2}}a_0\right](k,\tau) = -|k|^{\frac{s}{2}}\mathcal{F}\left[a_0\right](k,\tau)$$

In particular, the function $\mathcal{G}$ is given by

$$\mathcal{G}(x,\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx} e^{-i\mathcal{C}_{\frac{s}{2}}|k|^{\frac{s}{2}}\tau}\, dk.$$

and we then get from eq. (3.15) the following expression for the solution to eq. (3.14)

$$a_0(x,\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{ikx} e^{-i\mathcal{C}_{\frac{s}{2}}|k|^{\frac{s}{2}}\tau}\, dk\right) g_0(x-y)\, dy.$$

Step 2. $j = 1$

Once the expression of a_0 is determined, the next component in the expansion, corresponding to $j = 1$ in eq. (3.12), is obtained solving the non-homogeneous equation

$$\begin{cases} i\partial_\tau a_1 + \mathcal{C}_{\frac{s}{2}}\mathcal{D}^{\frac{s}{2}} a_1 = h, & (x,\tau) \in \mathbb{R}\times(0,+\infty) \\ a_1(x,0) = g_1(x), \end{cases} \tag{3.17}$$

where we indicated with h the function

$$h = h(x,\tau) = -\mathcal{C}_s \mathcal{D}^s a_0(x,\tau).$$

Moreover, also the solution of eq. (3.17) can be obtained explicitly employing classical splitting techniques and writing $a_1(x,t) = a_{1,1}(x,t) + a_{1,2}(x,t)$ with

$$\begin{cases} i\partial_\tau a_{1,1} + \mathcal{C}_{\frac{s}{2}}\mathcal{D}^{\frac{s}{2}} a_{1,1} = h, & (x,\tau) \in \mathbb{R}\times(0,+\infty) \\ a_{1,1}(x,0) = 0 \end{cases} \tag{3.18}$$

and

$$\begin{cases} i\partial_\tau a_{1,2} + \mathcal{C}_{\frac{s}{2}}\mathcal{D}^{\frac{s}{2}} a_{1,2} = 0, & (x,\tau) \in \mathbb{R}\times(0,+\infty) \\ a_{1,2}(x,0) = g_1(x). \end{cases} \tag{3.19}$$

Notice that the solution to eq. (3.18) can be obtained through the variation of constants formula, while the solution to eq. (3.19) is computed as in Step 1.

Step 3. $j \geq 2$

Starting from $j = 2$, in the equations determining a_j appears terms in the variable θ, coming from the remainders of the fractional Taylor expansion. Notice, however, that the support of θ is the interval $[x, x + \varepsilon q/\xi_0]$ and that we are interested in analyzing the behavior of the quasi-solutions as $\varepsilon \to 0^+$. Hence, without introducing significative errors, we can assume $\theta = x$. Therefore, we obtain that each a_j, $j \geq 2$, is the solution to the non-homogeneous problem

$$\begin{cases} i\partial_\tau a_j + \mathcal{C}_{\frac{s}{2}}\mathcal{D}^{\frac{s}{2}} a_j = H_j, & (x,\tau) \in \mathbb{R}\times(0,+\infty) \\ a_j(x,0) = g_j(x), \end{cases} \tag{3.20}$$

where the right hand sides H_j are determined in terms of the functions a_i, $i = 0, \ldots, j-1$. In particular, eq. (3.20) can be solved again as we did fro a_1 in Step 2, and we thus obtain explicit expressions for all the functions a_j, $j \geq 0$.

4 Localization of the quasi-solutions along rays

This section is devoted to showing that the quasi-solutions that can be computed by using the ansatz that we obtained are in fact localized along rays.

