A moment approach to solve scalar nonlinear hyperbolic PDEs

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Scalar nonlinear hyperbolic PDEs

Scalar nonlinear hyperbolic PDEs model numerous physical phenomena (fluid mechanics, traffic flow, nonlinear acoustics)...

Scalar nonlinear hyperbolic PDE

$$\begin{cases} \frac{\partial}{\partial t} y(t,x) + \frac{\partial}{\partial x} f(y(t,x)) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ y(0,x) = y_0(x), \end{cases}$$

with $y_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The flux function f is assumed to be polynomial.

In particular, one retrieves the Burgers equation setting

$$f(y) = \frac{y^2}{2}.$$

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with $y_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The flux function f is assumed to be polynomial.

Objective

We aim at providing a new numerical scheme based on polynomial optimization to solve the solution in a window $\mathbf{T} \times \mathbf{X} \times \mathbf{Y} \subset \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$. Bounds on the variable y due to the maximum principle.

Scalar nonlinear hyperbolic PDEs

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Scalar nonlinear hyperbolic PDE

$$\begin{cases} \dfrac{\partial}{\partial t} y(t,x) + \dfrac{\partial}{\partial x} f(y(t,x)) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ y(0,x) = y_0(x), \end{cases}$$

with $y_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The flux function f is assumed to be polynomial.

Challenge

(Localizing shocks) Smooth initial conditions might produce discontinuities.

Approximation scheme

$< \mu, y > = \int y \, d\mu$

Nonlinear PDE on functions

$$\frac{\partial}{\partial t} y + \frac{\partial}{\partial x} f(y) = 0$$

Relaxation →

← No gap of relaxation

Linear PDE on measures

$$\frac{\partial}{\partial t} < \mu, y > + \frac{\partial}{\partial t} < \mu, f(y) > = 0$$

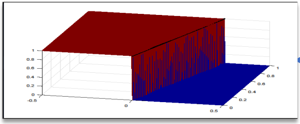
↓ Moments extraction

Moments of the measure

$$z_\alpha = \int \int \int t^{\alpha_1} x^{\alpha_2} y^{\alpha_3} dt dx \, d\mu$$

$$= \int \int t^{\alpha_1} x^{\alpha_2} y^{\alpha_3}(t, x) dt dx$$

← Reconstruction



Objective

Because of the absence of gap : $\mu := \delta_{y(t,x)}$. We want to obtain :

$$z_\alpha := \int_{\mathbf{T}} \int_{\mathbf{X}} \int_{\mathbf{Y}} t^{\alpha_1} x^{\alpha_2} \mathrm{y}^{\alpha_3} dt dx d\mu, \quad \alpha_1 + \alpha_2 + \alpha_3 \leq d,$$
$$= \int_{\mathbf{T}} \int_{\mathbf{X}} t^{\alpha_1} x^{\alpha_2} y(t,x)^{\alpha_3} dt dx, \quad \alpha_1 + \alpha_2 + \alpha_3 \leq d$$

with $d \in \mathbb{N}$ fixed. These quantities, called moments, are of interest, because of the Riesz-Haviland theorem.

Objective

To obtain these moments, we will transform the nonlinear hyperbolic equation as moment constraints.

Weak solutions

Even with regular initial conditions, solutions to nonlinear hyperbolic PDE might produce discontinuities (shocks).

Weak solutions

$$\forall \varphi \in C_c^1(\mathbb{R}_+ \times \mathbb{R}),$$
$$\int_{\mathbb{R}_+ \times \mathbb{R}} \frac{\partial \varphi}{\partial t} y + \frac{\partial \varphi}{\partial x} f(y) dx dt + \int_{\mathbb{R}} \varphi(0, x) y_0(x) dx = 0.$$

Such a notion guarantees existence, but not uniqueness.

Entropy solutions

Definition : Entropy pair

A pair of functions $\eta, q \in C^1(\mathbb{R})$ is called an entropy pair if η is strictly convex and $q' = f'\eta'$.

Definition : Entropy solution

Entropy solutions are weak solutions satisfying

$\forall$ (nonnegative) $\varphi_2 \in C_c^1(\mathbb{R}_+ \times \mathbb{R})$ and $\forall$ entropy pair (η, q),

$$\int_{\mathbb{R}_+\times\mathbb{R}} \frac{\partial \varphi_2}{\partial t}\eta(y) + \frac{\partial \varphi_2}{\partial x} q(y)\,dxdt + \int_{\mathbb{R}} \varphi_2(0,x)\eta(y_0(x))dx \geq 0.$$

Entropy inequalities guarantee uniqueness of the solution ($y \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$) and select a solution with a physical meaning.

