Spatial discretisation of dynamical systems

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with *Phil Diamond, Peter Imkeller, Jamie Mustard, Alexei Pokrovskii*

• P. Diamond, P.E. Kloeden and A. Pokrovskii,

Interval stochastic matrices, a combinatorial lemma, and the computation of invariant measures, *J. Dynamics & Diff. Eqns.* **7** (1995), 341–364.

• P. Imkeller and P.E. Kloeden,

On the computation of invariant measures in random dynamical systems, *Stochastics & Dynamics* **3** (2003), 247–265.

Spatial discretisation

Consider a continuous mapping

$$f : X \to X \tag{1}$$

on a compact metric space (X, d).

The difference equation

$$x_{n+1} = f(x_n) \tag{2}$$

generates a discrete time dynamical system on X.

Consider a finite subset X_h of X with grid fineness

$$\Delta_h := \sup_{x \in X} \inf_{x_h \in X_h} d(x, x_h)$$

Examples

- $X = [0,1]$, $\qquad X_h = 2^N$-bit computer numbers in $[0,1]$
- $X = [0,1]$, $\qquad X_h = \left\{ \frac{j}{2^N} : j = 0, 1, \ldots, N \right\}$

N-dyadic numbers

Consider a "projection" $P_h : X \to X_h$, e.g. round-off operator

The mapping $f_h := P_h \circ f : X_h \to X_h$ generates a discrete time dynamical system on X_h through the difference equation

$$\boxed{x_{n+1}^{(h)} = f_h\left(x_n^{(h)}\right)} \tag{3}$$

What is the relationship between the dynamical behaviour of the original dynamical system (2) and the spatially discretised system (3) as

$$\Delta_h \to 0 \ ?$$

Plan

• the effect of spatial discretisation on attractors

• the effect of spatial discretisation on chaos

• the approximation of Lebesgue measure preserving maps on a torus by permutations

• approximation by Markov chains of invariant measures of spatial discretised

i) deterministic difference equations

ii) random difference equations

Spatial discretisation of attractors

• P. Diamond and P. E. Kloeden,

Spatial discretisation of mappings, *J. Computers Math. Applns.* **26** (1993), 85-94.

• P. E. Kloeden and J. Lorenz,

Stable attracting sets in dynamical systems and in their one-step discretisations,

SIAM J. Numer. Analysis **23** (1986), 986-995.

Assume that

- $f : X \to X$ is Lipschitz with constant $K > 0$
- the projection $P_h : X \to X_h$ satisfies for a constant $M > 0$

$$d\left(P_h(x), x\right) \leq Mh$$

Theorem 1 [Diamond & Kloeden]

Suppose that a nonempty compact subset L of X is uniformly asymptotically stable (UAS) for the dynamical system f on X.

Then there exists a nonempty compact subset L_h of X_h which is UAS for the dynamical system $f_h := P_h \circ f$ on X_h such that the Hausdorff distance

$$H(L_h, L) \to 0 \quad \text{as} \quad h \to 0+$$

Sketch of proof

The UAS of the set L for the system f implies that there exists a Lyapunov function

$$V : X \to \mathbb{R}^{+},$$

which is Lipschitz continuous, and a constant $0 < q < 1$ such that

$$V(f(x)) \leq q\,V(x), \qquad \forall x \in X.$$

Then the discretised system satisfies the key inequality

$$\boxed{V\left(f_h(x_h)\right) \le q\,V(x_h) + KMh} \qquad \forall\, x_h \in X_h$$

Define

$$L_h := \left\{ x_h \in X_h \,:\, V\left(x_h\right) \le \frac{2KMh}{1-q} \right\},$$

which is a nonempty, compact subset of X_h for all $h > 0$.

The key inequality and other properties of the Lyapunov function V imply that L_h is UAS for f_h on X_h and satisfies the convergence asserted in the theorem.

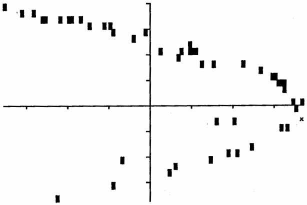

Figure 1. Λ_h : $h = 0.025$.