Theorem 4.1. *Let $u_{in} \in L^2(\mathbb{R})$ and let z^ε be constructed employing the expansion eq.* (3.12)*, with initial data $g_j \in L^2(\mathbb{R})$. Then, for any $\varepsilon > 0$ we have:*

1. *The functions z^ε are approximate solutions to eq.* (1.1)*:*

$$\|u_0(x) - z^\varepsilon(x,0)\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^{\frac{1}{2}}), \tag{4.1}$$

$$\|u(x,t) - z^\varepsilon(x,t)\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^{\frac{1}{2}}). \tag{4.2}$$

2. *The initial energy of z^ε remains bounded as $\varepsilon \to 0$, i.e.*

$$\|z^\varepsilon(x,0)\|^2_{H^s(\mathbb{R})} \approx 1. \tag{4.3}$$

3. *The energy of z^ε is exponentially small off the ray $(t, x(t))$:*

$$\int_{|x-x(t)|>\varepsilon^{\frac{1}{4}}} \left|(-d_x^2)^{\frac{s}{2}} z^\varepsilon(x,t)\right|^2 \, dx = \mathcal{O}(\varepsilon^{\frac{1}{4}}). \tag{4.4}$$

Proof. **Step 1: Approximation of the real solution.** First of all, from the definition eq. (3.12) of z^ε we have

$$z^\varepsilon(x,0) = e^{i\frac{\xi_0}{\varepsilon}x} \sum_{j\geq 0} \varepsilon^{\frac{s}{2}j} a_j(x,0) = e^{i\frac{\xi_0}{\varepsilon}x} \sum_{j\geq 0} \varepsilon^{\frac{s}{2}j} g_j(x).$$

By means of the above expression we obtain

$$\begin{aligned}
\|u_0 - z^\varepsilon(x,0)\|_{L^2(\mathbb{R})} \quad &= \left\|u_0 - e^{i\frac{\xi_0}{\varepsilon}x}\left(g_0 + \mathcal{O}(\varepsilon^{\frac{s}{2}})\right)\right\|_{L^2(\mathbb{R})} = \left\|e^{i\frac{\xi_0}{\varepsilon}x}\Big[u_{\text{in}} - \left(g_0 + \mathcal{O}(\varepsilon^{\frac{s}{2}})\right)\Big]\right\|_{L^2(\mathbb{R})} \\
&\leq \left(P.V. \int_{\mathbb{R}} \left|e^{i\frac{\xi_0}{\varepsilon}x}\right| \, dx\right)^{\frac{1}{2}} \left(\|u_{\text{in}} - g_0\|_{L^2(\mathbb{R})} + \mathcal{O}(\varepsilon^{\frac{s}{2}})\right) \\
&= \mathcal{O}(\varepsilon^{\frac{1}{2}}) \left(\|u_{\text{in}} - g_0\|_{L^2(\mathbb{R})} + \mathcal{O}(\varepsilon^{\frac{s}{2}})\right).
\end{aligned}$$

Therefore, since both u_{in} and g_0 belong to $L^2(\mathbb{R})$, we immediately have eq. (4.1). In order to obtain eq. (4.2), let us firstly remark that, by means of classical PDE techniques, we can obtain the following energy estimate for the solution to eq. (1.1) (see, e.g., [7])

$$\|u(t)\|_{L^2(\mathbb{R})} \leq C\left(\|u_0\|_{L^2(\mathbb{R})} + \|\mathcal{P}_s u(t)\|_{L^2(\mathbb{R})}\right). \tag{4.5}$$

Moreover, notice that since $\mathcal{P}_s$ is linear, by construction of z^ε we have

$$\mathcal{P}_s(u - z^\varepsilon) = \mathcal{P}_s u - \mathcal{P}_s z^\varepsilon = -\mathcal{P}_s z^\varepsilon = \mathcal{O}(\varepsilon^\infty).$$

Therefore, applying eq. (4.5) we immediately get

$$\|u(x,t)-z^{\varepsilon}(x,t)\|_{L^2(\mathbb{R})} \leq C\left(\|u_0-z^{\varepsilon}(x,0)\|_{L^2(\mathbb{R})}+\|\mathcal{P}_s(u-z^{\varepsilon})(t)\|_{L^2(\mathbb{R})}\right)=\mathcal{O}(\varepsilon^{\frac{1}{2}})+\mathcal{O}(\varepsilon^{\infty}),$$

and this clearly yields eq. (4.2).

Step 2: Initial energy estimate. Repeating the computations developed in Section 3 for deriving the appropriate value of the normalization constant $c(\varepsilon)$, we can easily check that

$$\|z^{\varepsilon}(x,0)\|^2_{L^2(\mathbb{R})} \approx \|g_0\|^2_{H^s(\mathbb{R})}.$$

Hence, up to a rescaling in the initial datum $g_0 \mapsto g_0/\|g_0\|_{H^s(\mathbb{R})}$, we have eq. (4.3).