Measure-valued (MV) solutions

Young measures

A Young measure on a Euclidean space $\mathcal{X}$ is a map

$$\mu : \mathcal{X} \subset \mathbb{R}_+ \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$$
$$(t, x) \mapsto \mu_{(t,x)}$$

such that, for all $g \in C_0(\mathbb{R})$, the function $(t, x) \mapsto \int_{\mathbb{R}} g(\mathrm{y})\mu_{(t,x)}(dy)$ is measurable.

We use the following notation :

$$\langle \mu_{(t,x)}, g(\mathrm{y}) \rangle := \int_{\mathbb{R}} g(\mathrm{y})\mu_{(t,x)}(dy), \quad \forall g \in C(\mathbb{R})$$

Measure-valued (MV) solutions

Entropy measure-valued solutions (DiPerna, 1985)

$$\forall \varphi_1 \in C_c^1(\mathbb{R}_+ \times \mathbb{R}),$$
$$\int_{\mathbb{R}_+\times\mathbb{R}} \frac{\partial \varphi_1}{\partial t}\langle \mu_{(t,x)}, \mathrm{y}\rangle + \frac{\partial \varphi_1}{\partial x}\langle \mu_{(t,x)}, f(\mathrm{y})\rangle dxdt$$
$$+ \int_{\mathbb{R}} \varphi_1(0,x)\langle \mu_{0,x}, \mathrm{y}\rangle dx = 0.$$

$\forall$ (nonnegative) $\varphi_2 \in C_c^1(\mathbb{R}_+ \times \mathbb{R})$ and $\forall$ convex pair (η, q),

$$\int_{\mathbb{R}_+\times\mathbb{R}} \frac{\partial \varphi_2}{\partial t}\langle \mu_{(t,x)}, \eta(\mathrm{y})\rangle + \frac{\partial \varphi_2}{\partial x}\langle \mu_{(t,x)}, q(\mathrm{y})\rangle dxdt$$
$$+ \int_{\mathbb{R}} \varphi_2(0,x)\langle \mu_{0,x}, \eta(\mathrm{y}))\rangle dx \geq 0.$$

LINEAR formulation

Measure-valued (MV) solutions

With the particular choice of

$$\mu_{t,x} = \delta_{y(t,x)}, \quad \mu_{0,x} = \delta_{y_0(x)}, \text{ for a.e. } t, x$$

one retrieves entropy solutions. Indeed,
$\langle \delta_{y(t,x)}, \mathrm{y} \rangle = y(t,x)$, for a.e. t, x.

entropy solutions $\subseteq$ entropy MV solutions

Question

Under which conditions, these two kinds of solutions coincide ?

A concentration theorem

Theorem (DiPerna, 1985)

Let C be the Lipschitz constant of the function f. Let y be an entropy solution and μ be an entropy MV solution. Then, for all $T \geq 0$ and all $r \geq 0$

$$\int_{|x|\leq r} \langle \mu_{t,x}, |\mathrm{y} - y(T,x)|\rangle dx \leq \int_{|x|\leq r+CT} \langle \mu_{0,x}, |\mathrm{y} - y_0(x)| dx.$$

In particular, one has

$$\mu_{0,x} = \delta_{y_0(x)} \Rightarrow \mu_{t,x} = \delta_{y(t,x)}, \qquad \text{for a.e. } t \in [0,T],\ x \in [-r,r].$$

Entropy measure-valued solutions are NOT a relaxation when $\mu_{0,x}$ is concentrated on the graph of the initial condition.

MV solutions on compact sets

Entropy MV solutions on the compact set $\mathbf{T} \times \mathbf{X}$

$$\forall \varphi_1 \in C^1(\mathbf{T} \times \mathbf{X}),$$

$$\int_{\mathbf{T}\times\mathbf{X}} \frac{\partial \varphi_1}{\partial t} \langle \mu_{(t,x)}, \mathrm{y} \rangle + \frac{\partial \varphi_1}{\partial x} \langle \mu_{(t,x)}, f(\mathrm{y}) \rangle dxdt + B.C. = 0.$$

$\forall$ (nonnegative) $\varphi_2 \in C^1(\mathbf{T} \times \mathbf{X})$ and $\forall$ convex pair (η, q),

$$\int_{\mathbb{R}_+\times\mathbb{R}} \frac{\partial \varphi_2}{\partial t} \langle \mu_{(t,x)}, \eta(\mathrm{y}) \rangle + \frac{\partial \varphi_2}{\partial x} \langle \mu_{(t,x)}, q(\mathrm{y}) \rangle dxdt + B.C. \geq 0.$$

From the maximum principle, we know that

$$\underbrace{-\|y_0\|_{L^\infty(\mathbb{R})}}_{:=\underline{y}} \leq \mathrm{y} \leq \underbrace{\|y_0\|_{L^\infty(\mathbb{R})}}_{:=\bar{y}} \Rightarrow \mathrm{y} \in \mathbf{Y} := [\underline{y}, \bar{y}].$$

