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Abstract

We study the null controllability of linear shadow models for reaction-diffusion systems arising as singular limits when the diffusivity of some of the components is very high. This leads to a coupled PDE-ODE system where one component solves a parabolic partial differential equation (PDE) and the other one an ordinary differential equation (ODE). This reduced system contains the essential dynamics of the original one.

We analyze these shadow systems from a controllability perspective and prove two types of results. First, by employing Carleman inequalities, ODE arguments and regularity results, we prove that the null controllability of the shadow model holds. This result, together with the effectiveness of the controls of the shadow system to control the full original dynamics for large values of the diffusivity parameter, is then illustrated by numerical simulations.

We also obtain a uniform Carleman estimate for the reaction-diffusion equations which allows to obtain the null control for the shadow system as a limit when the diffusivity tends to infinity in one of the equations.

These results justify the efficiency of shadow systems not only for modeling the dynamics of reaction-diffusion system in the large diffusivity limit of one of the equations, but at the control level too.

Keywords: Shadow model, reaction-diffusion systems, hybrid systems, null controllability, Carleman estimates, Neumann boundary conditions.

1 Introduction and main results

Reaction-diffusion systems with Neumann boundary conditions of the following form arise systematically in the study of biological, chemical, and ecological problems (see, for instance, [35])

$$\begin{cases} u_t - d_1\Delta u = \bar{f}(u,v) & \text{in } \Omega\times(0,T), \\ v_t - d_2\Delta v = \bar{g}(u,v) & \text{in } \Omega\times(0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega\times(0,T), \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, $T>0$ is given and $\Omega\subset\mathbb{R}^N$ is a nonempty and bounded set with smooth boundary $\partial\Omega$, while ν denotes the unit outward normal vector to $\partial\Omega$. Hereinafter, we set $Q:=\Omega\times(0,T)$ and $\Sigma:=\partial\Omega\times(0,T)$.

In (1.1), the terms $\bar{f}$ and $\bar{g}$ are two (smooth) nonlinear functions, commonly referred as reaction terms, and d_1 and d_2 are two positive constants representing the diffusion rates of the state variables $u=u(x,t)$ and $v=v(x,t)$.

A major concern regarding these systems is to understand the mechanism for the creation of patterns, that is, solutions to (1.1) which are spatially dependent in a complex manner. At this respect we refer the reader to the seminal paper by Turing [42] or to [30] for a recent survey.

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As pointed out in [22], the qualitative behavior of (1.1) depends on the different values of the diffusion coefficients d_1 and d_2. In [34], the author studied a limiting case, the so-called *shadow* system, which is obtained by letting d_2 tend to $+\infty$ and consists of a reaction-diffusion equation coupled with an ordinary differential equation (ODE).

More precisely, in the singular limit as $d_2 \to \infty$, (1.1) reduces to

$$\begin{cases} u_t - d_1 \Delta u = \bar{f}(u, \vartheta), & \text{in } Q, \\ \vartheta' = |\Omega|^{-1} \displaystyle\int_\Omega \bar{g}(u, \vartheta)\, \mathrm{d}x, & \text{in } (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \Sigma, \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \quad \vartheta(0) = \vartheta_0, \end{cases} \tag{1.2}$$

where $|\Omega|$ is the measure of Ω and $\vartheta = \vartheta(t)$ is the limit of $v(x,t)$ as $d_2 \to +\infty$. In [22] it was proved that the existence and stability of equilibria for the shadow system were already reflected in the original pair of reaction-diffusion equations.

The idea of addressing the "reduced" system (1.2) (since the second equation is now an ODE, the second component of the state only depending on t) to deduce properties for the complete system (1.1) has been developed successfully, as seen for instance in [18, 33]. But this strategy exhibits also some limitations since discrepancies may arise between (1.1) and (1.2) for some fundamental aspects, such as global existence and finite-time blow up, see [25].

The main objective of this paper is to analyze the suitability of this reduction in the context of controllability focusing in the linear case. Thus we study the controllability properties of a linear version of (1.2) and its connections with the control properties of the full system.

To formulate the problem under consideration more precisely, let $\omega \subset \Omega$ be a nonempty (small) open set. As usual, χ_ω will denote the characteristic function of ω.

We consider the controlled linear shadow system

$$\begin{cases} u_t - \Delta u + a\, u + b\, v = \chi_\omega f, & \text{in } Q \\ v' = c \left(|\Omega|^{-1} \displaystyle\int_\Omega u \,\mathrm{d}x \right) + d\, v, & \text{in } (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \quad v(0) = v_0. \end{cases} \tag{1.3}$$

In (1.3), $u = u(x,t)$ and $v = v(t)$ are the state variables, while $f = f(x,t)$ is the control that is exerted on the system. We assume that $a, b, c, d \in \mathbb{R}$ are the coupling constants and, without loss of generality, $d_1 \equiv 1$.

We analyze the problem of controllability, the goal being to steer the state to a null final target by a suitable choice of the control function f. More precisely,

Definition 1.1. *System* (1.3) *will be said to be* **null-controllable at time** T *if for any* $(y_0, v_0) \in L^2(\Omega) \times \mathbb{R}$, *there exists a control* $f \in L^2(\omega \times (0,T))$ *such that the associated states* (u, v) *satisfy*

$$y(\cdot, T) = 0 \text{ in } \Omega \quad \text{and} \quad v(T) = 0.$$

Notice that we are applying the control only in the first equation of the system. This means that the first equation is controlled directly by the action of the control, while the other one is being controlled indirectly, through the coupling.

There is an extensive literature on the control of parabolic systems with less controls than equations (see, for instance, [3] and the references therein). But the case where parabolic equations and ODE's are coupled together has not been addressed so far, as far as we know.

The idea of analyzing controllability problems for differential equations subject to a limit process has been successfully addressed in various frameworks. The control of parabolic equations degenerating into hyperbolic ones has been studied in [9, 20], while the opposite case was analyzed in [26, 27]. More recently, the controllability of parabolic systems degenerating into parabolic-elliptic ones has been addressed in [5, 6, 7]. However, the present case has not been treated so far.

The model under consideration differs from the systems coupling PDEs and ODEs that arise when dealing with evolution equations involving *memory terms* ([21, 8, 29]). In that case, the memory term can be rewritten as an infinite-dimensional ODE (i.e., an ODE distributed in space). While, formally, these are PDE-ODE systems too, in that case the ODE depends on the continuous space parameter x and, accordingly, evolves in an infinite-dimensional space. The control of these systems is therefore more complex and requires the support of the control ω to move in time.

But the models we consider here are of different nature since the ODE entering in it is really a scalar model, not depending on any other parameter. Thus, the shadow system under consideration is just a perturbation of the classical heat equation by the addition of an ODE.

Among other related works, we can also mention [40] where the stabilization of a cascade PDE-ODE system is studied by the backstepping approach and [41] where the controllability of coupled systems in abstract form (with the possibility of one component being finite dimensional) is addressed.

The first main result of this paper is the following.

Theorem 1.2. *System (1.3) is null-controllable at time T if and only if $c \neq 0$.*

Using the well-known equivalence between null controllability and observability (see, e.g., [32]), the proof of the main result will consist in obtaining a suitable observability inequality for the corresponding adjoint system. This will be done firstly by obtaining an auxiliary Carleman inequality and then by exploiting the fact that the second equation on the adjoint system verifies an ODE, while the first one has zero Neumann boundary conditions. These allows to derive a closed system of two first order ODEs coupling the spatial mean of the u-component and the v component.

Since (1.3) appears as the limit case of a reaction-diffusion system of the form

$$\begin{cases} u_t - \Delta u + a\,u + b\,v = \chi_\omega f, & \text{in Q} \\ z_t - K\Delta z = c\,u + d\,z, & \text{in Q,} \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & \text{on } \Sigma, \\ u(x,0) = u_0(x), \quad z(x,0) = z_0(x) & \text{in } \Omega, \end{cases} \tag{1.4}$$

i.e., as $K \to +\infty$, the following two questions arise naturally:

- The control of the reduced shadow system (1.3) can be employed to control somehow the complete system (1.4) too?
- Is the control of the shadow system the limit of controls of the reaction-diffusion systems (1.4) as $K \to +\infty$?

Concerning the first question, under suitable conditions, by employing a null-control for (1.3), we may recover an approximate controllability result for (1.4). The result is given in the following theorem.

Theorem 1.3. *Fix any arbitrary initial data $u_0, z_0 \in L^2(\Omega)$ and $T > 0$. Let $f_S \in L^2(\omega \times (0,T))$ be a null control (in time T) for the shadow system (1.3) with initial data u_0 and $v_0 = \int_\Omega z_0 \,\mathrm{d}x$. Then, for given $\varepsilon > 0$, there exists K large enough such that*

$$\|u(T)\|_{L^2(\Omega)} + \|z(T)\|_{L^2(\Omega)} \leq \varepsilon, \tag{1.5}$$

where (u,z) is the solution to the reaction-diffusion system (1.4). Moreover,

$$\varepsilon = \mathcal{O}(K^{-\mu}), \quad \text{with } \mu = \min\{1, N/2\}.$$

Concerning the second question, in the spirit of [5, 7], the third result of this paper shows that system (1.4) can be controlled uniformly with respect to the diffusion K:

Theorem 1.4. *Assume that $c \neq 0$. Then, system (1.4) is uniformly (on $K \geq 1$) null controllable.*

More precisely, there exists $f = f(x,t;K) \in L^2(\omega \times (0,T))$ such that

$$(u(x,T;K), z(x,T;K)) \equiv (0,0). \tag{1.6}$$

Moreover, we have the following uniform bound on the controls

$$\|f(K)\|_{L^2(\omega\times(0,T))} \leq C\left(\|u_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}\right), \tag{1.7}$$

where C is positive constant that does not depend on K, u_0 and v_0.

The controllability of coupled parabolic PDE systems has been thoroughly studied in the recent past, leading to important progress. We refer the reader to [3] for an exhaustive survey on this topic. However, it is noteworthy that most of the literature is devoted to the case of Dirichlet boundary conditions. But, despite some technical difficulties, such results, that commonly rely on Carleman inequalities, can be adapted to the case of Neumann boundary conditions (see, e.g., [6, 23]).

Adapting [3, Theorem 4.1] one can prove that, for each $K \geq 1$, system (1.4) is null controllable at time T. But, in principle the obtained constants in (1.7) diverge as $K \to \infty$ (see Remark 5.3). Here, combining sharp Carleman estimates and weighted energy inequalities we are able to obtain an uniform Carleman inequality for the adjoint system to (1.4), which in turn yields Theorem 1.4.

As a further consequence, to answer to the second question above, we can prove the following result on the limiting behaviour of controls:

Theorem 1.5. *Let $T > 0$ be given and fix any arbitrary initial data $u_0, z_0 \in L^2(\Omega)$.*

The controls (of minimal L^2-norm) and controlled states of (1.4) converge as $K \to +\infty$, in the sense that

$$\begin{aligned} & f^K \rightharpoonup f \quad \text{weakly in } L^2(\omega \times (0,T)) \\ & u^K \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)), \\ & u^K \to u \quad \text{strongly in } L^2(Q), \\ & \frac{1}{|\Omega|}\int_\Omega z^K dx \to v \quad \text{strongly in } L^2(0,T). \end{aligned}$$

and the limit control f is also the one of minimal L^2-norm assuring the null controllability of the solution (u,v) of the shadow system (1.3) with initial data $(u_0, \frac{1}{|\Omega|}\int_\Omega v_0 dx)$.

The rest of the paper is organized as follows. Section 2 will be devoted to prove the observability inequality needed to establish the controllability of the shadow system. In Section 3, we discuss and illustrate the numerical implementation of the control problem for the shadow system. Then, we devote Section 4 to prove the approximate controllability result presented in Theorem 1.3, while in Section 5 we present the proofs of Theorems 1.4 and 1.5. Finally, some additional comments and open questions are indicated in Section 6.

2 Null control of the shadow system

As already mentioned, the null controllability of (1.3) will be reformulated in terms of the observability of the adjoint system, which is given by

$$\begin{cases} -\varphi_t - \Delta\varphi + a\,\varphi = c\,\theta, & \text{in } Q \\ -\theta' = -b\left(|\Omega|^{-1}\displaystyle\int_\Omega \varphi\,\mathrm{d}x\right) + d\,\theta, & \text{in } (0,T), \\ \dfrac{\partial\varphi}{\partial\nu} = 0 & \text{on } \Sigma \\ \varphi(x,T) = \varphi_T(x) \quad \text{in } \Omega, \quad \theta(T) = \theta_T. \end{cases} \tag{2.1}$$

The main task will reduce to prove the following result

Proposition 2.1. *Assume that $c \neq 0$. Then, there exists a constant $C = C(c) > 0$, such that, for all $(\varphi_T, \theta_T) \in L^2(\Omega) \times \mathbb{R}$, the solution (φ, θ) of (2.1) verifies*

$$\|\varphi(0)\|^2_{L^2(\Omega)} + |\theta(0)|^2 \leq C \iint_{\omega\times(0,T)} |\varphi|^2\,\mathrm{d}x\mathrm{d}t. \tag{2.2}$$

Furthermore, this inequality fails when $c = 0$.