Step 3: Localization along rays. Following the same approach that we used in precedence, and employing the change of variables $\varepsilon^{\frac{1}{4}}(x-x(t)) \mapsto z$, we have

$$\begin{aligned}
&\int_{|x-x(t)|>\varepsilon^{\frac{1}{4}}} \left|(-d_x^2)^{\frac{s}{2}} z^{\varepsilon}\right|^2 \, dx \\
&\qquad \approx \varepsilon^{\frac{1}{4}} \int_{|z|>1} \left| e^{i\left[\frac{\xi_0}{\varepsilon}\left(x(t)+\varepsilon^{\frac{1}{4}}z\right)+\left(\frac{|\xi_0|}{\varepsilon}\right)^{2s}t\right]} \left(|\xi_0|^s a_0 + c(s)\mathcal{D}^s a_0 + \varepsilon^s(-d_x^2)^{\frac{s}{2}}a_0\right)\right|^2 \, dz \\
&\qquad \leq \varepsilon^{\frac{1}{4}} \max\left\{|\xi_0|^s, c(s)\right\}\|a_0\|_{H^s(\mathbb{R})} + \varepsilon^{\frac{1}{4}+s}\|a_0\|_{H^s(\mathbb{R})} = \mathcal{O}(\varepsilon^{\frac{1}{4}}).
\end{aligned}$$

This concludes the proof. □

5 Application to the analysis of control properties

It is by now well known (see, e.g., [2, 23]) that geometric optics constructions for wave-like equations can be used for deriving controllability properties. In this Section, we present an informal discussion on the application of the WKB construction obtained in this paper to the null-controllability of equation eq. (1.1).

According to Theorem 4.1, the quasi-solution z^{ε} to our original system are concentrated along the rays. Therefore, they propagate with the group velocity of the plane wave solutions. Moreover, this group velocity can be analyzed in terms of s and of the frequency ξ_0. Before doing that, let us first recall that the rays for our equation have been defined in Section 2 as

$$x(t) = x_0 \pm 2s|\xi_0|^{2s-1}t.$$

Moreover, without losing generality, we may assume here that $x_0 = 0$.

We recall that the group velocity of the plane wave u^{ε} can be easily computed as follows: first of all, rewrite

$$u^{\varepsilon}(x,t) = e^{i\left[\xi_0\varepsilon^{-1}x+|\xi_0|^{2s}\varepsilon^{-2s}t\right]} = e^{it\left[\xi_0\varepsilon^{-1}\left(\frac{x}{t}\right)+|\xi_0|^{2s}\varepsilon^{-2s}\right]} = e^{it\phi(x,t,\xi_0,\varepsilon)}.$$

Hence, $v := |x/t|$ is obtained solving the equation

$$\frac{\partial\phi}{\partial\xi_0} = 0.$$

In our particular case, we have

$$\frac{\partial \phi}{\partial \xi_0} = \varepsilon^{-1}\frac{x}{t} + 2s\varepsilon^{-2s}|\xi_0|^{2s-1}\text{sgn}(\xi_0),$$

and we immediately find that

$$v = \left|\frac{x}{t}\right| = 2s\varepsilon^{1-2s}|\xi_0|^{2s-1}.$$

Let us now analyze the behavior of v with respect to s. Firstly, we immediately see that for $s = 1/2$ we have

$$v = 1,$$

i.e. the velocity is constant and independent of the frequency ξ_0. For $s \in (0, 1/2)$, instead, we have that $1 - 2s > 0$. Hence, taking $\varepsilon < 1$, we easily get

$$v < |\xi_0|^{2s-1}.$$

Finally, for $s \in (1/2, 1)$ the situation is the opposite. We have $1 - 2s < 0$ and, for $\varepsilon < 1$,

$$v > |\xi_0|^{2s-1}.$$

In view of these behaviors, we can conclude that:

- For $s > 1/2$, the group velocity increases with the frequency.
- For $s = 1/2$, the group velocity remains constant.
- For $s < 1/2$, the group velocity decreases with the frequency.