All the variables lie in compact sets

Dynamic and entropy constraints

Let us define :

$$d\nu(t,x,\mathrm{y}) = \mu_{t,x}(d\mathrm{y})dtdx.$$

Constraints

1. **Dynamic**

$$\forall \varphi_1 \in C^1(\mathbf{T}\times\mathbf{X}),$$
$$\int_{\mathbf{T}\times\mathbf{X}\times\mathbf{Y}} \frac{\partial \varphi_1}{\partial t}\mathrm{y} + \frac{\partial \varphi_1}{\partial x} f(\mathrm{y}) d\nu + B.C. = 0.$$

2. **Entropy inequalities**

$\forall$ (nonnegative) $\varphi_2 \in C^1(\mathbf{T}\times\mathbf{X})$ and $\forall$ convex pair (η, q),

$$\int_{\mathbf{T}\times\mathbf{X}\times\mathbf{Y}} \frac{\partial \varphi_2}{\partial t}\eta(\mathrm{y}) + \frac{\partial \varphi_2}{\partial x} q(\mathrm{y}) d\nu + B.C. \geq 0.$$

Dynamic and entropy constraints

We express the data $(\varphi_1, \varphi_2, \eta, q)$ as polynomials.

MV solutions as moments constraints

Moment constraints

1. **Dynamic**

$$\forall \alpha_1, \alpha_2 \in \mathbb{N},$$
$$\int_{\mathbf{T}\times\mathbf{X}\times\mathbf{Y}} \left(\alpha_1 t^{\alpha_1 - 1} x^{\alpha_2} \mathrm{y} + t^{\alpha_1} x^{\alpha_2 - 1} f(\mathrm{y}) \right) d\nu + B.C. = 0.$$

2. **Entropy inequalities**

$$\forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{N}$$
$$\int_{\mathbf{T}\times\mathbf{X}\times\mathbf{Y}} \left(\frac{\partial g^{\alpha}(t,x)}{\partial t} \eta(\mathrm{y}) + \frac{\partial g^{\alpha}(t,x)}{\partial x} q(\mathrm{y}) \right) d\nu + B.C. \geq 0.$$

with $g^{\alpha} := t^{\alpha_1}(T-t)^{\alpha_2}(L-x)^{\alpha_3}(x-R)^{\alpha_4}$.

The solution $\nu = dtdx\delta_{y(t,x)}$ (with y entropy solution) is the UNIQUE solution to these moment constraints.

MV solutions as moments constraints

Obtaining the right solution needs us to impose an infinite number of constraints. Numerically, it is however impossible.

Numerical solution

We will truncate the number of contraints up to an order d.

Truncated moment problem

Truncated moment constraints (order d)

1. **Dynamic**

$$\forall \alpha_1, \alpha_2 \in \mathbb{N} \text{ such that } |\alpha| \leq d,$$

$$\int_{\mathbf{T}\times\mathbf{X}\times\mathbf{Y}} \left(\alpha_1 t^{\alpha_1 - 1} x^{\alpha_2} \mathrm{y} + \alpha_2 t^{\alpha_1} x^{\alpha_2 - 1} f(\mathrm{y}) \right) d\nu + B.C. = 0.$$

2. **Entropy inequalities**

$$\forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{N} \text{ such that } |\alpha| \leq d$$

$$\int_{\mathbf{T}\times\mathbf{X}\times\mathbf{Y}} \left(\frac{\partial g^\alpha(t,x)}{\partial t} \eta(\mathrm{y}) + \frac{\partial g^\alpha(t,x)}{\partial x} q(\mathrm{y}) \right) d\nu + B.C. \geq 0.$$

with $g^\alpha := t^{\alpha_1}(T-t)^{\alpha_2}(L-x)^{\alpha_3}(x-R)^{\alpha_4}$.

Truncated moment problem

The moment formulation with all the moments leads to a unique solution. But, when truncating, it is not the case.

Solution

We optimize an objective function. The sum of the right hand-side of the entropy inequalities leads in practice to good numerical results.