By adapting classical arguments (see, for instance, [32]), the observability inequality (2.1) implies the null controllability of (1.3). The methodology to prove (2.2) is as follows. First, using a Carleman inequality for the heat equation we will get global weighted estimates on φ from observations in ω and some remaining term depending on θ. On the other hand, when $c \neq 0$, the first equation leads to local information on θ out of local information on φ. Furthermore, averaging the heat equation satisfied by φ, one can obtain an ODE system coupling the average of φ for θ. This allows to recover full information on θ out of local one. Overall, when $c \neq 0$ we get global weighted norms on φ and θ out of local measurements on φ. Of course this argument and the observability inequality fail for $c = 0$ since, in that case, local information on φ does not lead to any measurement on θ.

Before proving Proposition 2.1, we recall below a global Carleman estimate for heat equations with homogeneous Neumann boundary conditions. For this, we introduce a special function whose existence is guaranteed by the following result [17, Lemma 1.1].

Lemma 2.2. *Let $\mathcal{B} \subset\subset \Omega$ be a nonempty open subset. Then, there exists $\eta^0 \in C^2(\overline{\Omega})$ such that $\eta^0 > 0$ in Ω, $\eta^0 = 0$ on $\partial\Omega$, and $|\nabla\eta^0| > 0$ in $\overline{\Omega \setminus \mathcal{B}}$.*

Then, for $\lambda > 0$ a parameter, we introduce the weight functions

$$\alpha(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0(x)}}{t(T-t)}, \quad \xi(x,t) = \frac{e^{\lambda\eta^0(x)}}{t(T-t)},$$
$$\alpha^\star(t) = \max_{x\in\overline{\Omega}} \alpha(x,t), \quad \widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t), \quad \widehat{\xi}(t) = \max_{x\in\overline{\Omega}}(x,t), \quad \xi^\star(t) = \min_{x\in\overline{\Omega}} \xi(x,t). \tag{2.3}$$

To abridge the presentation of the estimates below we will use the notation

$$I_{s,\lambda}(q) := s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla q|^2 \,\mathrm{d}x\mathrm{d}t + s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|q|^2 \,\mathrm{d}x\mathrm{d}t$$
$$+ s^{-1} \iint_Q e^{-2s\alpha}\xi^{-1}\left(|q_t|^2 + |\Delta q|^2\right) \mathrm{d}x\mathrm{d}t$$

for some positive parameters s and λ.

The Carleman inequality reads as follows.

Lemma 2.3 ([13, Lemma 1])**.** *Let $g \in L^2(Q)$ be given. There exist λ^*, σ^* and C only depending on Ω and $\mathcal{B}$ such that, for any $\lambda \geq \lambda^*$, any $s \geq s^*(\lambda) = \sigma^*(e^{2\lambda\|\eta^0\|_\infty}T + T^2)$ and any $q_T \in L^2(\Omega)$, the weak solution to*

$$\begin{cases} -q_t - \Delta q = g(x,t), & \text{in } Q \\ \dfrac{\partial q}{\partial \nu} = 0 & \text{on } \Sigma \\ q(x,T) = q_T(x) & \text{in } \Omega, \end{cases}$$

satisfies

$$I_{s,\lambda}(q) \leq C\left(\iint_Q e^{-2s\alpha}|g|^2 \,\mathrm{d}x\mathrm{d}t + \iint_{\mathcal{B}\times(0,T)} e^{-2s\alpha}s^3\lambda^4\xi^3|q|^2 \,\mathrm{d}x\mathrm{d}t\right). \tag{2.4}$$

Proof of Theorem 2.1. For the sake of clarity, we have divided the proof in three steps.

Step 1. An auxiliary estimate. We apply inequality (2.4) to φ solution to the PDE of system (2.1), with $g = c\,\theta - a\,\varphi$ and $\mathcal{B} = \omega$. We readily get

$$I_{s,\lambda}(\varphi) \leq C\left(\iint_Q e^{-2s\alpha}|c\,\theta - a\,\varphi|^2 \,\mathrm{d}x\mathrm{d}t + \iint_{\omega\times(0,T)} e^{-2s\alpha}s^3\lambda^4\xi^3|\varphi|^2 \,\mathrm{d}x\mathrm{d}t\right)$$
$$\leq C\left(\iint_Q e^{-2s\alpha}|c\,\theta|^2 \,\mathrm{d}x\mathrm{d}t + \iint_{\omega\times(0,T)} e^{-2s\alpha}s^3\lambda^4\xi^3|\varphi|^2 \,\mathrm{d}x\mathrm{d}t\right),$$

where we have taken s sufficiently large to absorb the lower order term depending on φ in the left-hand side. Since θ is x-independent and using the definition of the weight function (2.3), we see that

$$I_{s,\lambda}(\varphi) \leq C\left(|\Omega|\int_0^T e^{-2s\widehat{\alpha}}|\theta|^2 \mathrm{d}t + \iint_{\omega\times(0,T)} e^{-2s\alpha}s^3\lambda^4\xi^3|\varphi|^2 \,\mathrm{d}x\mathrm{d}t\right). \tag{2.5}$$

We set λ to a fixed large enough value. Using the properties of the Carleman weight, it can be seen that for every $a > 0$, $b, c \in \mathbb{R}$,

$$s^b e^{-as\alpha}(\xi)^c \leq C \quad \text{in } Q,$$

for s sufficiently large. Note that the same holds, instead, when α is replaced by $\widehat{\alpha}$. Therefore, we can bound the weights in right-hand side of (2.5), obtaining the following

$$I_{s,\lambda}(\varphi) \leq C \left(\int_0^T |\theta|^2 \mathrm{d}t + \iint_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\mathrm{d}t \right). \tag{2.6}$$

Inequality (2.6) together with some well-known estimates to eliminate the singularity of the Carleman weight at $t = 0$ (see, e.g., [16, Section 6]), allow us to conclude that

$$\int_0^T\int_\Omega e^{-\frac{2C_0}{T-t}} |\varphi|^2 \,\mathrm{d}x\mathrm{d}t \leq C_1 \left(\int_0^T |\theta|^2 \mathrm{d}t + \iint_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\mathrm{d}t \right), \tag{2.7}$$

for all$(\varphi_T, \theta_T) \in L^2(\Omega) \times \mathbb{R}$, for some positive constants $C_0(\Omega, \omega, T)$ and $C_1 = \exp\left[C(\Omega, \omega, a, b)(1 + \frac{1}{T})\right]$.

Step 2. A regularity result for θ. In order to obtain (2.2), we need to eliminate the term corresponding to θ in the right-hand side of (2.7). For this, we will take advantage of the fact that θ is independent of x and solves an ODE.

Notice that if $\varphi \in L^2(\omega \times (0, T))$, by applying the heat operator $\mathcal{L}_a = -\partial_t - \Delta + a$ we necessarily have $\mathcal{L}_a\varphi \in H^{-2}(\omega \times (0, T))$. Moreover, we have the estimate

$$\| - \varphi_t - \Delta\varphi + a\varphi\|_{H^{-2}(\omega\times(0,T))} \leq C\|\varphi\|_{L^2(\omega\times(0,T))}$$

for some constant C only depending on ω and a. Observe that the expression on the left-hand side verifies the PDE in (2.1), thus

$$\|c\,\theta\|_{H^{-2}(\omega\times(0,T))} \leq C\|\varphi\|_{L^2(\omega\times(0,T))}$$

and since θ does not depend on x, we have

$$\|\theta\|_{H^{-2}(0,T)} \leq C\|\varphi\|_{L^2(\omega\times(0,T))} \tag{2.8}$$

for a constant C only depending on ω, c and a.

By a bootstrapping argument we will show that the norm in the left-hand side of the previous inequality (2.8) can be replaced by a more regular one. To this end, define

$$\zeta(t) := |\Omega|^{-1} \int_\Omega \varphi \,\mathrm{d}x.$$

Integrating over Ω in the equation satisfied by φ (see eq. (2.1)), using the zero Neumann boundary conditions, we can obtain the system of differential equations satisfied by ζ and θ :

$$\begin{cases} -\zeta' + a\zeta = c\,\theta, & \text{in } (0,T) \\ -\theta' = -b\,\zeta + d\,\theta, & \text{in } (0,T) \end{cases} \tag{2.9}$$

and, consequently,

$$-\theta'' - d\,\theta' + bc\,\theta = ab\,\zeta \quad \text{in } (0,T).$$

Substituting in the above expression the equation satisfied by θ, we get

$$-\,\theta'' - (a+d)\,\theta' + (bc - ad)\,\theta = 0 \quad \text{in } (0,T). \tag{2.10}$$

With this new expression, we estimate higher order norms of θ on the whole interval $[0, T]$. More precisely, we have the following

Lemma 2.4. *Let $s \geq 1$ be any integer. Then, there is a positive constant $C_s > 0$ such that the norm of θ in $C^s([0, T])$ can be estimated in terms of the $H^{-2}(0, T)$ norm of θ as follows:*

$$||\theta||_{C^s([0,T])} \leq C_s||\theta||_{H^{-2}(0,T)}.$$

The proof of this result can be found in Appendix A.1.

We readily deduce that

$$\int_0^T |\theta|^2 \mathrm{d}t \leq C\|\theta\|^2_{H^{-2}(0,T)} \leq C\|\varphi\|^2_{L^2(\omega\times(0,T))} \tag{2.11}$$

for some universal constant C only depending on ω, T, and the coupling coefficients a, b, c, d. Putting together estimates (2.7) and (2.11) we obtain

$$\int_0^T\int_\Omega e^{-\frac{2C_0}{T-t}}|\varphi|^2\,\mathrm{d}x\mathrm{d}t \leq C\left(\iint_{\omega\times(0,T)}|\varphi|^2\,\mathrm{d}x\mathrm{d}t\right) \tag{2.12}$$

valid for every $(\varphi_T, \theta_T) \in L^2(\Omega)\times\mathbb{R}$.

Step 3. Conclusion. To conclude, we would like to estimate the terms $|\theta(0)|^2$ and $\|\varphi(0)\|^2_{L^2(\Omega)}$ in the left-hand side of (2.12). We follow well-known arguments (see, e.g., [14]) allowing to get pointwise estimates (for t fixed) out of time-averaged estimates. Of course this can only be done in the correct sense of time in view of the parabolic nature of the system.

Let us introduce a function $\eta \in C^1([0,T])$ such that

$$\eta = 1 \text{ in } [0,T/2], \quad \eta = 0 \text{ in } [3T/4,T], \quad |\eta'(t)| \leq C/T.$$

Using classical energy estimates for $\eta\varphi$, where φ is the solution to the PDE in (2.1), we obtain

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C\left(\frac{1}{T^2}\|\varphi\|^2_{L^2(T/2,3T/4;L^2(\Omega))} + c^2\|\theta\|^2_{L^2(0,3T/4)}\right)$$

Since the weight function appearing on the left-hand side of (2.12) is bounded away from $t = T$, we can introduce it in the expression above, hence

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C\left(\iint_{\Omega\times(T/2,3T/4)} e^{-\frac{2C_0}{T-t}}|\varphi|^2\,\mathrm{d}x\mathrm{d}t + \int_0^{3T/4}|\theta|^2\mathrm{d}t\right)$$

This inequality together with (2.11) and (2.12) yields

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C\left(\iint_{\omega\times(0,T)}|\varphi|^2\,\mathrm{d}x\mathrm{d}t\right) \tag{2.13}$$

On the other hand, defining $\Psi := \begin{pmatrix} \zeta & \theta \end{pmatrix}^\top$, we can rewrite (2.9) as

$$-\Psi' + \underbrace{\begin{pmatrix} a & -c \\ b & -d \end{pmatrix}}_{:=A^\star}\Psi = 0, \quad \text{in } (0,T) \tag{2.14}$$

We are dealing with an ODE system, and $B^\star = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $(A^\star, B^\star)$ fulfill the classical rank condition that ensures observability, in any time interval. Thus,

$$|\Psi(0)|^2 \leq C\int_0^T |B^\star\Psi|^2\mathrm{d}t, . \tag{2.15}$$

Similarly, for $\delta > 0$ small enough, with a constants $C = C(\delta)$ we also have

$$|\Psi(0)|^2 \leq C\int_0^{T-\delta}|B^\star\Psi|^2\mathrm{d}t.$$

From the definition of Ψ, it is not difficult to see that

$$|\theta(0)|^2 \leq C\int_0^{T-\delta}\left(|\Omega|^{-1}\int_\Omega\varphi\,\mathrm{d}x\right)^2 \leq C\iint_{\Omega\times(0,T-\delta)}|\varphi|^2\,\mathrm{d}x\mathrm{d}t.$$

Here we have used Jensen's inequality in the second line. Arguing as before, since we are away from the singularity of the weight, we obtain

$$|\theta(0)|^2 \leq C \iint_Q e^{-\frac{2C_0}{T-t}} |\varphi|^2 \, \mathrm{d}x\mathrm{d}t \leq C \left(\iint_{\omega\times(0,T)} |\varphi|^2 \, \mathrm{d}x\mathrm{d}t \right) \tag{2.16}$$

Putting together (2.13) and (2.16) we obtain the desired inequality. This concludes the proof. □

Remark 2.5. *One can also prove Proposition 2.1 by means of a compactness-uniqueness argument. This kind of argument is rarely used in the framework of diffusion equations because of the very strong time irreversibility, but has been recently applied to study the controllability of nonlocal and coupled parabolic systems, see [15, 28].*

In fact, developing further from (2.5), one may obtain an estimate of the form

$$\begin{aligned}\int_0^T\int_\Omega e^{-\frac{2C_0}{T-t}} |\varphi|^2 \, \mathrm{d}x\mathrm{d}t + \int_0^T e^{-\frac{2C_0}{T-t}} |\theta|^2 \mathrm{d}t \\ \leq C \left(|\theta_T|^2 + \int_0^T \left(|\Omega|^{-1} \int_\Omega \varphi \, \mathrm{d}x \right)^2 \mathrm{d}t + \iint_{\omega\times(0,T)} |\varphi|^2 \, \mathrm{d}x\mathrm{d}t \right)\end{aligned}$$

At this point, to conclude, one needs to prove the following inequality

$$|\theta_T|^2 + \int_0^T \left(|\Omega|^{-1} \int_\Omega \varphi \, \mathrm{d}x \right)^2 \mathrm{d}t \leq C \iint_{\omega\times(0,T)} |\varphi|^2 \, \mathrm{d}x\mathrm{d}t. \tag{2.17}$$

Arguing by contradiction, assuming that it does not hold and using a unique continuation argument together with the fact that $\int_\Omega \varphi$ and θ verify (2.9), one reaches a contradiction. The drawback to this method is that the observability constant is not explicit.