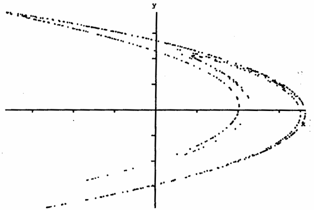

Figure 2. Λ_h : $h = 0.005$.

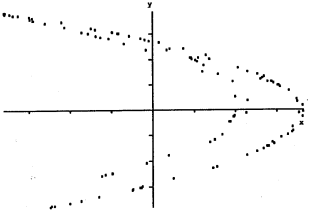

Figure 3. Λ_h : $h = 0.0005$.

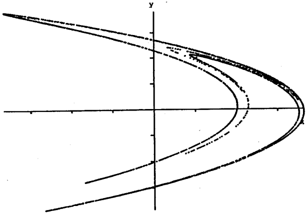

Figure 4. Λ_h : $h =$ double precision.

Fig. 1 consists of stable cycles of periods 4, 11 and 33

Fig. 3 consists of stable cycles of periods 30 and 78

Complications

• a fixed point $f(\bar{x}) = \bar{x} \in X$ need not belong to X_h

• if such a fixed point $\bar{x} \in X_h$, then it need not be a fixed point of f_h.

• f_h may have spurious cycles in X_h, i.e. periodic solutions which do not correspond to periodic solutions of f.

In fact, the dynamics of f_h on X_h is always eventually periodic

Moreover, the convergence $H(L_h, L) \to 0$ as $h \to 0$ is deceptive

• the attracting set L_h of f_h may contains transients as well as limit points and cycles

• it is better to consider the omega set of limiting values

$$L_h^* := \bigcap_{j \geq 1} \overline{\bigcup_{n \geq 1} f_h^j(L_h)},$$

i.e. the global attractor, which may be a proper subset of L_h.

Without additional assumptions about the dynamics of f on L such as hyperbolicity, we only have the weaker convergence in the Hausdorff semi-distance

$$H^*(L_h^*, L) := \max_{x_h \in L_h^*} d(x_h, L) \to 0 \quad \text{as} \quad h \to 0+$$

the effect can be extreme

Example

$\star$ Consider the extended tent mapping $f : [0,2] \to [0,2]$ defined by

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2(1-x) & \text{if } \frac{1}{2} \le x \le 1 \\ 0 & \text{if } 1 \le x \le 2 \end{cases}$$

which has the chaotic attractor $L = [0,1]$.

* Consider the N-dyadics

$$X_h := \left\{ \frac{j}{2^N}, 1 + \frac{j}{2^N} \ : \ j = 0, 1, \ldots, N \right\}, \qquad h = 2^{-N}.$$

Since $f : X_h \to X_h$, here we take $f_h \equiv f$.

$$\boxed{f_h^N(x_h) = 0, \quad \forall\, x_h \in X_h} \qquad \Longrightarrow \qquad L_h^* = \{0\}$$

the chaos has collapsed onto trivial behaviour

This collapsing effect is not exceptional

Theorem 2 [Diamond, Kloeden & Pokrovskii]

For any continuous $f : X \to X$ *and any cycle* $\{c_1, \ldots, c_p\}$ *of* f *there exists a finite subset* X_h *of* X *which contains* $\{c_1, \ldots, c_p\}$ *and a mapping* $f_h : X_h \to X_h$ *for* $h \to 0$ *such that the dynamics of* f_h *collapses on* $\{c_1, \ldots, c_p\}$.

P. Diamond, P.E. Kloeden und A. Pokrovskii,
Cycles of spatial discretisations of shadowing dynamical systems,
Mathematische Nachrichten **171** (1995), 95–110.