Optimization problem

Truncated moment problem as an optimization problem

Setting $|\alpha| \leq d$, one then obtains the following optimization problem.

$$\rho_d := \sup_{z_\alpha} \sum_{|\alpha| \leq d} \int_{\mathbf{K}} h_2(t, x, \mathrm{y}) d\nu \quad \text{(objective)}$$

$$\text{s.t.} \int_{\mathbf{K}} h_1(t, x, \mathrm{y}) d\nu + B.C = 0, \ \text{(Dynamic)} \ \forall |\alpha| \leq d$$

$$-\int_{\mathbf{K}} h_2(t, x, \mathrm{y}) d\nu + B.C \leq 0 \ \text{(Entropies)} \ \forall |\alpha| \leq d$$

$$h_1(t, x, \mathrm{y}) = \alpha_1 t^{\alpha_1 - 1} x^{\alpha_2} \mathrm{y} + \alpha_2 t^{\alpha_1} x^{\alpha_2 - 1} f(\mathrm{y})$$

$$h_2(t, x, \mathrm{y}) = \frac{\partial g^\alpha}{\partial t} \eta(\mathrm{y}) + \frac{\partial g^\alpha}{\partial x} q(\mathrm{y})$$

$$z_\alpha = \int_{\mathbf{T} \times \mathbf{X} \times \mathbf{Y}} t^{\alpha_1} x^{\alpha_2} \mathrm{y}^{\alpha_3} d\nu.$$

Optimization problem

Numerical implementation

Implementable with the toolbox *Globtipoly*.

Some nice properties

Similarly, we may define the full moment optimization problem, which is the same optimization problem, satisfied $\forall\alpha$

Full moment problem

$$\rho^\star := \sup_{\nu \in \mathcal{M}(\mathbf{K})_+} \sum_\alpha \int_{\mathbf{K}} h_2(t,x,\mathrm{y}) d\nu \quad \text{(objective function)}$$

$$\text{s.t.} \int_{\mathbf{K}} h_1(t,x,\mathrm{y}) d\nu + B.C = 0, \ \text{(Dynamic)}\ \forall\alpha$$

$$-\int_{\mathbf{K}} h_2(t,x,\mathrm{y}) d\nu + B.C \leq 0 \ \text{(Entropies)}\ \forall\alpha$$

with $\nu^\star = dtdx\delta_{y(t,x)}$

Some nice properties

Similarly, we may define the full moment optimization problem, which is the same optimization problem, satisfied $\forall \alpha$

Theorem (Lasserre, 2008)

One has $\lim_{\rightarrow_{d \rightarrow +\infty}} \rho_d = \rho^\star$. We have moreover :

$$\rho_d \leq \rho_{d+1}.$$

Concluding remarks

Originality

Compared to existing numerical schemes, our alternative numerical scheme

1. does NOT rely on time/space discretization ;
2. allows to compute the solution GLOBALLY in a window $\mathbf{T} \times \mathbf{X} \subset \mathbb{R}_+ \times \mathbb{R}$

Burgers equation and Riemann problem

$\mathbf{T} = [0, 1]$, $\mathbf{X} = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $f = \frac{1}{4}y^2$.

$$y_0(x) := \begin{cases} \mathrm{y}_l, & x < 0 \\ \mathrm{y}_r, & x > 0 \end{cases}$$

Two possible situations

1. $\mathrm{y}_l > \mathrm{y}_r$: propagating shock
2. $\mathrm{y}_l < \mathrm{y}_r$: rarefaction wave.

Shock wave (d=6)

Analytic vs computed : $\int \mathrm{y}^k d\nu$

1.00000000	0.99999999
0.62500000	0.62500001
0.62500000	0.62500001
0.62500000	0.62500001
0.62500000	0.62500002
0.62500000	0.62500002
0.62500000	0.62500002
0.62500000	0.62500002
0.62500000	0.62500002
0.62500000	0.62500002
0.62500000	0.62500002
0.62500000	0.62500003
0.62500000	0.62500003

Shock wave (d=6)

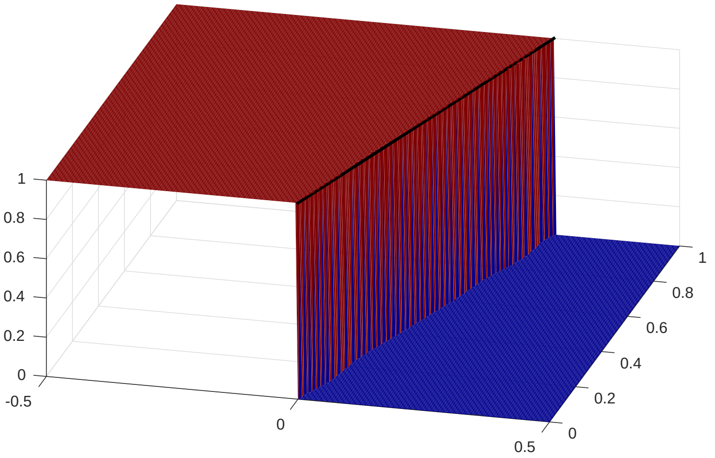
We use an algorithm inspired by [Pauwels, Lasserre, 2017] to reconstruct the solution from moments data.