Following the proof presented above, we can keep track of this constant to see that the constant appearing in (2.2) is of the form

$$C = T^{-2} \exp\left[C_1 \left(1 + \frac{1}{T} + T \right) \right],$$

where C_1 only depends on Ω, ω and the coupling coefficients a, b, c, d.

3 Numerical results for the shadow system

In this section, we illustrate numerically the controllability result of the previous section. We adapt the classical Hilbert Uniqueness Method (HUM) introduced in [19] to obtain a numerical validation of Theorem 1.2.

For any parameter $\varepsilon > 0$, we compute the control that minimizes the penalized HUM functional F_ε given by

$$F_\varepsilon(f) := \frac{1}{2} \iint_{\omega\times(0,T)} |f|^2 \, \mathrm{d}x\mathrm{d}t + \frac{1}{2\varepsilon} \left(\|u(T)\|^2_{L^2(\Omega)} + |v(T)|^2 \right), \tag{3.1}$$

where (u, v) is the solution to (1.3). Since (3.1) is continuous, coercive and strictly convex, the existence of a unique minimizer, that we denote by f_ε, is guaranteed.

Using Fenchel-Rockafellar theory (see, e.g., [11]), we can identify an associated dual functional to (3.1): for any $\varepsilon > 0$, $\varphi_T \in L^2(\Omega)$ and $\theta_T \in \mathbb{R}$, we introduce

$$J_\varepsilon(\varphi_T, \theta_T) := \frac{1}{2} \iint_{\omega\times(0,T)} |\varphi|^2 \, \mathrm{d}x\mathrm{d}t + \frac{\varepsilon}{2} \left(\|\varphi_T\|^2_{L^2(\Omega)} + |\theta_T|^2 \right) + \int_\Omega \varphi(0) u_0 \, \mathrm{d}x + \langle \theta(0), v_0 \rangle \tag{3.2}$$

where (φ, θ) is the solution to (2.1) associated to the initial data (φ_T, θ_T). It is not difficult to see that (3.2) is continuous and strictly convex. Moreover, thanks to the observability inequality (2.2), we can prove that (3.2) is coercive in $L^2(\Omega) \times \mathbb{R}$, and hence the existence and uniqueness of a minimizer $(\varphi_{T,\varepsilon}, \theta_{T,\varepsilon})$ is also guaranteed.

Using well-known arguments, it can be readily seen that the minimizers f_ε and $(\varphi_{T,\varepsilon}, \theta_{T,\varepsilon})$ are related through the formulas

$$f_\varepsilon = \varphi_\varepsilon|_\omega, \quad y_\varepsilon(T) = -\varepsilon\varphi_{T,\varepsilon}, \quad v_\varepsilon(T) = -\varepsilon\theta_{T,\varepsilon} \tag{3.3}$$

where φ_ε is taken from $(\varphi_\varepsilon, \theta_\varepsilon)$ solution to (2.1) with initial data $(\varphi_{T,\varepsilon}, \theta_{T,\varepsilon})$ and $(y_\varepsilon, v_\varepsilon)$ stands for the solution to (1.3) with control f_ε.

A straightforward adaptation of [4, Theorem 1.7] allow us to relate the null controllability property for system (1.3) with the behavior of these minimizers with respect to ε. In more detail, we have

Proposition 3.1. *System (1.3) is null controllable if and only if*

$$\mathcal{M}^2 := \sup_{\varepsilon>0} \left(\inf_{L^2(\omega\times(0,T))} F_\varepsilon \right) < +\infty. \tag{3.4}$$

In this case, we have,

$$\|f_\varepsilon\|_{L^2(\omega\times(0,T))} \leq \mathcal{M} \quad \text{and} \quad \|y(T), v(T)\|_{L^2(\Omega)\times\mathbb{R}} \leq \mathcal{M}\sqrt{\varepsilon}$$

Remark 3.2. *Some remarks are in order:*

- *In Section 2, inequality (2.2) together with duality arguments between observability and controllability led to the null controllability of (1.3). However, no explicit expression for the control was obtained. Here, we will exploit the fact that the control can be computed from a convenient minimization problem to obtain numerical approximations.*
- *For $\varepsilon_1 > \varepsilon_2 > 0$ and $f \in L^2(\omega \times (0,T))$ we have $F_{\varepsilon_2}(f) \geq F_{\varepsilon_1}(f)$. It follows that the supremum in (3.4) is actually the limit as $\varepsilon \to 0$ of $\inf_{L^2(\omega\times(0,T))} F_\varepsilon$.*
- *The control f_ε can be computed directly by minimizing (3.1), but the space where the minimization is carried out depends on the time variable. From a practical point of view, it is easier to minimize the dual functional (3.2) and then use the identities (3.3) to study the behavior of the minimizer with respect to ε.*

Since the functional (3.2) is convex, quadratic and coercive, the conjugate gradient algorithm is a natural and simple choice to minimize it. A straightforward computation yields to

$$\nabla J_\varepsilon(\varphi_T, \theta_T) = \Lambda(\varphi_T, \theta_T) + \varepsilon(\varphi_T, \theta_T) + (\overline{u}(T), \overline{v}(T)) \tag{3.5}$$

with the Gramiam operator Λ defined as follows

$$\begin{array}{rcl} \Lambda : L^2(\Omega)\times\mathbb{R} & \to & L^2(\Omega)\times\mathbb{R}, \\ (\varphi_T, \theta_T) & \mapsto & (w(T), z(T)), \end{array}$$

where $(w(T), z(T))$ can be found from the solution to the forward-backward systems

$$\begin{cases} -\varphi_t - \Delta\varphi + a\,\varphi = c\,\theta, & \text{in } Q \\ -\theta' = -b\left(|\Omega|^{-1}\displaystyle\int_\Omega \varphi\,\mathrm{d}x\right) + d\,\theta, & \text{in } (0,T), \\ \dfrac{\partial\varphi}{\partial\nu} = 0 & \text{on } \Sigma \\ \varphi(x,T) = \varphi_T(x) \quad \text{in } \Omega, \quad \theta(T) = \theta_T. \end{cases} \tag{3.6}$$

and

$$\begin{cases} w_t - \Delta w + a\,w + b\,z = \chi_\omega\varphi, & \text{in } Q \\ z' = c\left(|\Omega|^{-1}\displaystyle\int_\Omega w\,\mathrm{d}x\right) + d\,z, & \text{in } (0,T), \\ \dfrac{\partial w}{\partial\nu} = 0 & \text{on } \Sigma \\ w(x,0) = 0 \quad \text{in } \Omega, \quad z(0) = 0. \end{cases} \tag{3.7}$$

and where the pair $(\overline{u}(T), \overline{v}(T))$ stands for the *free* solution to (1.3), that is, the solution at time T starting at initial data (u_0, v_0) and control $f \equiv 0$.

The minimizer f_ε ca be obtained through $(\varphi_{T,\varepsilon}, \theta_{T,\varepsilon})$, solution to the linear problem

$$(\Lambda + \varepsilon I)(\varphi_T, \theta_T) = -\left(\overline{u}(T), \overline{v}(T)\right) \tag{3.8}$$

and (3.3). According to Proposition 3.1, the expected controllability result can be tested analysing the behaviour of these controls.

For the numerical tests, we will set $T = 5$, $\Omega = (0,1)^2$ and $y_0 = \sin(2\pi x)\sin(3\pi y)$. We consider the coupling constant coefficients $a = -1, \quad b = -3, \quad c = -1, \quad d = -0.1$.

Systems (1.3) and (3.6)-(3.7) are discretized in time by the standard implicit Euler scheme. For the space discretization of the PDEs, the classical 5-point stencil for the Laplacian on uniform meshes is employed (see, for instance, [24]), that is, we take constant discretization steps in each direction of size $h = 1/(N+1)$, where N is the number of steps.

The linear system of algebraic equations derived from the finite difference discretization in $N \times N$ grid points for an elliptic operator has N^2 unknowns, so the associated stiffness matrix has $N^2 \times N^2$ entries. Once an implementation for the Laplacian with Neumann boundary conditions is known, the integral involved in the ODE can be carried out by adding up the average at each node of the mesh. By handling the coupling coefficients, one can write the PDE-ODE system in a single vector of N^2+1 equations together with a $(N^2+1)\times(N^2+1)$ coupling matrix.

Remark 3.3. *From here, we observe that the computational effort of obtaining a numerical approximation of a control for the shadow system is not far from the case of a single parabolic PDE, since we are just adding one component to the discretized system.*

To numerically illustrate these results, we follow the penalized HUM methodology developed in [4]. We denote by E_h, U_h and $L^2_{\delta t}(0,T;U_h)$ the discrete spaces associated to $L^2(\Omega)\times\mathbb{R}$, $L^2(\omega)$ and $L^2(\omega\times(0,T))$, respectively. We denote by $F_\varepsilon^{h,\delta t}$ the discretization of the functional F_ε and by $(u_\varepsilon^{h,\delta t}, v_\varepsilon^{h,\delta t}, f_\varepsilon^{h,\delta t})$ the solution to the corresponding minimization problem.

To connect the discretization scheme to the control problem, we use the penalization parameter $\varepsilon = \phi(h) = h^4$. This choice is consistent with the order of approximation of the finite difference scheme. We refer the reader to [4, Section 4] for a more detailed discussion on the selection of the function $\phi(h)$ in the context of the null-controllability of some parabolic problems.

We begin by plotting the solution to the shadow system (1.3) without any control. In Figures 1 and 2, we observe the time evolution of the uncontrolled PDE and ODE, respectively. From here, we see that the PDE component quickly converges to a uniform spatial configuration which is in accordance with the behavior of the shadow systems, but neither of the components is damped over time. A closer inspection allow us to conclude that the stability of the systems is dictated by the eigenvalues of the corresponding matrix of the simplified ODE system (see, for instance,(2.14). This can be seen also by writing the solution of the PDE in Fourier series). For this particular case, we have that the two eigenvalues are complex conjugate with positive real part, hence the oscillatory and growing behavior shown in Figures 1-2 is expected.

In Figures 3 and 4, we plot the solutions (u,v) obtained with the HUM control computed by the algorithm described in (3.5)-(3.8). We observe that, due to the action of the control, both of the components of the state move towards zero at the prescribed time $T = 5$. For these experiments, we have chosen $\omega = (0.3,0.8)\times(0.3,0.8)$.