Therefore, the high frequency solutions are traveling faster and faster, for $s > 1/2$, and slower and slower, for $s < 1/2$. As a consequence,

- For $s > 1/2$, the velocity of propagation of the rays allow them to be observable in any finite time $T > 0$.
- For $s = 1/2$, the velocity of propagation being constant, a minimum observation time T_0 is needed.
- for $s < 1/2$ the high frequency rays may not reach the control region, thus implying the failing of controllability properties.

This, in particular, confirms the already known results presented in [3].

6 Numerical results

We conclude this paper with some simulations showing the propagation of solutions to the fractional Schrödinger equation eq. (1.1) corresponding to initial data in the form eq. (1.2).

For the numerical resolution of the equation, we employed a uniform mesh in the space variable and a FE discretization of the fractional Laplacian, obtained following the methodology presented in [4]. Moreover, we used a Crank-Nicholson scheme in time, which is known to be stable for the Schrödinger equation (see, e.g., [1]). The initial data u_0 has been chosen as

$$u_0(x) = e^{-\frac{\gamma}{2}(x-x_0)^2} e^{i\frac{\xi_0}{\varepsilon}x},$$

where the profile $u_{\text{in}}(x)$ is given by a Gaussian with standard deviation measured in terms of the parameter γ, which is related to the mesh size h. In particular we chose $\gamma = h^{0.9}$. Finally, for the oscillations we considered frequencies $\xi_0 = \pi^2/16$ and $\xi_0 = 2\pi^2$.

In Figure 1, we show the plots for $\xi_0 = 2\pi^2$ and different values of $s \in (0,1)$. The space domain has been chosen to be the interval $(-1,1)$, while we considered a time interval of 5 seconds.

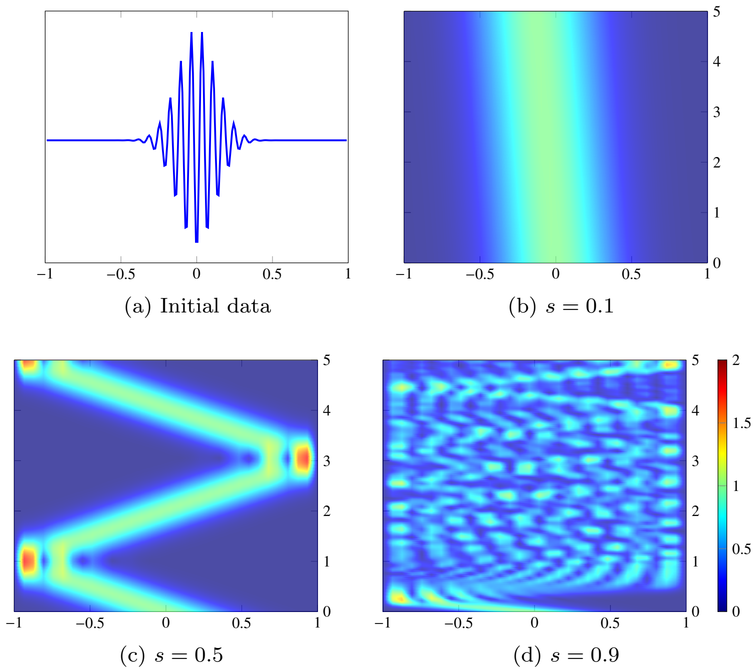


Figure 1: Propagation of the solution for $\xi_0 = 2\pi^2$ and different values of s. Available in color online.

It is seen there that, for small values of s, say $s = 0.1$, the solution remains concentrated along rays which propagate only in the vertical direction. In other words, there is no propagation in space and, as we mentioned before, this implies that it will not be possible to control these solutions, no matter how one places the controls. For $s = 0.5$, instead, the plots show that the solutions propagate along rays which reach the boundary of the space domain in finite time, and are reflected according to the laws of optics. This translate in the fact that, provided that the time is large enough, it will be possible to control these solutions, acting with a distributed in a neighborhood ω of the boundary. The case of high values of the power s of the fractional Laplacian is the most puzzling one. For instance, for $s = 0.9$ our plots seem to show a lost of concentration of the solution along the ray, while our theoretical results would suggest that this concentration is preserved. On the other hand, we believe that what the simulations are showing is not in contradiction with the theory. We are going to discuss this issue in more details later.