Idea

From this algorithm, we extract polynomials $p_d^i(t, x, \mathrm{y})$ satisfying $p_d^i(t, x, \mathrm{y}) = 0$ and the approximated function we obtain is

$$\hat{y}^d(t, x) = \mathrm{argmin}_{\mathrm{y} \in \mathbf{Y}} \sum_i p_d^i(t, x, \mathrm{y}).$$

Shock wave (d=6)

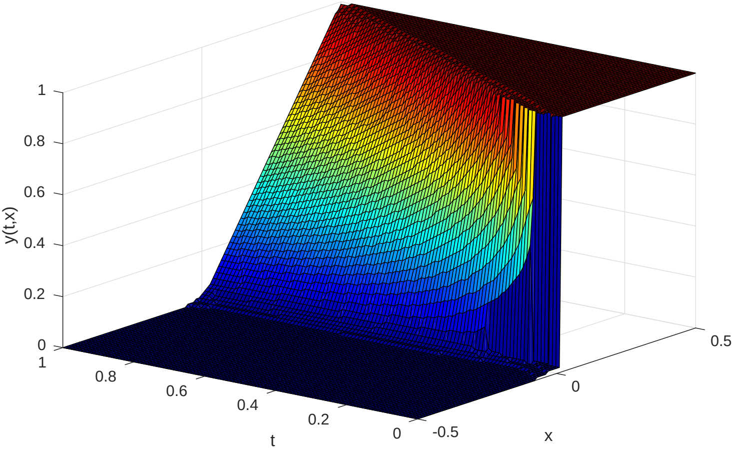


Rarefaction wave

Analytic vs computed : $\int \mathrm{y}^k d\nu$

1.00000000	1.00000000
0.37500000	0.37499990
0.33333333	0.33333312
0.31250000	0.31249968
0.30000000	0.29999957
0.29166667	0.29166612
0.28571429	0.28571362
0.28125000	0.28124922
0.27777778	0.27777688
0.27500000	0.27499899
0.27272727	0.27272614
0.27083333	0.27083209

Rarefaction wave



Further research lines

Inverse design problem

The Burgers equation is **irreversible**. A continuum of initial conditions might lead to a given final state $y(T,x) = \phi(x)$.
How can one compute this set with a moment approach ?

Class of approximated function

Some ideas

Some polynomials p_i^d solve the following equation

$$\int_{\mathbf{T}} \int_{\mathbf{X}} \int_{\mathbf{Y}} p_i^d(t, x, \mathrm{y})^2 d\nu = 0$$

$$\Rightarrow p_i^d(t, x, \mathrm{y}) = 0, \quad \text{on } \mathrm{supp}(\nu).$$

Therefore, since $\nu = dtdx\delta_{y(t,x)}$:

$$(t, x, y(t, x)) \subset \{(t, x, \mathrm{y}) \mid p_i^d(t, x, \mathrm{y}) = 0, \ \forall i\}.$$

Numerically, we take :

$$y^d(t, x) = \mathrm{argmin}_{\mathrm{y} \in \mathbf{Y}} \sum_{p_i^d} p_i^d(t, x, \mathrm{y})^2$$

A simpler example

The following nonconvex and nonlinear optimization problem of finite dimension

$$f^\star = \sup_{\mathbf{w}} f(\mathbf{w})$$
$$\text{s.t. } \mathbf{w} \in \mathbf{K} \subset \mathbb{R}^n$$

is equivalent to the convex and linear optimization problem of infinite-dimension

$$\rho = \sup_{\nu \in \mathcal{M}(K)_+} \int_{\mathbf{K}} f(\mathbf{w}) d\nu(d\mathbf{w})$$
$$\text{s.t. } \int_{\mathbf{K}} d\nu = 1.$$

where $\mathcal{M}(K)_+$ denotes the set of Borel measures, which is a convex set. Solution of the latter optimization problem is $\nu = \delta_{w^\star}$. But what is the advantage ?

A simpler example

Answer : if the data (f and $\mathbf{K}$) are expressed with polynomials, you can propose a numerical scheme !

From measures to moments

Assume that f is a polynomial. Then :

$$\int_{\mathbf{K}} f(\mathbf{w})d\nu = \int_{\mathbf{K}} \sum_{\alpha\in\mathbb{N}^n} f_\alpha \mathbf{w}^\alpha d\nu$$
$$= \sum_{\alpha\in\mathbb{N}^n} f_\alpha \underbrace{\int_{\mathbf{K}} \mathbf{w}^\alpha d\nu}_{\text{moments of the measure}} .$$

Moments are **quantities of interest** for a measure.
For univariate measure ($n = 1$) :

$$\text{0th order moment } \left(\int_{\mathbf{K}} d\nu\right) : \text{ Average,}$$
$$\text{First order moment } \left(\int_{\mathbf{K}} \mathbf{w} d\nu\right) : \text{ Mean,}$$

We will compute the moments of a measure (that are real numbers) instead of the measure itself.