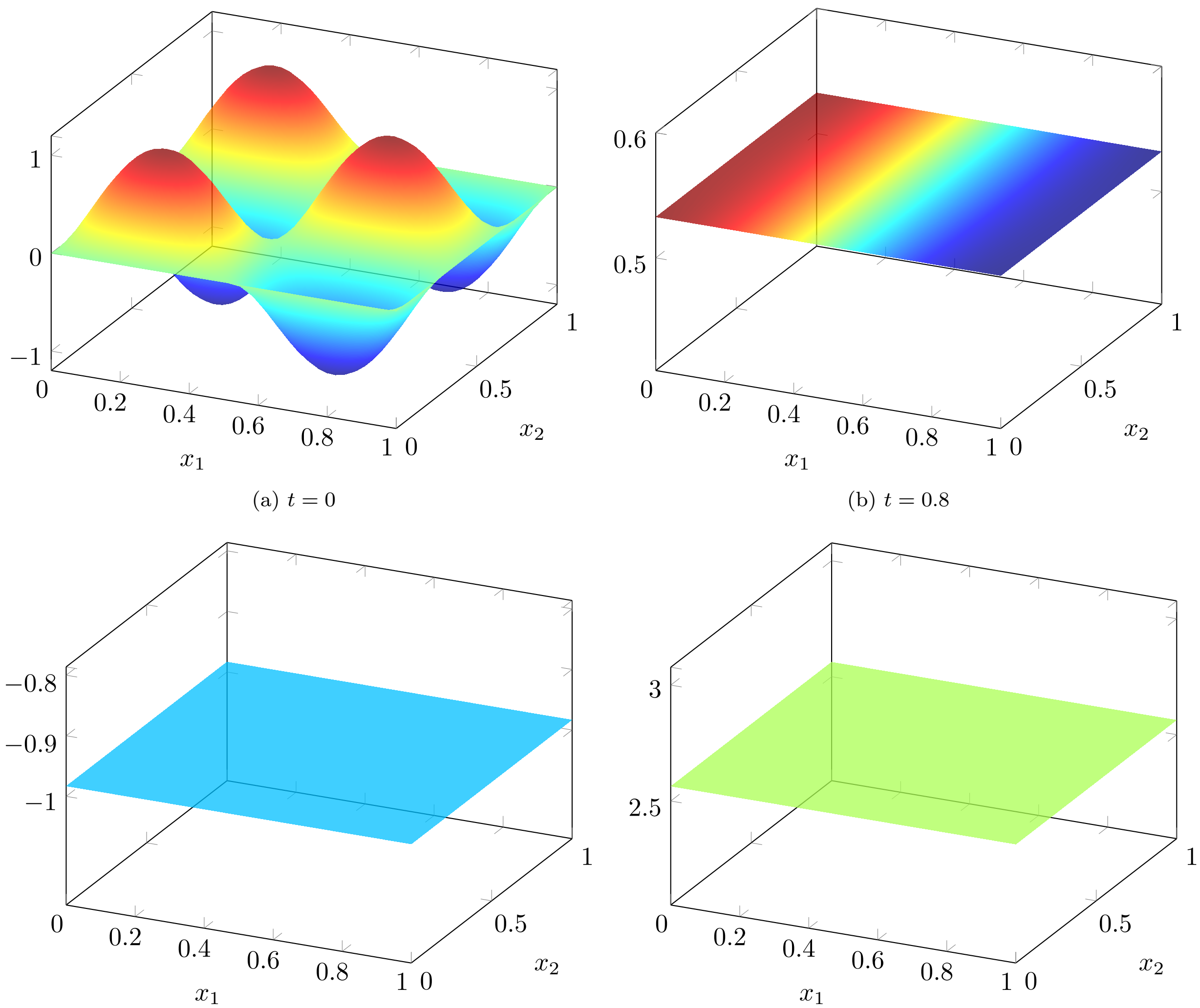
As far as the asymptotic behavior of the method is concerned, we present in Figure 5 the behavior of various quantities of interest when the mesh size goes to 0. More precisely, we observe that the control cost $\|f^{h,\delta t}_{\phi(h)}\|_{L^2_{\delta t}(0,T;U_h)}$ (—◆—) as well as the optimal energy $\inf F^{h,\delta t}_{\phi(h)}$ (—▲—) both remain bounded as $h \to 0$. In the meantime, we see that the norm of the controlled state $\|(u^{h,\delta t}(T), v^{\delta t}(T)\|_{E_h}$ (—■—) behaves like $\sim C\sqrt{\phi(h)} = Ch^2$ as predicted by Proposition 3.1

4 Approximate controllability for the original equations

As mentioned above, shadow models allow, to some extent, reducing the dynamics of reaction-diffusion systems. Here, we will use the controllability result for the shadow system (1.3) to deduce control properties for the original one.

For this, let us consider the following system of coupled PDE

$$\begin{cases} u_t - \Delta u + a\,u + b\,v = \chi_\omega f, & \text{in Q} \\ z_t - K\Delta z = c\,u + d\,z, & \text{in Q,} \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & \text{on } \Sigma, \\ u(x,0) = u_0(x), \quad z(x,0) = z_0(x) & \text{in } \Omega. \end{cases} \tag{4.1}$$



(a) $t = 0$ (b) $t = 0.8$ (c) $t = 2.5$ (d) $t = 5$

Figure 1: Evolution in time of the PDE component $u = u(x, t)$ of the uncontrolled shadow system.

where f is a control function and $K > 0$ is a given diffusion coefficient. Letting $K \to \infty$, one recovers system (1.3) and, thanks to Theorem 1.2, the corresponding shadow system is null controllable at time T. On the other hand, adapting the proof of [3, Theorem 4.1] to the case of Neumann conditions (see also Remark 5.3), one can prove similarly that (4.1) is null controllable at time T if and only if $c \neq 0$ and for any $K > 0$.

Here, we will use the fact that the existence of a null control for the shadow system (1.3) is known to prove that for K sufficiently large, system (4.1) is approximately controllable at time T (see Theorem 1.3).

In the remainder of this section, without loss of generality and to simplify notation, we consider $|\Omega| = 1$. We begin by recalling some well-known properties of the heat semigroup.

Lemma 4.1. *Let $\{e^{tK\Delta}\}_{t\geq 0}$ be the heat semigroup with Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary. We have*

a) $e^{tK\Delta}C = C$ *for all* $C \in \mathbb{R}$ *and all* $t \geq 0$.

b) For every $w_0 \in L^1(\Omega)$ *there exists a positive constant* $C = C(\|w_0\|_{L^1(\Omega)})$, *independent of* K, *such*

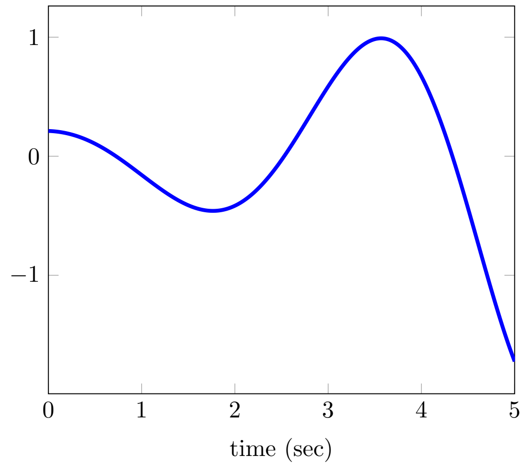


Figure 2: Evolution in time of the ODE component $v = v(t)$ of the uncontrolled shadow system.

that

$$t^{N/2}\|e^{tD\Delta}(w_0 - \int_\Omega w_0 \,\mathrm{d}x)\|_{L^2(\Omega)} \leq CK^{-N/2}, \quad \forall t > 0. \tag{4.2}$$

The proof of this lemma is standard and can be found in Appendix A.2. Now, we are in position to prove our result.

Proof of Theorem 1.3. In what follows, $C > 0$ will denote a constant that may vary from line to line, that may depend on the data of the problem but which is independent of the diffusion coefficient K.

The proof of this result is inspired by some arguments presented in [31]. Hereinafter, we denote by (u^K, z^K) the solutions to (4.1) with diffusion coefficient K and control $f = f_S$. Then, for $t \in [0, T]$, (u^K, z^K) can be represented in their integral forms as follows

$$u^K(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left(\chi_\omega f_S(s) - a\, u^K(s) - b\, z^K(s)\right) \mathrm{d}s, \tag{4.3}$$

$$z^K(t) = e^{tK\Delta}z_0 + \int_0^t e^{(t-s)K\Delta}\left(c\, u^K(s) + d\, v^K(s)\right) \mathrm{d}s, \tag{4.4}$$

Likewise, we write the solutions to the corresponding shadow system as

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left(\chi_\omega f_S(s) - au(s) - bv(s)\right) \mathrm{d}s, \tag{4.5}$$

$$v(t) = v_0 + \int_0^t \left[c\int_\Omega u(s)\,\mathrm{d}x + d\, v(s)\right] \mathrm{d}s. \tag{4.6}$$

Subtracting (4.3) from (4.5) and computing the L^2-norm, we obtain

$$\|u^K(t) - u(t)\|_{L^2(\Omega)} \leq \int_0^t \|e^{(t-s)\Delta}\left[a\left(u^K(s) - u(s)\right) + b\left(z^K(s) - v(s)\right)\right]\|_{L^2(\Omega)}\,\mathrm{d}s \tag{4.7}$$

Respectively, subtracting (4.4) from (4.6) we get

$$\begin{aligned}\|z^K(t) - v(t)\|_{L^2(\Omega)} &\leq \|e^{tK\Delta}\left(v_0 - \int_\Omega v_0\,\mathrm{d}x\right)\|_{L^2(\Omega)} \\ &\quad + \int_0^t \|e^{(t-s)K\Delta}\left[c\left(u^K(s) - \int_\Omega u^K(s)\,\mathrm{d}x\right) + d\left(z^K(s) - \int_\Omega z^K(s)\,\mathrm{d}x\right)\right]\|_{L^2(\Omega)} \\ &\quad + \int_0^t \int_\Omega \left|c\left(u^K(s) - u(s)\right) + d\left(z^K(s) - v(s)\right)\right| \mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{4.8}$$

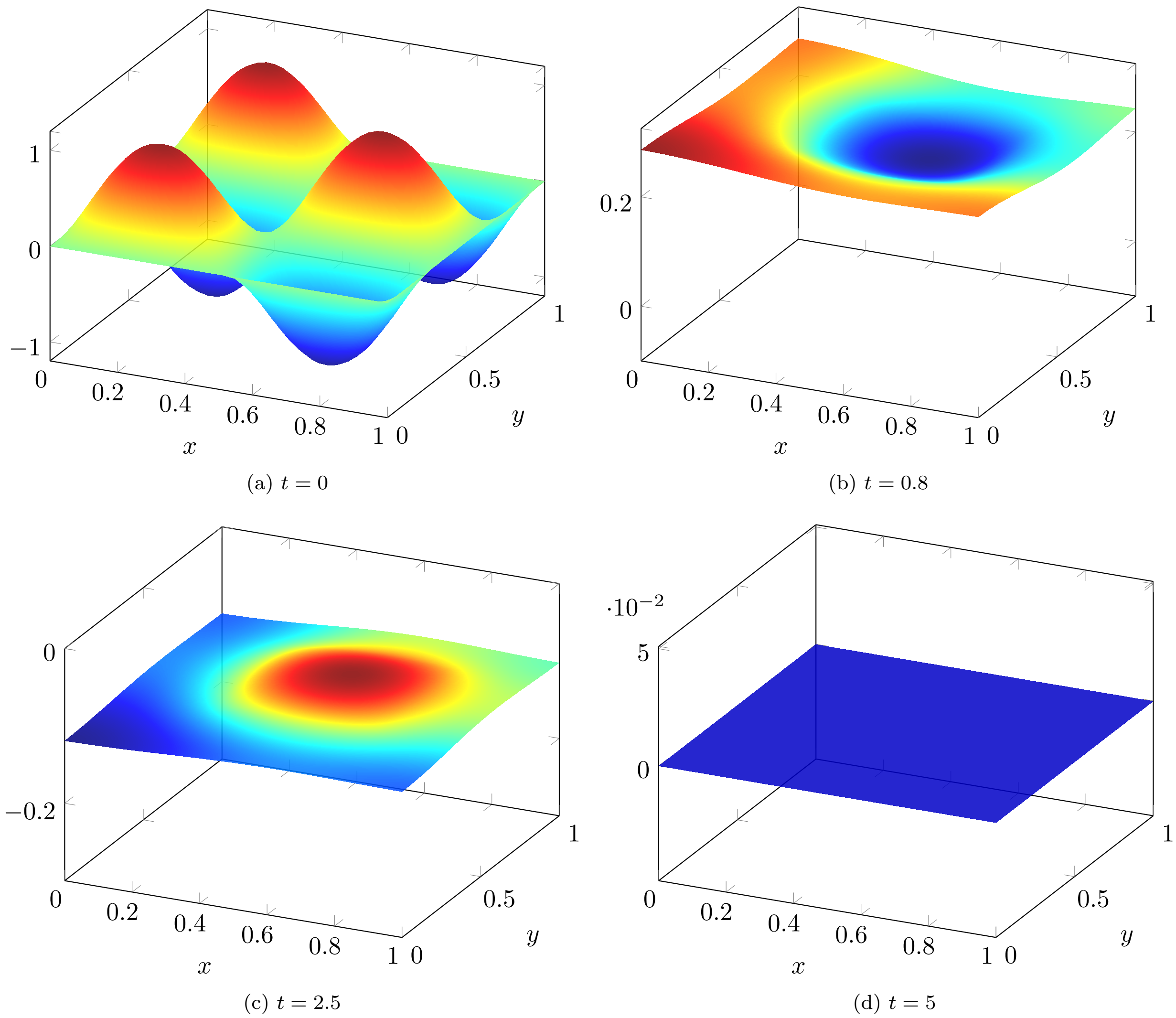


Figure 3: Evolution in time of the PDE component $u = u(x, t)$ of the controlled shadow system.

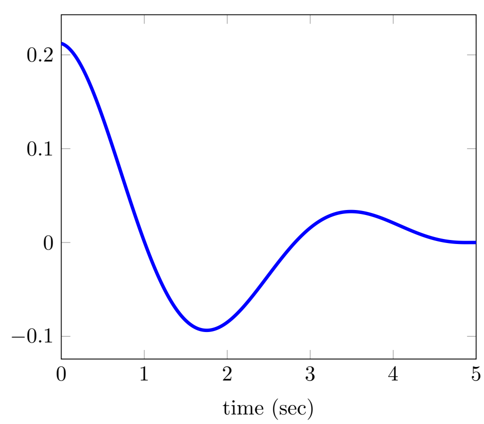


Figure 4: Evolution in time of the ODE component $v = v(t)$ of the controlled shadow system.