Invariant measures

- allow us to circumvent some of the above difficulties with attractors and cycles

- are robuster for approximation and comparison

A measure μ *on* X *is called* *f-invariant* *if*

$$\mu(B) = \mu\left(f^{-1}(B)\right), \quad \forall\, B \in \mathcal{B}(X),$$

for the Borel subsets $\mathcal{B}(X)$ of X, where

$$f^{-1}(B) := \{x \in X \,:\, f(x) \in B\}$$

• *Can we always approximate an invariant measure μ of f on X by an invariant measure μ_h of f_h on X_h?*

<u>how?</u>

SPECIAL CASE: mappings on a torus

Consider

- $\diamond$ a d-dimensional torus $\mathbb{T}^d$, where $d \geq 1$,
- $\diamond$ a <u>measurable</u> mapping $f : \mathbb{T}^d \to \mathbb{T}^d$;
- $\diamond$ a uniform $\frac{1}{N}$ partition $\mathbb{T}^d_N$ of $\mathbb{T}^d$.

How should we construct a mapping f_N on $\mathbb{T}^d_N$ to approximate f?

Theorem 3 [Kloeden & Mustard]
Suppose that the Lebesgue measure on $\mathbb{T}^d$ is f-invariant.
Then there exists a permutation $P_N(f)$ on $\mathbb{T}^d_N$ with

$$H^* \left(\mathrm{Gr}(P_N(f)), \mathrm{Gr}(f)\right) \le \frac{1}{N}$$

where H^* is the Hausdorff semi-distance on $\mathbb{T}^d \times \mathbb{T}^d$ and $\mathrm{Gr}(f)$ is the graph of f defined by

$$\mathrm{Gr}(f) := \left\{(x, y) \in \mathbb{T}^d \times \mathbb{T}^d \; : \; y = f(x)\right\}$$

P.E. Kloeden and J. Mustard,

Construction of permutations approximating Lebesgue measure preserving dynamical systems under spatial discretisation.

J. Bifurcation & Chaos **7** (1997), 401–406.

Comments

- f can be non-injective here, i.e. not 1 to 1
- the inverse of the theorem holds if f is continuous
- Peter Lax has an theorem about permutations approximating area-preserving diffeomorphisms

Outline of proof

- enumerate $\mathbb{T}_N^d = \{x_1, \ldots, x_M\}$, where $M = N^d$
- define the $\frac{1}{N}$-band about the graph $\mathrm{Gr}(f)$ of f, i.e.

$$S_N(f) := \left\{(x,y) \in \mathbb{T}_N^d \times \mathbb{T}_N^d \,:\, \mathrm{dist}\left((x,y), \mathrm{Gr}(f)\right) \leq \frac{1}{N}\right\}$$

The following problems are equivalent by the f-invariance of the Lebesgue measure and a combinatorial theorem of Frobenius and König, *(1) construct a permutation* $P_N(f)$ *on* $\mathbb{T}^d_N$ *with* $Gr(P_N(f)) \subseteq S_N(f)$.

(2) choose a diagonal (possibly permuted) without zeros of the $M \times M$ *matrix* $A_N(f) = [a_{i,j}]$ *defined by*

$$a_{i,j} = \begin{cases} 1 & \text{if } (x_i, x_j) \in S_N(f) \\ 0 & \text{otherwise} \end{cases}$$

reformulate the problem as an optimal assignment LP problem

GENERAL CASE: using Markov chains

Consider a finite subset $X_N = \{x_1^{(N)}, \ldots, x_N^{(N)}\}$ of a compact metric space (X, d) with fineness parameter

$$h_N := \Delta_N := \sup_{x \in X} \inf_{x_j^{(N)} \in X_N} d\left(x, x_j^{(N)}\right) \to 0 \quad \text{as } N \to \infty$$

How do we construct an approximation f_N on X_N of a function $f : X \to X$?

The choice is usually not unique: there may be several nearest grid points to an $f(x_j^{(N)}) \notin X_N$.

There are two ways to handle the problem:

1) setvalued: use a setvalued mapping

$$F_N(x_j^{(N)}) := \left\{ \text{nearest points in } X_N \text{ to } f(x_j^{(N)}) \right\}$$

and then consider the setvalued dynamical system $x_{n+1} \in F_N(x_n)$ on X_N.