From measures to moments

Suppose that you have an infinite sequence $(z_\alpha)_{\alpha\in\mathbb{N}}$. You want to know whether this sequence represents a measure. Focus on the following linear functional

$$f(w) = \sum_{\alpha\in\mathbb{N}^n} f_\alpha \mathbf{w}^\alpha \mapsto L_z(f) = \sum_{\alpha\in\mathbb{N}} f_\alpha z_\alpha$$

Riesz-Haviland Theorem (1935/1936)

Let $z = (z_\alpha)_{\alpha\in\mathbb{N}^n}$ and $\mathbf{K}$ closed. There exists $\mu \in \mathcal{M}_+(\mathbf{K})$ with

$$z_\alpha := \int_{\mathbf{K}} w^\alpha d\nu$$

if and only if

$L_z(p) \geq 0$ for every nonnegative polynomials p

From measures to moments

An univariate example

Set $n = 1$. Then

$$L_z(1 + 3w_1 + w_1^2) = z_0 + 3z_1 + z_2$$

A multivariate example

Set $n = 2$. Then

$$L_z(1 + 2w_1 w_2 + 5w_2^2 + 4w_1^4 w_2) = z_{00} + 2z_{11} + 5z_{02} + 4z_{41}.$$

Moment and localization matrices

Moment and localization matrices

If $\mathbf{K}$ is expressed with polynomials

$$\mathbf{K} := \{\mathbf{w} \in \mathbb{R}^n \mid g_j(\mathbf{w}) \geq 0,\ j = 1, \dots, m\},$$

$L_z(\cdot) \geq 0$ can be imposed by LMIs (Linear Matrix Inequalities)

$$\underbrace{L_z(\mathbf{v}_d \mathbf{v}_d^\top)}_{\text{moment matrix}} \succeq 0,\ \underbrace{L_z(\mathbf{v}_d \mathbf{v}_d^\top g_j)}_{\text{localizing matrix}} \succeq 0,\ j = 1, \dots, m,\ d = 1, 2, \dots$$

where $\mathbf{v}_d := (\mathbf{w}^\alpha)_{|\alpha| \leq d} \in \mathbb{R}[\mathbf{w}]^{s(d)}$, with $s(d) := \binom{n+d}{d}$

Moment matrix ensures that you are considering a measure. The localizing matrix ensures that the support of this measure is $\mathbf{K}$.

Example of moment matrices

First moment matrix

$n = 2$, $d = 1$. $v_d := \begin{bmatrix} 1 & w_1 & w_2 \end{bmatrix}^\top$,

$$L_z(\mathbf{v}_1 \mathbf{v}_1^\top) = \begin{bmatrix} \int_{\mathbf{K}} 1 d\nu & \int_{\mathbf{K}} w_1 d\nu & \int_{\mathbf{K}} w_2 d\nu \\ \int_{\mathbf{K}} w_1 d\nu & \int_{\mathbf{K}} w_1^2 d\nu & \int_{\mathbf{K}} w_1 w_2 d\nu \\ \int_{\mathbf{K}} w_2 d\nu & \int_{\mathbf{K}} w_1 w_2 d\nu & \int_{\mathbf{K}} w_2^2 d\nu \end{bmatrix} = \begin{bmatrix} z_{00} & z_{10} & z_{01} \\ z_{10} & z_{20} & z_{11} \\ z_{01} & z_{11} & z_{02} \end{bmatrix}.$$

Example of moment matrices

Second moment matrix

$n = 1$, $d = 2$. $v_d := \begin{bmatrix} 1 & w_1 & w_2 & w_1^2 & w_1 w_2 & w_2^2 \end{bmatrix}^\top$,

$$L_z(\mathbf{v}_2\mathbf{v}_2^\top) = \left[\begin{array}{cccccc} \color{red}{z_{00}} & \color{red}{z_{10}} & \color{red}{z_{01}} & z_{20} & z_{11} & z_{02} \\ \color{red}{z_{10}} & \color{red}{z_{20}} & \color{red}{z_{11}} & z_{30} & z_{21} & z_{12} \\ \color{red}{z_{01}} & \color{red}{z_{11}} & \color{red}{z_{02}} & z_{21} & z_{12} & z_{03} \\ z_{20} & z_{30} & z_{21} & z_{40} & z_{31} & z_{22} \\ z_{11} & z_{21} & z_{12} & z_{31} & z_{22} & z_{13} \\ z_{02} & z_{12} & z_{03} & z_{22} & z_{13} & z_{04} \end{array}\right]$$

SDP

The polynomial optimization problem reduces to the following SDP problem :

$$\rho = \min_z L_z(p)$$

$$\text{s.t. } z_0 = 1$$

$$\underbrace{L_z(\mathbf{v}_d \mathbf{v}_d^\top)}_{\text{moment matrix}} \succeq 0, \; \underbrace{L_z(\mathbf{v}_d \mathbf{v}_d^\top g_j)}_{\text{localizing matrix}} \succeq 0, \; j = 1, \ldots, m,$$

It can be solved numerically with SDP solver such as Sedumi.