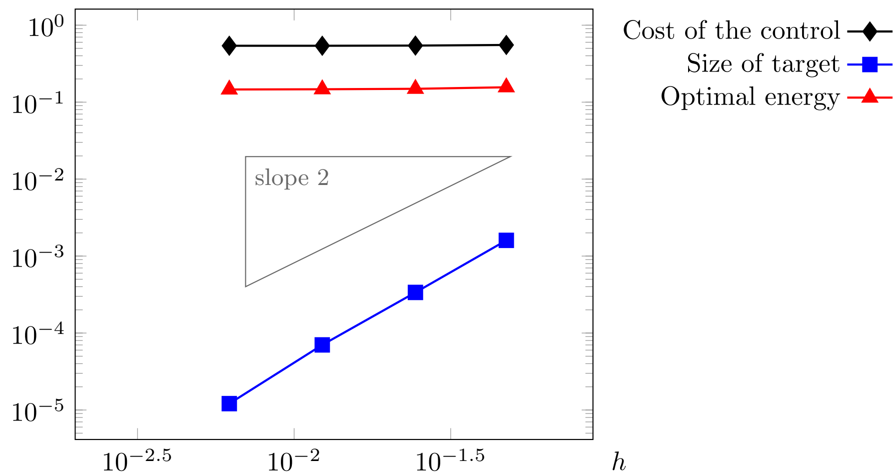


Figure 5: Convergence properties of the method for the control of shadow systems.

Note that here we have used repeatedly property *a)* from Lemma 4.1. Moreover, multiplying by $t^{N/2}$ on both sides of (4.7) and (4.8), adding them and applying Gronwall lemma, we obtain

$$
\begin{aligned}
&t^{N/2}\left(\|u^K(t)-u(t)\|_{L^2(\Omega)}+\|z^K(t)-v(t)\|_{L^2(\Omega)}\right)\\
&\quad\le C\left(t^{N/2}\|e^{tK\Delta}\left(v_0-\int_\Omega v_0\,\mathrm{d}x\right)\|_{L^2(\Omega)}+t^{N/2}\int_0^t\|e^{(t-s)K\Delta}\left[c\left(u^K(s)-\int_\Omega u^K(s)\,\mathrm{d}x\right)\right]\|_{L^2(\Omega)}\,\mathrm{d}s\right.\\
&\qquad\left.+t^{N/2}\int_0^t\|e^{(t-s)K\Delta}\left[c\left(z^K(s)-\int_\Omega z^K(s)\,\mathrm{d}x\right)\right]\|_{L^2(\Omega)}\,\mathrm{d}s\right).
\end{aligned}
\tag{4.9}
$$

Observe that the first term on the right-hand side of the above inequality can be bounded by using property *b)* from Lemma 4.1. To deal with the second and third one, we apply estimate (A.3) with $p=q=2$, more precisely

$$
\begin{aligned}
&t^{N/2}\left(\int_0^t\|e^{(t-s)K\Delta}\left[c\left(u^K(s)-\int_\Omega u^K(s)\,\mathrm{d}x\right)\right]\|_{L^2(\Omega)}\,\mathrm{d}s\right.\\
&\qquad\left.+\int_0^t\|e^{(t-s)K\Delta}\left[d\left(z^K(s)-\int_\Omega z^K(s)\,\mathrm{d}x\right)\right]\|_{L^2(\Omega)}\,\mathrm{d}s\right)\\
&\qquad\qquad\le Ct^{N/2}\int_0^t e^{-(t-s)K\lambda_1}\left(\|U(K,s)\|_{L^2(\Omega)}+\|Z(K,s)\|_{L^2(\Omega)}\right)\,\mathrm{d}s
\end{aligned}
\tag{4.10}
$$

with

$$
U(D,s)=u^K(s)-\int_\Omega u^K(s)\,\mathrm{d}x\quad\text{and}\quad Z(K,s)=z^K(s)-\int_\Omega z^K(s)\,\mathrm{d}x,
$$

and where $\lambda_1>0$ denotes the first nonzero eigenvalue of $-\Delta$ (with Neumann boundary conditions). Here, $C>0$ does not depend on t, K, U or Z.

Applying Hölder inequality on the right-hand side of (4.10), we get

$$
\begin{aligned}
&t^{N/2}\int_0^t e^{-(t-s)K\lambda_1}\left(\|U(K,s)\|_{L^2(\Omega)}+\|Z(K,s)\|_{L^2(\Omega)}\right)\,\mathrm{d}s\\
&\quad\le t^{N/2}\left(\int_0^t e^{-(t-s)K\lambda_1}\,\mathrm{d}s\right)\left(\operatorname*{ess\,sup}_{0\le s\le t}\left\{\|U(K,s)\|_{L^2(\Omega)}+\|Z(K,s)\|_{L^2(\Omega)}\right\}\right)
\end{aligned}
\tag{4.11}
$$

We will bound the term inside the second parenthesis by a uniform constant independent of K. For this, we will take advantage of the regularity of (u^K,z^K) (cf. [10, pp. 524]). Multiplying (4.1) by (u^K,z^K) in

$[L^2(\Omega)]^2$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(|u^K|^2 + |z^K|^2 \,\mathrm{d}x\right) + \int_\Omega \left(|\nabla u^K|^2 \; + K|\nabla z^K|^2 \,\mathrm{d}x\right) \\ \leq C\left[\int_\omega |f_S|^2 \,\mathrm{d}x + \int_\Omega \left(|u^K|^2 + |z^K|^2\right) \,\mathrm{d}x\right]$$

where C only depends on a, b, c, d. Dropping the second term in the left-hand side and applying Gronwall Lemma, we deduce that

$$\|u^K(s)\|^2_{L^2(\Omega)} + \|z^K(s)\|^2_{L^2(\Omega)} \leq C\left(\|u_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)} + \int_0^t \int_\omega |f_S|^2 \,\mathrm{d}x\mathrm{d}t\right).$$

Thanks to the constructive method presented in the previous section (in particular Proposition 3.1), we can build the control f_S in such way that it is uniformly bounded by the observability constant (see eq. (2.2)). Hence, we obtain the estimate

$$\max_{0\leq s\leq t}\left\{\|u^K(s)\|_{L^2(\Omega)} + \|z^K(s)\|_{L^2(\Omega)}\right\} \leq C, \tag{4.12}$$

for some uniform $C > 0$ depending on T, a, b, c, d, $\|u_0\|_{L^2(\Omega)}$ and $\|z_0\|_{L^2(\Omega)}$, but which is independent of K. Using (4.12), together with triangle and Jensen inequalities, we deduce from (4.11) that

$$t^{N/2}\left(\int_0^t \|e^{(t-s)K\Delta}\left[c\left(u^K(s) - \int_\Omega u^K(s)\,\mathrm{d}x\right)\right]\|_{L^2(\Omega)}\,\mathrm{d}s \right. \\ \left. + \int_0^t \|e^{(t-s)K\Delta}\left[d\left(z^K(s) - \int_\Omega z^K(s)\,\mathrm{d}x\right)\right]\|_{L^2(\Omega)}\,\mathrm{d}s\right) \leq Ct^{N/2}\int_0^t e^{-(t-s)K\lambda_1}\,\mathrm{d}s$$

On the other hand, by a change of variables and taking the supremum in $0 \leq t \leq T$ on the right-hand side of the above expression, we get

$$t^{N/2}\left(\int_0^t \|e^{(t-s)K\Delta}\left[c\left(u^K(s) - \int_\Omega u^K(s)\,\mathrm{d}x\right)\right]\|_{L^2(\Omega)}\,\mathrm{d}s \right. \\ \left. + \int_0^t \|e^{(t-s)K\Delta}\left[d\left(z^K(s) - \int_\Omega z^K(s)\,\mathrm{d}x\right)\right]\|_{L^2(\Omega)}\,\mathrm{d}s\right) \leq CK^{-1} \tag{4.13}$$

for a constant C not depending on K.

Putting together (4.9), (4.13) and using property *b)* from Lemma 4.1, we deduce

$$t^{N/2}\left(\|u^K(t) - u(t)\|_{L^2(\Omega)} + \|z^K(t) - v(t)\|_{L^2(\Omega)}\right) \leq CK^{-\mu}, \quad t > 0$$

where $\mu = \min\{1, N/2\}$. Finally, by setting $t = T$ in the above estimate and since f_S is a null control for (u, v) solution to (1.3), we get

$$\|u^K(T)\|_{L^2(\Omega)} + \|z^K(T)\|_{L^2(\Omega)} \leq CK^{-\mu} \tag{4.14}$$

Therefore, for given $\varepsilon > 0$, we conclude the existence of K sufficiently large such that (1.5) holds. This concludes the proof. $\square$

5 Uniform null-controllability for the original reaction-diffusion system

5.1 A uniform observability inequality

As seen in the previous section, the control obtained for the shadow system (1.3) is good enough for the approximate controllability of (4.1) as long as K is large enough. In this section we show that it is possible to obtain the null controllability of (4.1), uniformly with respect to the diffusion parameter K.

Let us introduce the adjoint system to (4.1)

$$\begin{cases} -\varphi_t - \Delta\varphi + a\,\varphi = c\,\psi, & \text{in Q} \\ -\psi_t - K\Delta\psi - d\,\psi = -b\,\varphi, & \text{in Q}, \\ \dfrac{\partial\varphi}{\partial\nu} = \dfrac{\partial\psi}{\partial\nu} = 0, & \text{on } \Sigma, \\ \varphi(x,T) = \varphi_T(x), \quad \psi(x,T) = \psi_T(x) & \text{in } \Omega. \end{cases} \tag{5.1}$$

Here, we will prove a uniform inequality of the form

$$\int_\Omega |\varphi(0)|^2\,\mathrm{d}x + \int_\Omega |\psi(0)|^2\,\mathrm{d}x \le C \iint_{\omega\times(0,T)} |\varphi|^2\,\mathrm{d}x\mathrm{d}t, \tag{5.2}$$

valid for every $K \ge 1$ and every $(\varphi_T, \psi_T) \in [L^2(\Omega)]^2$.

We will prove (5.2) from an appropriate Carleman inequality for (5.1). For this, we will need a refined version of the Carleman estimate (2.4), which reads as follows.

Lemma 5.1. *There exists $C = C(\Omega,\mathcal{B})$ and $\lambda_0 = \lambda_0(\Omega,\mathcal{B})$ such that, for every $\lambda \ge \lambda_0$, there exists $s_0 = s_0(\Omega,\mathcal{B},\lambda)$ such that, for any $s \ge s_0(T+T^2)$, any $q_T \in L^2(\Omega)$ and any $g \in L^2(Q)$, the weak solution to*

$$\begin{cases} -\sigma q_t - \Delta q = g(x,t), & \text{in } Q \\ \dfrac{\partial q}{\partial\nu} = 0 & \text{on } \Sigma \\ q(x,T) = q_T(x) & \text{in } \Omega, \end{cases}$$

satisfies

$$\tilde{I}(s,\sigma;q) \le C\left(\iint_Q e^{-2s\alpha}|f|^2\,\mathrm{d}x\mathrm{d}t + s^3\iint_{\mathcal{B}\times(0,T)} e^{-2s\alpha}\xi^3|q|^2\,\mathrm{d}x\mathrm{d}t\right), \tag{5.3}$$

for any $0 < \sigma \le 1$ and where $\tilde{I}(s,\sigma;z)$ stands for

$$\tilde{I}(s,\sigma;q) := s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}\left(\sigma^2|q_t|^2 + \sum_{i,j=1}^N \left|\frac{\partial^2 q}{\partial x_i\partial x_j}\right|^2\right)\mathrm{d}x\mathrm{d}t$$
$$+ s\iint_Q e^{-2s\alpha}\xi|\nabla q|^2\,\mathrm{d}x\mathrm{d}t + s^3\iint_Q e^{-2s\alpha}\xi^3|q|^2\,\mathrm{d}x\mathrm{d}t.$$

The proof of Lemma 5.1 can be deduced from the Carleman inequality for the heat equation with homogeneous Neumann boundary conditions given in [13] or [17] and arguing as in [5] (see also [6]).

The main result of this section is the following.

Theorem 5.2. *Assume that $c \neq 0$. Given $K \ge 1$, there exist $C = C(\Omega,\omega)$ and $\lambda_1 = \lambda_1(\Omega,\omega)$ such that, for every $\lambda \ge \lambda_1$, there exists $s_1 = s_1(\Omega,\omega,\lambda,a,b,c,d)$ such that, for any $s \ge s_1(T+T^2)$, any $(\varphi_T,\psi_T) \in [L^2(\Omega)]^2$, the solution to (5.1) satisfies*

$$s^3\iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,\mathrm{d}x\mathrm{d}t + s^3\iint_Q e^{-2s\alpha}\xi^3|\psi|^2\,\mathrm{d}x\mathrm{d}t \le Cs^{15}\iint_{\omega\times(0,T)} \left(e^{-2s\widehat{\alpha}} + e^{-4s\widehat{\alpha}+3s\alpha^\star}\right)(\widehat{\xi})^{15}|\varphi|^2\,\mathrm{d}x\mathrm{d}t. \tag{5.4}$$

Proof. For the purpose of our proof, let us consider $\omega_i \subset \Omega$, $i = 0,1$ such that $\omega_0 \subset\subset \omega_1 \subset\subset \omega$ and let us assume for the moment that $\varphi_T, \psi_T \in C_0^\infty(\Omega)$. In what follows, C will denote a generic constant only depending on Ω and ω that may change from line to line. For clarity, we have divided the proof in three steps.