2) stochastic: use a Markov chain P_N on X_N with transition probabilities

$$p_{i,j}^{(N)} = \begin{cases} > 0 & \text{if } x_i^{(N)} \text{ in a neighbourhod of } f(x_j^{(N)}) \\ 0 & \text{otherwise} \end{cases}$$

Distances

- between a Markov chain P_N on $X_N \subset X$ and a mapping $f : X \to X$

$$D(P_N, f) := \max_{1 \leq i \leq N} \sum_{j=1}^{N} p_{i,j}^{(N)} \text{dist}\left(\left(x_i^{(N)}, x_j^{(N)}\right), \text{Gr}(f)\right)$$

- between a probability vector p_N on X_N and a probability measure μ on X

Prokhorov metric $\quad \rho(\mu_N, \mu)$

where μ_N is the extension of p_N to a measure on X.

Let $f : X \to X$ be <u>Borel measurable</u> and consider the generalized inverse

$$\widetilde{f^{-1}}(B) := \left\{ x \in X \,:\, \exists\, y \in \overline{B} \text{ with } (x,y) \in \overline{\mathrm{Gr}(f)} \right\}$$

A Borel measure μ on X is called <u>f-semi-invariant</u> if

$$\boxed{\mu(B) \leq \mu\left(\widetilde{f^{-1}}(B)\right), \qquad \forall\, B \in \mathcal{B}(X)}$$

f <u>continuous</u> $\quad \Longrightarrow \quad$ f-semi-invariant $\quad \equiv \quad$ f-invariant

Theorem 4 [Diamond, Kloeden & Pokrovskii]
A probability measure μ on X is f-semi-invariant if and only if it is stochastically approachable, i.e. for each N there exist

1) a grid X_N with fineness $\Delta_N \to 0$ as $N \to \infty$

2) a Markov chain P_N on X_N

3) probability measure μ_N on X corresponding to an equilibrium probability vector $\bar{p}_N$ of P_N on X_N, such that

$$D(P_N, f) \to 0, \quad \rho(\bar{\mu}_N, \mu) \to 0 \qquad \text{as} \quad N \to \infty$$

P. Diamond, P.E. Kloeden and A. Pokrovskii,
Interval stochastic matrices, a combinatorial lemma, and the computation of invariant measures, *J. Dynamics & Diff. Eqns.* **7** (1995), 341–364.

Key idea in the proof: interval stochastic matrices

An $N \times N$ matrix $C = [c_{i,j}]$ with nonnegative components is called

$$\left.\begin{array}{c}\text{substochastic}\\ \text{stochastic}\\ \text{superstochastic}\end{array}\right\} \text{ if } \sum_{j=1}^{N} c_{i,j} \quad \begin{cases} \leq & 1 \\ = & 1 \\ \geq & 1 \end{cases} \qquad \text{for } i = 1, \ldots, N.$$

Let $A = [a_{i,j}]$ be substochastic and $B = [b_{i,j}]$ be superstochastic. Then

$$\widehat{AB} := \{P \text{ stochastic } : a_{i,j} \leq p_{i,j} \leq b_{i,j}, \quad \forall\, i,j = 1,\ldots,N\}$$

is called an interval stochastic matrix with boundaries A and B.

- The (j, I)-flow of an interval stochastic matrix $\widehat{AB}$ is defined by

$$H_j\left(I, \widehat{AB}\right) := \min\left\{\sum_{i\in I} b_{i,j}, 1 - \sum_{i\notin I} a_{i,j}\right\},$$

where $j \in \{1,\ldots,N\}$ and $I \subset \{1,\ldots,N\}$

- A probability vector p_N on X_N is called <u>$\widehat{AB}$-semi-invariant</u> if the inequalities

$$\sum_{j=1}^{N} p_j H_j\left(I, \widehat{AB}\right) \geq \sum_{j=1}^{N} p_j$$

for every subset $I \subset \{1, \ldots, N\}$.