Issue

You do not know in advance the size of the moment matrix and the localization matrix.

Lasserre solution (2001)

Truncate the moments up to an order d, prescribed.

An example of hierarchy of SDP problems

An example

Here, we have $n = 2$.

$$\rho = \min_{w} p(\mathbf{w}) := -w_2$$

$$\text{s.t } \underbrace{3 - 2w_2 - w_1^2 - w_2^2}_{:=g_1(w)} \geq 0$$

$$\underbrace{-w_1 - w_2 - w_1 w_2 \geq 0}_{:=g_2(w)} \geq 0$$

$$\underbrace{1 + w_1 w_2}_{:=g_3(w)} \geq 0.$$

The global optimal value of this problem is -1.680.

An example of hierarchy of SDP problems

First relaxation $d = 1$

$$
\begin{aligned}
\rho_1 = \min_z & -z_{01} \\
\text{s.t} & \begin{bmatrix} 1 & z_{10} & z_{01} \\ z_{10} & z_{20} & z_{11} \\ z_{01} & z_{11} & z_{02} \end{bmatrix} \succeq 0 \\
& 3 - 2z_{01} - z_{20} - z_{02} \geq 0 \\
& -z_{10} - z_{01} - z_{11} \geq 0 \\
& 1 + z_{11} \geq 0.
\end{aligned}
$$

After resolution with the toolbox *GlobtiPoly* : solution of this first relaxation : -2.

An example of hierarchy of SDP problems

Second order relaxation

$$\rho_2 = \min_z -z_{01}$$

$$\text{s.t.} \quad \begin{bmatrix} z_{00} & z_{10} & z_{01} & z_{20} & z_{11} & z_{02} \\ z_{10} & z_{20} & z_{11} & z_{30} & z_{21} & z_{12} \\ z_{01} & z_{11} & z_{02} & z_{21} & z_{12} & z_{03} \\ z_{20} & z_{30} & z_{21} & z_{40} & z_{31} & z_{22} \\ z_{11} & z_{21} & z_{12} & z_{31} & z_{22} & z_{13} \\ z_{02} & z_{12} & z_{03} & z_{22} & z_{13} & z_{04} \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} 3-2z_{01}-z_{20}-z_{02} & 3z_{10}-2z_{11}-z_{30}-z_{12} & 3z_{01}-2z_{02}-z_{21}-z_{03} \\ 3z_{10}-2z_{11}-z_{30}-z_{12} & 3z_{20}-2z_{21}-z_{40}-z_{22} & 3z_{11}-2z_{12}-z_{31}-z_{13} \\ 3z_{01}-2z_{02}-z_{21}-z_{03} & 3z_{11}-2z_{12}-z_{31}-z_{13} & 3z_{02}-2z_{03}-z_{22}-z_{04} \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} -z_{10}-z_{01}-z_{11} & -z_{20}-z_{11}-z_{21} & -z_{11}-z_{02}-z_{12} \\ -z_{20}-z_{11}-z_{21} & -z_{30}-z_{21}-z_{31} & -z_{21}-z_{12}-z_{22} \\ -z_{11}-z_{02}-z_{12} & -z_{21}-z_{12}-z_{22} & -z_{12}-z_{03}-z_{13} \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} 1+z_{11} & z_{10}+z_{21} & z_{01}+z_{12} & & \\ z_{10}+z_{21} & z_{20}+z_{31} & z_{11}+z_{22} z_{01}+z_{12} & z_{11}+z_{22} & z_{02}+z_{13} \end{bmatrix} \succeq 0.$$

We obtain the optimal global value -1.680.