Step 1. First estimate for the coupled system

We apply estimate (2.4) to the first equation of (5.1) with $\mathcal{B} = \omega_0$ and $g = c\psi - a\varphi$, thus

$$I_{s,\lambda}(\varphi) \le C\left(\iint_Q e^{-2s\alpha}|c\psi - a\varphi|^2\,\mathrm{d}x\mathrm{d}t + s^3\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\lambda^4\xi^3|\varphi|^2\,\mathrm{d}x\mathrm{d}t\right).$$

Now, we divide over K the second equation of (5.1) and apply inequality (5.3) with $g = K^{-1}(d\psi - b\varphi)$ and $\mathcal{B} = \omega_0$, which yields

$$\tilde{I}(s, K^{-1}; \psi) \leq C \left(\iint_Q e^{-2s\alpha} |K^{-1}(d\psi - b\varphi)|^2 \,\mathrm{d}x\mathrm{d}t + s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} \xi^3 |\psi|^2 \,\mathrm{d}x\mathrm{d}t \right).$$

Since $K^{-1} \leq 1$ and choosing $\lambda_1 = \max\{\lambda^*, \lambda_0\}$ (see the conditions from Lemmas 2.3 and 5.1), we can add the previous inequalities and absorb the lower order terms. More precisely, we obtain

$$I_{s,\lambda}(\varphi) + \tilde{I}(s, K^{-1}; \psi) \leq C \left(s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} \xi^3 |\varphi|^2 \,\mathrm{d}x\mathrm{d}t + s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} \xi^3 |\psi|^2 \,\mathrm{d}x\mathrm{d}t \right), \quad (5.5)$$

for every $\lambda \geq \lambda_1$ and $s \geq s_1(T + T^2)$, where s_1 is a constant only depending on Ω, ω_0 and the coefficients a, b, c, d.

Step 2. Estimate of the local integral of ψ

In this step, we estimate the local integral of ψ in the right-hand side of (5.5). This will be done by employing the equation verified by φ in (5.1). Consider a function $\eta \in C_0^\infty(\omega_1)$ satisfying $0 \leq \eta \leq 1$ and $\eta(x) \equiv 1$ for all $x \in \omega_0$. We have

$$\begin{aligned} s^3 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} \xi^3 |\psi|^2 \,\mathrm{d}x\mathrm{d}t &\leq s^3 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \xi^3 \, c\, |\psi|^2 \eta \,\mathrm{d}x\mathrm{d}t \\ &= \frac{1}{c} s^3 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \xi^3 \psi(-\varphi_t - \Delta\varphi + a\varphi) \eta \,\mathrm{d}x\mathrm{d}t \\ &:= M_1 + M_2 + M_3. \end{aligned}$$

Observe that at this point is crucial to have $c \neq 0$.

Let us estimate each term M_i, $1 \leq i \leq 3$. Using Hölder and Young inequalities it can be readily seen that

$$|M_3| \leq \delta s^3 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \xi^3 |\psi|^2 \eta \,\mathrm{d}x\mathrm{d}t + C_\delta s^3 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \xi^3 |\varphi|^2 \eta \,\mathrm{d}x\mathrm{d}t, \quad (5.6)$$

for any $\delta > 0$.

On the other hand, integrating by parts in the space variable and using the fact that that $|\nabla(e^{-2s\alpha} s^3 \xi^3)| \leq Ce^{-2s\alpha} s^4 \xi^4$, it is not difficult to see that

$$\begin{aligned} |M_2| \leq \delta &\left(s \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \xi |\nabla\psi|^2 \,\mathrm{d}x\mathrm{d}t + s^3 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \xi^3 |\psi|^2 \,\mathrm{d}x\mathrm{d}t \right) \\ &+ C_\delta s^5 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \xi^5 |\nabla\varphi|^2 \,\mathrm{d}x\mathrm{d}t. \end{aligned} \quad (5.7)$$

Finally, we estimate M_1 as follows

$$|M_1| \leq \delta s^3 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \xi^3 |\psi|^2 \,\mathrm{d}x\mathrm{d}t + C_\delta s^3 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \xi^3 |\varphi_t|^2 \,\mathrm{d}x\mathrm{d}t. \quad (5.8)$$

Putting together the estimates (5.5)–(5.8) and taking $\delta > 0$ sufficiently small, we can absorb the terms corresponding to ψ in the left-hand side and obtain

$$\begin{aligned} I_{s,\lambda}(\varphi) + \tilde{I}(s, K^{-1}; \psi) \leq C &\left(s^5 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \xi^5 (|\varphi|^2 + |\nabla\varphi|^2) \,\mathrm{d}x\mathrm{d}t \right. \\ &\left. + s^3 \iint_{\omega_1 \times (0,T)} e^{-2s\alpha} \xi^3 |\varphi_t|^2 \,\mathrm{d}x\mathrm{d}t \right). \end{aligned} \quad (5.9)$$

To eliminate the local integral of $\nabla\varphi$, we consider a cut-off function $\tilde\eta \in C_0^\infty(\omega)$ satisfying $0 \le \tilde\eta \le 1$, $\tilde\eta \equiv 1$ on ω_1. Integrating by parts yields

$$s^5 \iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^5\tilde\eta|\nabla\varphi|^2\,\mathrm{d}x\mathrm{d}t = -\, s^5 \iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^5\tilde\eta\Delta\varphi\varphi\,\mathrm{d}x\mathrm{d}t \\ - \frac{1}{2}s^5 \iint_{\omega\times(0,T)} \Delta(\tilde\eta e^{-2s\alpha}\xi^5)|\varphi|^2\,\mathrm{d}x\mathrm{d}t$$

Since $|\Delta(\tilde\eta e^{-2s\alpha}\xi^5)| \le Cs^2\xi^7 e^{-2s\alpha}$ in $\omega\times(0,T)$ and using Hölder and Young inequalities, we have

$$s^5 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^5|\nabla\varphi|^2\,\mathrm{d}x\mathrm{d}t \le \delta s^{-1} \iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^{-1}|\Delta\varphi|^2\,\mathrm{d}x\mathrm{d}t \\ + C_\delta s^{15} \iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^{15}|\varphi|^2\,\mathrm{d}x\mathrm{d}t, \tag{5.10}$$

for all $\delta > 0$. Combining estimates (5.9)–(5.10) together with the definitions on the weight functions (2.3) and since $\omega_1 \subset\subset \omega$, we get

$$I_{s,\lambda}(\varphi) + \tilde I(s, K^{-1}; \psi) \le C\left(s^{15} \iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^{15}|\varphi|^2\,\mathrm{d}x\mathrm{d}t + s^3 \iint_{\omega\times(0,T)} e^{-2s\widehat\alpha}(\widehat\xi)^3|\varphi_t|^2\,\mathrm{d}x\mathrm{d}t\right). \tag{5.11}$$

Remark 5.3. *Observe that if instead of bounding the term M_1 as in (5.8) we would have integrated by parts in the time variable, we would have obtained*

$$|M_1| \le \delta\left(s^{-1} \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^{-1}K^{-2}|\psi_t|^2\,\mathrm{d}x\mathrm{d}t + s^3 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}s^3|\psi|^2\,\mathrm{d}x\mathrm{d}t\right) \\ + C_\delta K^2 s^7 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^7|\varphi|^2\,\mathrm{d}x\mathrm{d}t \tag{5.12}$$

for all $\delta > 0$. Arguing as above, we can derive a Carleman inequality of the form

$$s^3 \iint_Q e^{-2s\alpha}\xi^3|\psi|^2\,\mathrm{d}x\mathrm{d}t + s^3 \iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,\mathrm{d}x\mathrm{d}t \le CK^2\left(s^{15} \iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^{15}|\varphi|^2\,\mathrm{d}x\mathrm{d}t\right) \tag{5.13}$$

From here, we can prove an observability inequality for (5.1) which would imply that system (4.1) is indeed null controllable for every fixed K, but whose cost of controllability will not remain bounded as $K \to +\infty$. By avoiding using (5.12), we will see that we can improve (5.13) in the sense that no powers of K appear on the right-hand side.

Step 3. Estimations on the term involving φ_t

In this step, we deal with the second term appearing on the right-hand side of (5.11). Integrating by parts in the time variable yields

$$s^3 \iint_{\omega\times(0,T)} e^{-2s\widehat\alpha}(\widehat\xi)^3|\varphi_t|^2 = -\, s^3 \iint_{\omega\times(0,T)} e^{-2s\widehat\alpha}(\widehat\xi)^3\varphi_{tt}\varphi\,\mathrm{d}x\mathrm{d}t \\ + \frac{s^3}{2} \iint_{\omega\times(0,T)} \left(e^{-2s\widehat\alpha}(\widehat\xi)^3\right)_{tt}|\varphi|^2\,\mathrm{d}x\mathrm{d}t, \tag{5.14}$$

and since

$$s^3 \iint_{\omega_\times(0,T)} e^{-2s\widehat\alpha}(\widehat\xi)^3\varphi_{tt}\varphi\,\mathrm{d}x\mathrm{d}t \le \frac{s^{-6}}{2} \iint_{\omega_\times(0,T)} e^{-2s\alpha^\star}(\xi^\star)^{-6}|\varphi_{tt}|^2\,\mathrm{d}x\mathrm{d}t \\ + \frac{s^{12}}{2} \iint_{\omega\times(0,T)} e^{-4s\widehat\alpha+2s\alpha^\star}(\xi^\star)^6(\widehat\xi)^6|\varphi^2|\,\mathrm{d}x\mathrm{d}t \tag{5.15}$$

it is enough to estimate the integral of φ_{tt} in the right-hand side of (5.15). Observe that we have introduced the smaller weight function $e^{-2s\alpha^\star}$ and that it only depends on time. This will facilitate to obtain an estimate of φ_{tt} and allows to prove that the weight $e^{-4s\widehat{\alpha}+2s\alpha^\star}$ remains bounded.

Using the equation verified by φ and ψ (see system (5.1)) we have that

$$-\varphi_{tt} - \Delta\varphi_t + a\varphi_t = -cK\Delta\psi - d\psi + b\varphi \quad \text{in } Q. \tag{5.16}$$

Multiplying both sides of (5.16) by $e^{-3s\alpha^\star}(\xi^\star)^{-6}\varphi_{tt}$ and integrating over Q, we obtain

$$\begin{aligned}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-6}|\varphi_{tt}|^2\,\mathrm{d}x\mathrm{d}t = &-\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-6}\Delta\varphi_t\varphi_{tt}\,\mathrm{d}x\mathrm{d}t + a\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-6}\varphi_t\varphi_{tt}\,\mathrm{d}x\mathrm{d}t\\ &+ cK\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-6}\Delta\psi\varphi_{tt}\,\mathrm{d}x\mathrm{d}t + d\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-6}\psi\varphi_{tt}\,\mathrm{d}x\mathrm{d}t\\ &- b\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-6}\varphi\varphi_{tt}\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{5.17}$$

Integrating by parts with respect to space and then to time, and since φ_t also satisfies homogeneous Neumann boundary conditions, we observe that

$$-\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-6}\Delta\varphi_t\varphi_{tt}\,\mathrm{d}x\mathrm{d}t = -\frac{1}{2}\iint_Q \left(e^{-2s\alpha^\star}(\xi^\star)^{-6}\right)_t|\nabla\varphi_t|^2\,\mathrm{d}x\mathrm{d}t.$$

Notice also that no terms at $t=0$ and $t=T$ appear due to the singularity of the weight function.