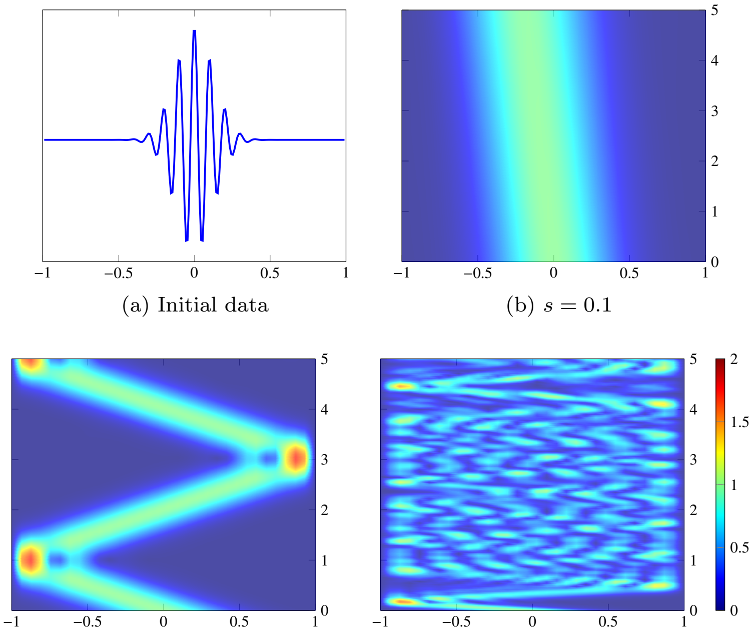
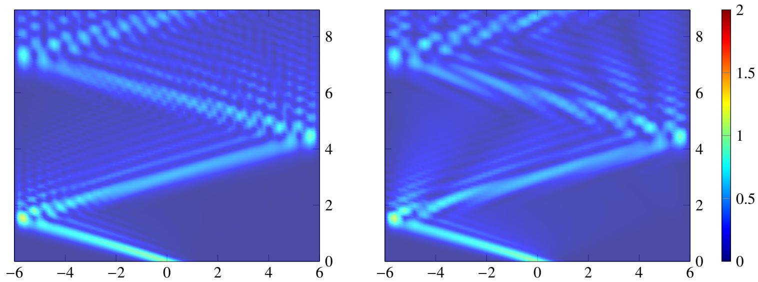


Figure 2: Propagation of the solution for $\xi_0 = \pi^2/16$ and different values of s. Available in color online.

In Figure 2, the simulations have been run with an initial datum with frequency $\xi_0 = \pi^2/16$. The plots obtained show a behavior which is totally analogous with what observed in Figure 1:

- For $s = 0.1$, the solutions are once again concentrated along vertical rays, without propagation in time and, therefore, without possibility of being controlled.
- For $s = 0.5$, we have propagation with constant velocity, and the ray reaches the boundary in finite time.
- For $s = 0.9$ the chaotic comportment is still present.

Once again, the most surprising case is the last one, for $s > 0.5$, in which the simulations seem to display dispersive features. Nevertheless, as we mentioned before, we retain that this does not contradict the results of Section 4. In our opinion, this strange phenomenon appearing in the plot can be explained with the accumulation of higher order terms in the asymptotic expansion of z^ε which, combined with the small size of the space interval considered, enhance a chaotic behavior. This interpretation is supported by the fact that, as it is shown in Figure 3, enlarging the space domain up to $(-6, 6)$ seems to fix the problem and the localization of the solution along the rays appears once again.



(a) $\xi_0 = 2\pi^2$ (b) $\xi_0 = \pi^2/16$

Figure 3: Propagation of the solution for $s = 0.9$ on the space interval $(-6, 6)$. Available in color online.

7 Acknowledgments

The authors wish to acknowledge Enrique Zuazua (Autonomous University of Madrid, DeustoTech and University of Deusto - Bilbao, and University Pierre and Marie Curie - Paris) for having suggested the topic of this research and for interesting discussions. A special thank goes to Aurora Marica (Polytechnic University of Bucharest) for her help for obtaining the simulations presented in this work. The second author (AA) thanks DeustoTech and the University of Deusto for hosting him in during summer of 2017, where part of this work was performed.

The work of both authors was supported by the Advanced Grant DYCON (Dynamic Control) of the European Research Council Executive Agency.

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