Generalized Moment Problem (GMP) formulation

Generalization : if your problem can be rephrased as follows

GMP formulation

$$\rho^\star := \sup_{\nu \in \mathcal{M}(\mathbf{K})_+} \int_{\mathbf{K}} f(\mathbf{w}) d\nu \quad \text{(objective function)}$$

$$\text{s.t.} \int_{\mathbf{K}} h_j(\mathbf{w}) d\nu \leqq \gamma_j, \; j \in \Gamma \quad \text{(constraints)}$$

then you are able to apply the Lasserre hierarchy, with ρ_d the value of the relaxed functional. Moreover,

Convergence (Lasserre (2008))

The solution of the moment-SOS hierarchy converges to the solution of the GMP. Moreover, one has

$$\rho_d \leq \rho_{d+1}.$$

Projection on polynomials for the dynamic

Key : Use the Stone-Weierstrass theorem !

$$\int_{\mathbf{T}\times\mathbf{X}\times\mathbf{Y}} \frac{\partial\varphi_1}{\partial t}\mathrm{y} + \frac{\partial\varphi_1}{\partial x} f(\mathrm{y}) d\nu + B.C. = 0 \Leftrightarrow$$

$$\sum_{\alpha_1,\alpha_2\in\mathbb{N}} c_\alpha \int_{\mathbf{T}\times\mathbf{X}\times\mathbf{Y}} \frac{\partial t^{\alpha_1} x^{\alpha_2}}{\partial t}\mathrm{y} + \frac{\partial t^{\alpha_1} x^{\alpha_2}}{\partial x} f(\mathrm{y}) d\nu + B.C. = 0.$$

Hence, the dynamic can be imposed with this moment constraint

$$\int_{\mathbf{T}\times\mathbf{X}\times\mathbf{Y}} \frac{\partial t^{\alpha_1} x^{\alpha_2}}{\partial t}\mathrm{y} + \frac{\partial t^{\alpha_1} x^{\alpha_2}}{\partial x} f(\mathrm{y}) d\nu + B.C. = 0,\ \forall \alpha_1,\alpha_2 \in \mathbb{N}$$

Test functions for the entropy inequalities

Stone-Weierstrass theorem does not work directly with nonnegative test function. Indeed,

$$\int_{\mathbf{T}\times\mathbf{X}\times\mathbf{Y}} \frac{\partial \varphi_2}{\partial t}\eta(\mathrm{y}) + \frac{\partial \varphi_2}{\partial x} q(\mathrm{y}) d\nu + B.C. \geq 0 \Leftrightarrow$$

$$\sum_{\alpha_1,\alpha_2\in\mathbb{N}} c_\alpha \frac{\partial t^{\alpha_1} x^{\alpha_2}}{\partial t}\eta(\mathrm{y}) + \frac{\partial t^{\alpha_1} x^{\alpha_2}}{\partial x} q(\mathrm{y}) d\nu + B.C. \geq 0$$

We cannot get rid of the coefficient c_α.
Key : use Handelman's Positivstellensatz (1988).

An example for a nonnegative test function

$\mathbf{T} = [0, T]$, $\mathbf{X} = [L, R]$. Hence, every nonnegative function in $\mathbf{T} \times \mathbf{X}$ can be written as follows

$$\varphi_2(t,x) = \sum_\alpha \underbrace{c_{\tilde{\alpha}}}_{>0} g^\alpha(t,x) := c_{\tilde{\alpha}}(t-T)^{\alpha_1} t^{\alpha_2} (L-x)^{\alpha_3} (x-R)^{\alpha_4}.$$

Entropy pair

Lax entropies

Using this special family of entropy pair

$$\eta_v := |y - v|,\ q_v := \mathrm{sign}(y - v)(f(y) - f(v)), \quad \forall v \in \mathbf{Y},$$

is equivalent to using any entropy pair.

Issues :

1. The functions are parametrized by any v $\Rightarrow$ introduce v as a new variable.
2. The absolute value and the sign functions are NOT polynomials $\Rightarrow$ double the number of measures

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Doubling measure strategy

Doubling measure strategy

$$\nu \in \mathcal{M}(\mathbf{T} \times \mathbf{X} \times \mathbf{Y})_+ \Rightarrow \begin{cases} \nu^+ \in \mathcal{M}(\mathbf{T} \times \mathbf{X} \times \{\mathbf{Y}^2 \mid \mathrm{y} \geq v\}) \\ \nu^- \in \mathcal{M}(\mathbf{T} \times \mathbf{X} \times \{\mathbf{Y}^2 \mid \mathrm{y} \leq v\}) \end{cases}$$

Hence,

$$\int_{\mathbf{T} \times \mathbf{X} \times \mathbf{Y}^2} \frac{\partial g^{\alpha,\beta}}{\partial t}(\mathrm{y} - v) + \frac{\partial g^{\alpha,\beta}}{\partial x} \underbrace{(f(\mathrm{y}) - f(v))}_{\text{assumed to be polynomial}} d\nu^+ +$$

$$\int_{\mathbf{T} \times \mathbf{X} \times \mathbf{Y}^2} \frac{\partial g^{\alpha,\beta}}{\partial t}(v - \mathrm{y}) + \frac{\partial g^{\alpha,\beta}}{\partial x}(f(v) - f(\mathrm{y}))d\nu^- + B.C. \geq 0$$