Using the above identity and applying repeatedly Hölder and Young inequalities in (5.17), it is straightforward to show that

$$\begin{aligned}&s^{-6}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-6}|\varphi_{tt}|^2\,\mathrm{d}x\mathrm{d}t\\ &\quad\leq Cs^{-6}\left(\iint_Q \left|\left(e^{-2s\alpha^\star}(\xi^\star)^{-6}\right)_t\right||\nabla\varphi_t|^2\,\mathrm{d}x\mathrm{d}t + \iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-6}\left(|\varphi_t|^2+|\theta|^2+|\varphi|^2\right)\,\mathrm{d}x\mathrm{d}t\right)\\ &\qquad + CK^2s^{-6}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-6}|\Delta\psi|^2\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{5.18}$$

Since

$$\left|\left(e^{-2s\alpha^\star}(\xi^\star)^\mu\right)_t\right| \leq Cs^2e^{-2s\alpha^\star}(\xi^\star)^{\mu+2}, \quad \forall\mu\in\mathbb{N}, \tag{5.19}$$

we can estimate the first term on the right-hand side of (5.18) as

$$s^{-6}\iint_Q \left|\left(e^{-2s\alpha^\star}(\xi^\star)^{-6}\right)_t\right||\nabla\varphi_t|^2\,\mathrm{d}x\mathrm{d}t \leq Cs^{-4}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}|\nabla\varphi_t|^2\,\mathrm{d}x\mathrm{d}t \tag{5.20}$$

Now, multiplying (5.16) by $e^{-2s\alpha^\star}(\xi^\star)^{-4}\varphi_t$ and integrating by parts in Q we readily obtain

$$\begin{aligned}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}|\nabla\varphi_t|^2\,\mathrm{d}x\mathrm{d}t = &-\frac{1}{2}\iint_Q \left(e^{-2s\alpha^\star}(\xi^\star)^{-4}\right)_t|\varphi_t|^2\,\mathrm{d}x\mathrm{d}t - a\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}|\varphi_t|^2\,\mathrm{d}x\mathrm{d}t\\ &- cK\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}\Delta\psi\varphi_t\,\mathrm{d}x\mathrm{d}t - d\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}\psi\varphi_t\,\mathrm{d}x\mathrm{d}t\\ &+ b\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}\varphi\varphi_t\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{5.21}$$

As before, we apply Hölder and Young inequalities together with estimate (5.19) in equation (5.21) to deduce

$$\begin{aligned}&s^{-4}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}|\nabla\varphi_t|^2\,\mathrm{d}x\mathrm{d}t\\ &\quad\leq Cs^{-2}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-2}|\varphi_t|^2\,\mathrm{d}x\mathrm{d}t + CK^2s^{-4}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}|\Delta\psi|^2\,\mathrm{d}x\mathrm{d}t\\ &\qquad + Cs^{-4}\left(\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}(|\psi|^2+|\varphi_t|^2+|\varphi|^2)\,\mathrm{d}x\mathrm{d}t\right)\end{aligned} \tag{5.22}$$

Combining estimates (5.18), (5.20) and (5.22), we finally deduce

$$
\begin{aligned}
&s^{-6}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-6}|\varphi_{tt}|^2\,\mathrm{d}x\mathrm{d}t \\
&\quad\leq CK^2 s^{-4}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}|\Delta\psi|^2\,\mathrm{d}x\mathrm{d}t + Cs^{-2}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-2}|\varphi_t|^2\,\mathrm{d}x\mathrm{d}t \\
&\qquad + Cs^{-4}\left(\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}(|\psi|^2+|\varphi_t|^2+|\varphi|^2)\,\mathrm{d}x\mathrm{d}t\right).
\end{aligned}
\tag{5.23}
$$

Step 4. Conclusion

From (5.14)–(5.15) and (5.23), we can write

$$
\begin{aligned}
&s^3\iint_{\omega\times(0,T)} e^{-2s\widehat{\alpha}}(\widehat{\xi})^3|\varphi_t|^2\,\mathrm{d}x\mathrm{d}t \\
&\quad\leq Cs^{12}\iint_{\omega\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^\star}(\widehat{\xi})^{12}|\varphi|^2\,\mathrm{d}x\mathrm{d}t + CK^2s^{-4}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}|\Delta\psi|^2\,\mathrm{d}x\mathrm{d}t \\
&\qquad + Cs^{-2}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-2}|\varphi_t|^2\,\mathrm{d}x\mathrm{d}t + Cs^{-4}\left(\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}(|\psi|^2+|\varphi_t|^2+|\varphi|^2)\,\mathrm{d}x\mathrm{d}t\right).
\end{aligned}
\tag{5.24}
$$

Observe that the we have estimated the local integral of φ_t by the observation of φ in $\omega\times(0,T)$ and some other global terms. Except for the one containing $\Delta\psi$, all of these terms are estimated uniformly with respect to K. Notice also that they have negative powers on the parameter s and the weight $(\xi^\star)$. A simple computation allows to prove that

$$
s^{-1}(\xi^\star)^{-1}\leq C \tag{5.25}
$$

for some $C>0$ only depending on Ω and ω.

Therefore, replacing (5.24) in (5.11) and since $e^{-2s\alpha^\star}\leq e^{-2s\alpha}$, we can use (5.25) and take the parameter s large enough to absorb the remaining terms. More precisely, we have

$$
\begin{aligned}
I_{s,\lambda}(\varphi)+\tilde{I}(s,K^{-1};\psi) \leq Cs^{15}&\iint_{\omega\times(0,T)}\left(e^{-4s\widehat{\alpha}+2s\alpha^\star}+e^{-2s\widehat{\alpha}}\right)(\widehat{\xi})^{15}|\varphi|^2\,\mathrm{d}x\mathrm{d}t \\
&+CK^2s^{-4}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}|\Delta\psi|^4\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{5.26}
$$

The last task is to estimate uniformly with respect to K the second term in the right-hand side of the above expression. To this end, we present the following lemma.

Lemma 5.4. *Given $K\geq 1$, the solution ψ to the second equation of (5.1) satisfies*

$$
K^2s^{-4}\iint_Q e^{-2s\alpha^\star}|\Delta\psi|^2(\xi^\star)^4\,\mathrm{d}x\mathrm{d}t \leq C\left(s^{-4}\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}|\varphi|^2\,\mathrm{d}x\mathrm{d}t + \iint_Q e^{-2s\alpha^\star}|\theta|^2\,\mathrm{d}x\mathrm{d}t\right)
\tag{5.27}
$$

for some constat $C>0$ independent of K.

We prove this result in Appendix A.3. From here, it is enough to substitute estimate (5.27) in (5.26) and, using again (5.25), one can absorb the remaining terms by taking s sufficiently large.

In this way, the Carleman inequality (5.4) follows immediately by using the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$. Thus, the proof is complete. □

For the sake of brevity, we will omit the proof of Theorem 1.4. In fact, from the Carleman estimate (5.4) and some energy inequalities it is standard to prove the observability inequality (5.2). Moreover, using well-known results of control theory for linear PDE (see, e.g., [16]) one can prove the existence of a control f and the uniform estimate 1.7 by means of a suitable minimization problem for each $K\geq 1$.

5.2 Proof of Theorem 1.5

Here, we will use the uniform controllability result presented in Theorem 1.4 to deduce a controllability result for the shadow system 1.3. The proof of Theorem 1.5 relies on several well-known arguments and, for completeness, we sketch it briefly.

For each $K > 0$, let us denote by f^K the control provided by Theorem 1.4 and let us write as (u^K, v^K) the corresponding controlled solution to (1.4) for any fixed $(u_0, z_0) \in [L^2(\Omega)]^2$.

It is classical (see, e.g., [10]) that there exists a unique weak solution to (1.4) verifying

$$
\begin{aligned}
&u^K, z^K \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\\
&u_t^K, z_t^K \in L^2(0,T;H^1(\Omega)').
\end{aligned}
$$

Moreover, using standard energy estimates and the uniform bound (1.7) for the control f^K, it is not difficult to see that

$$
\max_{0\le t\le T} \|u^K(t)\|^2_{L^2(\Omega)} + \iint_Q |\nabla u^K|^2 \,\mathrm{d}x\mathrm{d}t \le C \tag{5.28}
$$

$$
\max_{0\le t\le T} \|z^K(t)\|^2_{L^2(\Omega)} + K\iint_Q |\nabla z^K|^2 \,\mathrm{d}x\mathrm{d}t \le C \tag{5.29}
$$

for some uniform $C > 0$ independent of K. Then, up to a subsequence of $K \to +\infty$ we have

$$
\begin{aligned}
&f^K \rightharpoonup f \quad \text{weakly in } L^2(\omega \times (0,T)),\\
&u^K, z^K \rightharpoonup u, v \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)).
\end{aligned}
$$

Since u_t^K and z_t^K are also uniformly bounded in $L^2(0,T;(H^1(\Omega))')$, we can use classical compactness results (see, for instance, [39]) to deduce

$$
u^K, v^K \to u, v \quad \text{strongly in } L^2(Q). \tag{5.30}
$$

Clearly u solves the first equation of (1.3) in the usual variational sense. From the uniqueness of the weak solution and since both u^K and its limit u belong to $C([0,T];L^2(\Omega))$, we conclude that $u(\cdot,T) = 0$.

It remains to clearly identify the limit problem for v. First, from (5.29) we have

$$
\iint_Q |\nabla z^K|^2 \,\mathrm{d}x\mathrm{d}t \le CK^{-1}, \tag{5.31}
$$

and as $K \to +\infty$ we see that ∇v is a.e. zero, i.e., $v = v(t)$ is a function of t only.

Integrating the second equation of (1.4) in Ω and using the boundary conditions, we get

$$
\int_\Omega z_t^K \,\mathrm{d}x = \int_\Omega \left[c\, u^K + d\, z^K\right] \mathrm{d}x
$$

which yields for $0 < \delta < t \le T$

$$
\int_\Omega \left[z^K(t) - z^K(\delta)\right] \mathrm{d}x = \int_\delta^t \int_\Omega \left[c\, u^K + d\, z^K\right] \mathrm{d}x\mathrm{d}t.
$$

Since $z^K \in C([0,T];L^2(\Omega))$, letting first $\delta \to 0$ and then $K \to +\infty$, from the above expression we conclude using (5.30) that

$$
v(t) - |\Omega|^{-1} \int_\Omega z_0 \,\mathrm{d}x = c\int_0^t \left(|\Omega|^{-1}\int_\Omega u \,\mathrm{d}x\right) \mathrm{d}t + d\int_0^t v \,\mathrm{d}t
$$

where we have used the fact that v does not depend on $x \in \Omega$.

Since $u \in L^\infty(0,T;L^2(\Omega))$, we conclude that $v \in W^{1,\infty}(0,T)$ and that v solves the ODE in system (1.3) with initial datum $v_0 = |\Omega|^{-1}\int_\Omega z_0 \,\mathrm{d}x$. Integrating in Ω the limit equation verified by u (see (1.3)) and using an argument similar to the one in the proof of Theorem 1.2, we can deduce the additional regularity $v \in C^1([0,T];\mathbb{R})$ and consequently $v(T) = 0$.

From estimate (5.31) the strong convergence of v in $L^2(0,T;H^1(\Omega))$ follows.

The arguments above show that a subsequence of the controls convergence. But, in fact, this analysis can be enriched using classical Γ-convergence arguments as in [43]. Controls are then characterized as minimizers of the corresponding adjoint functionals. One can then pass to the limit as $K \to \infty$, and show that the whole sequence of controls and controlled solutions converges.

This concludes the proof.

6 Further remarks and open questions

In this paper, we have considered some simple linear shadow models. However, there are many interesting and challenging problems related to the controllability of shadow systems. Some of them are mentioned below.

6.1 Control of the nonlinear problem

As we have mentioned in Section 1, shadow models are often used to approximate reaction-diffusion equations when one of the diffusion rates is rather large compared with the others. Adapting the results in [2] or following the methodology in [23], one is able to proof that for $f, g \in C^\infty(\mathbb{R}^2;\mathbb{R})$ and initial data "small enough" the system

$$\begin{cases} u_t - d_1\Delta u = f(u,v) + \chi_\omega h & \text{in } \Omega\times(0,T), \\ v_t - d_2\Delta v = g(u,v) & \text{in } \Omega\times(0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & \text{in } \Omega. \end{cases}$$

is *locally null-controllable* as long as $\frac{\partial g}{\partial u}(0,0) \neq 0$.

Even though we can obtain the corresponding shadow model and linearize it, we will encounter some difficulties to deal with the controllability of the nonlinear model.

In our analysis we used in an essential way the fact that the coupling coefficients are constant, which allowed us to obtain a reduced "closed" system of ODE (see eq. (2.9)). Allowing more general coefficients depending on (x,t) in the linearised system would lead to new difficulties at this respect:

$$\begin{cases} u_t - \Delta u + a(x,t)u + b(x,t)v = \chi_\omega h & \text{in } Q, \\ v' = |\Omega|^{-1}\displaystyle\int_\Omega c(\cdot,t)u\,\mathrm{d}x + \left(|\Omega|^{-1}\int_\Omega d(\cdot,t)\,\mathrm{d}x\right)v & \text{in } Q, \\ \dfrac{\partial u}{\partial\nu} = 0 & \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \quad v(0) = v_0. & \end{cases} \tag{6.1}$$

From here, we can observe that we cannot longer define ζ as $\int_\Omega u\,\mathrm{d}x$, integrate on the first equation and compute a full system of ODEs. In particular, this is an impediment for us to apply our procedure and to study the observability of the linearized adjoint system.

Notice that the approach used in Theorem 1.5 to pass to the limit also fails. Recall that its main ingredient is the uniform estimate (1.7) which, in turn, is a consequence of the uniform Carleman inequality (5.4). To prove such inequality, we obtained weighted energy estimates from the equation verified by (5.16), which was obtained by differentiating the first equation of (5.1) with respect to time and then replacing the expression of the second one. Using this approach with more general coefficients depending on x and t would introduce additional difficulties since some terms depending on the time derivatives of the coefficients a and c would appear.

Therefore, analyzing the controllability of (6.1) or obtaining a uniform Carleman inequality for (5.1) with general coefficients belonging to $L^\infty(Q)$ will be a first step in understanding more general nonlinear shadow models. This remains as an open question.