Lemma

A probability vector p_N on X_N is $\widehat{AB}$-semi-invariant if and only $p_N = p_N P_N$ for some $P_N \in \widehat{AB}$

In the proof of Theorem 4 we use

$$a_{i,j} \equiv 0, \qquad b_{i,j} = \begin{cases} 1 & \text{if } \operatorname{dist}\left(\left(x_i^{(N)}, x_j^{(N)}\right), \operatorname{Gr}(f)\right) \leq \frac{1}{N} \\ 0 & \text{otherwise} \end{cases}$$

i.e. we consider only those $\left(x_i^{(N)}, x_j^{(N)}\right) \in S_N(f)$, a $\frac{1}{N}$-neighbourhood of $\operatorname{Gr}(f)$.

$$\Longrightarrow \qquad H_j\left(I, \widehat{AB}\right) = \begin{cases} 1 & \text{if } b_{i,j} = 1 \text{ for some } i \in I \\ 0 & \text{otherwise} \end{cases}$$

Moreover, a probability vector p_N on X_N is $\widehat{AB}$-semi-invariant if and only

$$\sum_{j \in J(I)}^{N} p_j \geq \sum_{j \in I}^{N} p_j$$

for all $I \subset \{1, \ldots, N\}$, where

$$J(I) := \{j \,:\, b_{i,j} = 1 \text{ for some } i \in I\}$$

Convergence follows from this choice of matrix components

Other technical details include weak convergence of measures, etc

Random difference equations

• probability space $(\Omega, \mathcal{F}, \mathbb{P})$, ergodic process $\theta : \Omega \to \Omega$

• compact metric space (X, d), measurable mapping $f : X \times \Omega \to X$

random difference equation $$\boxed{x_{n+1} = f(x_n, \theta^n(\omega))}$$

$\Longrightarrow$ skew product $$(x, \omega) \mapsto F(x, \omega) := \begin{pmatrix} f(x, \omega) \\ \theta(\omega) \end{pmatrix}$$

$\Longrightarrow$ invariant measure μ on $X \times \Omega$ $$\boxed{\mu = F^* \mu}$$

BUT we can only discretise the state space X, i.e. use a grid

$$X_N = \{x_1^{(N)}, \ldots, x_N^{(N)}\} \quad \text{with} \quad h_N \to 0 \quad \text{as } N \to \infty$$

We can decompose the invariant measure $\mu = F^* \mu$ as

$$\mu(B, \omega) \;=\; \mu_\omega(B)\,\mathbb{P}(d\omega) \qquad\qquad \forall\, B \in \mathcal{B}(X)$$

where the measures μ_ω on X are θ-invariant w.r.t. f, i.e.

$$\boxed{\mu_{\theta(\omega)}(B) \;=\; \mu_\omega\left(f^{-1}(B,\omega)\right), \quad \forall\, B \in \mathcal{B}(X),\ \omega \in \Omega}$$

On the deterministic grid X_N we now consider

- random Markov chains $\{P_N(\omega), \omega \in \Omega\}$
- random probability vectors $\{p_N(\omega), \omega \in \Omega\}$

$$p_{N,n+1}(\theta^{n+1}(\omega)) = p_{N,n}(\theta^n(\omega))P_N(\theta^n(\omega))$$

equilibrium probability vector $$\bar{p}_N(\theta(\omega)) = \bar{p}_N(\omega)P_N(\omega)$$

$\Longrightarrow$ random measure $\mu_{N,\omega}$ on X

Theorem 5 [Imkeller & Kloeden]

A random probability measure $\{\mu_\omega, \omega \in \Omega\}$ is θ-semi-invariant w.r.t. f on X if and only if it is randomly stochastically approachable, i.e. for each N there exist

1) a grid X_N with fineness $\Delta_N \to 0$ as $N \to \infty$
2) a random Markov chain $\{P_N(\omega), \omega \in \Omega\}$ on X_N
3) random probability measure $\{\mu_{N,\omega}, \omega \in \Omega\}$ on X corresponding to a random equilibrium probability vectors $\{\bar{p}_N(\omega), \omega \in \Omega\}$ of the $\{P_N(\omega), \omega \in \Omega\}$ on X_N with the expected convergences.

$$\mathbb{E}D\left(P_N(\omega), f(\cdot,\omega)\right) \to 0 \qquad \mathbb{E}\rho\left(\mu_{N,\omega}, \mu\right) \to 0$$