6.2 Original vs. shadow

Even if the shadow model seems easier to address, one has to be cautious with the discrepancies that might appear when dealing with nonlinear models. Indeed, one may consider the Gierer-Meinhardt system given by

$$\begin{cases} u_t - d_1\Delta u = -u + \dfrac{u^p}{v^q} & \text{in } \Omega\times(0,T), \\ \tau v_t - d_2\Delta v = -v + \dfrac{u^r}{v^s} & \text{in } \Omega\times(0,T), \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) > 0, & \text{in } \Omega. \end{cases} \tag{6.2}$$

where the exponents p, q, r are positive and s is nonnegative and verify $0 < (p-1)/r < q/(s+1)$. This system was proposed in 1972 in [18] to model the regeneration phenomena of *hydra*. As pointed out in [25], there exists a serious gap between the original system (6.2) and its corresponding shadow one. Whereas the original system has global in time solutions for some appropriate coefficients, the shadow one always has finite-time blow-up solutions for suitable choices of the initial data.

6.3 On the lack of controllability from the ODE component

We have seen that the shadow can be controlled from the PDE component as soon as $c \neq 0$.

However, this is not the case when the control is only exerted in the ODE component of the system. To illustrate this, let us consider the following simplified 1–D cascade version of (1.3)

$$\begin{cases} u_t - \partial_{xx} u + b\, v = 0, & \text{in } (0,1)\times(0,T) \\ v' = f & \text{in } (0,T), \\ \partial_x u = 0 & \text{on } \{0,1\}\times(0,T) \\ u(x,0) = u_0(x) \quad \text{in } (0,1), \quad v(0) = v_0, \end{cases} \tag{6.3}$$

where $b \neq 0$ and $f = f(t)$ is the control acting indirectly on the PDE through the coupling $b\, v$. The corresponding adjoint system reads as follows

$$\begin{cases} -\varphi_t - \partial_{xx}\varphi = 0, & \text{in } (0,1)\times(0,T) \\ -\theta' = -b\left(\displaystyle\int_0^1 \varphi\,\mathrm{d}x\right) & \text{in } (0,T), \\ \partial_x \varphi = 0 & \text{on } \{0,1\}\times(0,T) \\ \varphi(x,T) = \varphi_T(x) \quad \text{in } (0,1), \quad \theta(T) = \theta_T. \end{cases} \tag{6.4}$$

We will see that the observability of some eigenfunctions of the PDE fails to be true. To check this, it is enough to verify that the unique continuation for (6.4) fails. Assume that θ vanishes in the interval $[0,T]$, this implies that

$$b\int_0^1 \varphi\,\mathrm{d}x = 0. \tag{6.5}$$

Integrating by parts in $(0,1)\times(t,T)$ in the first equation of (6.4) and using the boundary conditions, it can be readily seen that

$$0 = \int_t^T\int_0^1 (-\varphi_t - \partial_{xx}\varphi)\,\mathrm{d}x\mathrm{d}t = -\int_0^1 \varphi(\cdot,T)\,\mathrm{d}x + \int_0^1 \varphi(\cdot,t)\,\mathrm{d}x,$$

and since $\varphi \in C([0,T]; L^2(0,1))$, we have

$$\int_0^1 \varphi(\cdot,t)\,\mathrm{d}x = \int_0^1 \varphi_T\,\mathrm{d}x, \quad \forall t \in [0,T]. \tag{6.6}$$

Combining (6.5) and (6.6), we observe that even if the integral over x of φ vanishes, this does not necessarily mean that $\varphi \equiv 0$ in $(0,1)\times[0,T]$. In fact, there are many nontrivial solutions for which its integral in the x variable vanishes.

In this particular case, it is not difficult to see that the eigenvalues and the normalized eigenfunctions for the Neumann Laplacian in the interval $(0,1)$ are

$$\lambda_k = (k-1)^2\pi^2, \quad k = 1,\dots; \quad e_k(x) = \begin{cases} 1, & k = 1, \\ \sqrt{2}\cos((k-1)\pi x), & k = 2,\dots. \end{cases}$$

Since the first eigenfunction is constant, by orthogonality, every other eigenfunction has zero mean in $(0,1)$. Therefore, the integral over x of the solution to the PDE with initial datum $\varphi_T = e_k(x)$, for $k > 1$, will vanish in the interval $[0,T]$. Even if the growth of the spectrum is well-behaved and therefore a candidate to apply the Moment Method (see [12]), there is an obstruction at the level of the eigenfunctions that prevents system (6.3) to be controllable.

6.4 Turnpike property

The turnpike theory roughly states that under suitable conditions, optimal control problems have the property that in long time horizons, the optimal trajectories and controls are exponentially close to the steady-state ones during most of the time interval. This property, which has been mainly investigated in the finite-dimensional case (see [36] and the references therein), is of interest also in hybrid PDE-ODE systems as the ones addressed in this paper.

As discussed in [36], a key point to get the convergence of finite horizon optimal control problems as T tends to infinity is to establish the role played by observability estimates: even if the underlying system is unstable or oscillatory (see [44]) the turnpike property will hold. Here we have proved the observability estimate (2.2) for the adjoint system (2.1), hence, following the spirit of [36], we expect that the turnpike property holds likewise for (1.3).

A Some technical results

A.1 Proof of Lemma 2.4

Observe that equation (2.10) can be regarded as a second order elliptic equation in one variable. Thus, using a general result on local elliptic regularity (see, e.g., [38, Chapter IV]), if $\theta \in H^{-2}(0,T)$ (and thus to $H^{-2}_{loc}(0,T)$, i.e., $H^2(t_0,t_1)$ for any $(t_0,t_1) \subset\subset (0,T)$) we have that θ has the interior regularity $\theta \in L^2(t_0,t_1)$, along with the estimate

$$\|\theta\|^2_{L^2(t_0,t_1)} \leq C\|\theta\|^2_{H^{-2}(t_0,t_1)},$$

for some constant C only depending on t_0, t_1 and the coefficients a, b, c, d but independent of θ. Since we have an improved regularity for θ (at least locally) we may apply once again this idea to deduce that $\theta \in H^2(t_0,t_1)$, with the new estimate

$$\|\theta\|^2_{H^2(t_0,t_1)} \leq C\|\theta\|^2_{L^2(t_0,t_1)} \leq C\|\theta\|^2_{H^{-2}(t_0,t_1)},$$

Notice that this can be done iteratively, gaining two derivatives in each step. We may repeat this process up to a fixed number m sufficiently large, in such way that

$$\|\theta^k\|^2_{H^{2m}(t_0,t_1)} \leq C\|\theta\|^2_{H^{-2}(t_0,t_1)} \tag{A.1}$$

Thanks to the series of embeddings of

$$H^j(t_0,t_1) \hookrightarrow H^1(t_0,t_1) \hookrightarrow C^0((t_0,t_1);\mathbb{R}), \quad j > 1,$$

we obtain that $\theta \in C^{2m-1}((t_0,t_1);\mathbb{R})$. Moreover, from estimate (A.1) we see that if θ is bounded in $H^{-2}(t_0,t_1)$, then it is also bounded in $C^{2m-1}((t_0,t_1);\mathbb{R})$. To conclude the proof, we will see that in fact the local regularity can be extended to the whole interval $[0,T]$. We rewrite the system (2.9) in the variable $\mathcal{X} := \begin{pmatrix} \theta & \theta' \end{pmatrix}^\top$, thus $\mathcal{X}$ satisfies the differential equation

$$-\mathcal{X}' + \underbrace{\begin{pmatrix} 0 & 1 \\ bc - ad & -(a+d) \end{pmatrix}}_{:=G} \mathcal{X} = 0.$$

Now, assume by contradiction that

$$\lim_{t\downarrow 0} |\mathcal{X}(t)| = +\infty. \tag{A.2}$$

This would imply that the regularity of θ cannot be extended up to $t = 0$. Since $\mathcal{X} \in C^1((t_0,t_1);\mathbb{R}^2)$, then for any $(t,t_1) \in (t_0,t_1)$ we have

$$\mathcal{X}(t) = \mathcal{X}(t_1) + \int_t^{t_1} A\mathcal{X}(\tau)\,\mathrm{d}\tau$$

Then,

$$|\mathcal{X}(t)| \leq |\mathcal{X}(t_1)| + \|A\| \int_t^{t_1} |\mathcal{X}(\tau)|\,\mathrm{d}\tau$$

where $\|A\| := \max_{|y|=1} |Ay|$. Applying Gronwall's Lemma, we readily obtain

$$|\mathcal{X}(t)| \leq |\mathcal{X}(t_1)| e^{-C(t-t_1)},$$

and, passing to the limit, (A.2) cannot hold. Since the equation is time reversible, the same argument can be used to study the limit as $t \to T$. This concludes the proof. □

A.2 Proof of Lemma 4.1

The proof of this Lemma is totally analogous to that on [31, Lemma A.1]. For completeness, we sketch it briefly.

Property *a)* is immediate, since it is well-known that every constant $C \in \mathbb{R}$ is a solution of the heat equation in a bounded domain with homogeneous Neumann boundary conditions.

For proving property *b)*, we recall the following estimate (see, e.g., [37, pp. 25]) for the heat semigroup with Neumann boundary conditions: for each $1 \leq q \leq p \leq \infty$ and every $z_0 \in L^q(\Omega)$ verifying $\int_\Omega z_0 \,\mathrm{d}x = 0$, we have

$$\|e^{tK\Delta} z_0\|_{L^p(\Omega)} \leq C \left(1 + (tK)^{-(N/2)(1/q-1/p)}\right) e^{-\lambda_1 K t} \|z_0\|_{L^q(\Omega)}, \tag{A.3}$$

for all $t > 0$, where λ_1 denotes the first nonzero eigenvalue of $-\Delta$ in Ω (with Neumann boundary conditions) and with $C > 0$ independent of t, K and z_0.

Using (A.3), with $p = 2$, $q = 1$ and $z_0 = w_0 - \int_\Omega w_0 \,\mathrm{d}x$ yields

$$\|e^{tK\Delta}\left(w_0 - \int_\Omega w_0 \,\mathrm{d}x\right)\|_{L^2(\Omega)} \leq C \left(1 + (tK)^{-N/4}\right) e^{-\lambda_1 K t} \|w_0 - \int_\Omega w_0 \,\mathrm{d}x\|_{L^1(\Omega)}$$

for all $t > 0$ and a constant $C > 0$ independent of t, K and w_0. We multiply by $t^{N/2}$ on both sides of the above inequality. Then, using the change of variables $s = tK$ and taking the supremum in the right-hand side we see that

$$\sup_{t>0} t^{N/2} \left(1 + (tK)^{-N/4}\right) e^{-\lambda_1 D t} = K^{-N/2} \sup_{s>0} (1 + s^{-N/4}) e^{-\lambda_1 s} < +\infty.$$

from which we readily obtain (4.2). Therefore the proof is complete.

A.3 Proof of Lemma 5.4

Multiplying the equation verified by ψ in (5.1) by $e^{-2s\alpha^\star}(\xi^\star)^{-4}\Delta\psi$ and integrating in Q, we have

$$\begin{aligned} K \iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}|\Delta\psi|^2 \,\mathrm{d}x\mathrm{d}t = &- \iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}\psi_t \Delta\psi \,\mathrm{d}x\mathrm{d}t \\ &+ \iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}(d\theta - b\varphi)\Delta\theta \,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{A.4}$$

Moreover, integrating by parts in time and space, we see that

$$\begin{aligned} - \iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}\psi_t \Delta\psi \,\mathrm{d}x\mathrm{d}t &= \iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}\nabla\psi_t \nabla\psi \,\mathrm{d}x\mathrm{d}t \\ &= \frac{1}{2}\iint_Q \left(e^{-2s\alpha^\star}(\xi^\star)^{-4}\right)_t |\nabla\psi|^2 \,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{A.5}$$

Combining the above identities and using Hölder and Young inequalities, we readily see that

$$\begin{aligned} Ks^{-4} \iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}|\Delta\psi|^2 \,\mathrm{d}x\mathrm{d}t \leq \delta &\left(Ks^{-4} \iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}|\Delta\psi|^2 \,\mathrm{d}x\mathrm{d}t\right) \\ &+ C_\delta K^{-1} s^{-4} \left(\iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-4}\left(|\varphi|^2 + |\psi|^2\right) \mathrm{d}x\mathrm{d}t\right) \\ &+ Cs^{-2} \iint_Q e^{-2s\alpha^\star}(\xi^\star)^{-2}|\nabla\psi|^2 \,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{A.6}$$

for some $\delta > 0$. Notice that we have used (5.19) to estimate the last term in (A.5) and that we have introduced a negative power of K in the second term on the right-hand side. Integrating with respect to x in the last term of the above expression and arguing as above we can prove that

$$\begin{aligned} s^{-2} \iint_Q & e^{-2s\alpha^\star} (\xi^\star)^{-2} |\nabla \psi|^2 \,\mathrm{d}x\mathrm{d}t \\ & \leq \delta \left(K s^4 \iint_Q e^{-2s\alpha^\star} (\xi^\star)^{-4} |\Delta \psi|^2 \,\mathrm{d}x\mathrm{d}t \right) + C_\delta K^{-1} \left(\iint_Q e^{-2s\alpha^\star} |\psi|^2 \,\mathrm{d}x\mathrm{d}t \right) \end{aligned} \tag{A.7}$$

Using (A.7) in (A.6) and taking δ small enough, we obtain the desired result.

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