

CONTROLLABILITY OF A 1-D FRACTIONAL HEAT EQUATION

THEORETICAL AND NUMERICAL ASPECTS

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INTRODUCTION AND MAIN THEORETICAL RESULTS

Introduction

Consider the following non-local parabolic equation

Fractional heat equation

$$\begin{cases} z_t + (-d_x^2)^s z = g \mathbf{1}_\omega, & (x, t) \in (-1, 1) \times (0, T) \\ z = 0, & (x, t) \in (-1, 1)^c \times (0, T) \\ z(x, 0) = z_0(x), & x \in (-1, 1) \end{cases} \quad (\mathcal{H})$$

- $\omega \subset (-1, 1)$.
- $(-d_x^2)^s$: **fractional Laplacian**.

We are interested in analyzing controllability properties, both from a theoretical and a numerical viewpoint.

Fractional Laplacian

For any function u sufficiently regular and for any $s \in (0, 1)$, the s -th power of the Laplace operator is given by

$$(-d_x^2)^s u(x) = c_s P.V. \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy.$$

$$c_s = \frac{s 2^{2s} \Gamma\left(\frac{1+2s}{2}\right)}{\sqrt{\pi} \Gamma(1-s)}$$

normalization constant ensuring
that $\lim_{s \rightarrow 1^-} (-d_x^2)^s = -d_x^2$.

NULL CONTROLLABILITY at time $T > 0$:

given any initial datum $z_0 \in L^2(-1, 1)$ there exists $g \in L^2(\omega \times (0, T))$ such that the corresponding solution z satisfies $z(x, T) = 0$.

APPROXIMATE CONTROLLABILITY at time $T > 0$:

given any $z_0, z_T \in L^2(-1, 1)$ and any $\varepsilon > 0$ there exists $g \in L^2(\omega \times (0, T))$ such that the corresponding solution z with initial datum $z(x, 0) = z_0$ satisfies $\|z(x, T) - z_T\|_{L^2(-1, 1)} \leq \varepsilon$.

Theorem (U.B. and V. Hernández-Santamaría, IMA J. Math. Control Inf., 2018)

The fractional heat equation $(\mathcal{H})$ is

- *null-controllable at time $T > 0$ if and only if $s > 1/2$.*
- *approximately controllable at time $T > 0$ for all $s \in (0, 1)$.*

Existence uniqueness and regularity of solutions

Definition

We say that $z \in L^2(0, T; H_0^s(-1, 1)) \cap C([0, T], L^2(-1, 1))$ with $z_t \in L^2(0, T; H^{-s}(-1, 1))$ is a weak solution for the parabolic problem $(\mathcal{H})$ with $g \in L^2(0, T; H^{-s}(-1, 1))$ and $z_0 \in L^2(-1, 1)$ if it satisfies

$$\int_0^T \int_{-1}^1 z_t w \, dx dt + \int_0^T a(z, w) \, dt = \int_0^T \langle f, w \rangle_{-s, s} \, dt,$$

for any $w \in L^2(0, T; H_0^s(-1, 1))$.

The bilinear form $a(\cdot, \cdot) : H_0^s(-1, 1) \times H_0^s(-1, 1) \rightarrow \mathbb{R}$ is defined as

$$a(u, v) = \frac{c_{1,s}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+2s}} \, dx dy.$$

Theorem (T. Leonori, I. Peral, A. Primo, and F. Soria, Discrete Contin. Dyn. Syst., 2015.)

Assume $f \in L^2(0, T; H^{-s}(-1, 1))$. Then for any $z_0 \in L^2(-1, 1)$, problem $(\mathcal{H})$ has a unique weak solution.

Theorem (U.B., M. Warma, and E. Zuazua, SEMA SIMAI Springer Series, 2018.)

Assume $f \in L^2((-1, 1) \times (0, T))$, $z_0 = 0$, and let $z \in L^2((0, T); H_0^s(-1, 1)) \cap C([0, T]; L^2(-1, 1))$ with $z_t \in L^2((0, T); H^{-s}(-1, 1))$ be the unique weak solution of system $(\mathcal{H})$. Then

$$z \in L^2((0, T); H_{loc}^{2s}(-1, 1)) \cap L^\infty((0, T); H_0^s(-1, 1))$$

and

$$z_t \in L^2((-1, 1) \times (0, T)).$$

Known controllability results

- S. Micu and E. Zuazua, 2006: in the 1-d setting the authors consider **spectral fractional Laplacian** which is defined by

$$(-d_x^2)_S^s u(x) = \sum_{k \geq 1} \langle u, \psi_k \rangle \lambda_k^s \psi_k(x),$$

They prove that for $s > 1/2$ null controllability holds. But

spectral $\neq$ integral¹

- L. Miller, 2006: the same result holds in multi-d setting using **Lebeau-Robbiano** strategy.

¹ R. Servadei and E. Valdinoci, Proc. R. Soc. Edinb. A, 2014

Proof of the null controllability (sketch)

By employing the classical moment method¹, the equation is null-controllable if and only if

- $\sum_{k \geq 1} \lambda_k^{-1} < +\infty$.
- $\lambda_{k+1} - \lambda_k \geq \gamma > 0, \quad \forall k \in \mathbb{N}$.

Eigenvalues of the fractional Laplacian on $(-1, 1)$ with Dirichlet B.C.²

$$\begin{cases} (-d_x^2)^s \varrho_k = \lambda_k \varrho_k, & x \in (-1, 1), \quad k \in \mathbb{N} \\ \varrho_k = 0, & x \in \mathbb{R} \setminus (-1, 1), \end{cases}$$

$$\lambda_k = \left(\frac{k\pi}{2} - \frac{(1-s)\pi}{4} \right)^{2s} + O\left(\frac{1}{k}\right).$$

Therefore, the two conditions above are both satisfied if and only if $s > 1/2$.

¹ Fattorini and Russell, Quart. Appl. Math., 1974. ² Kwaśnicki, J. Funct. Anal., 2012.

Proof of the approximate controllability (sketch)

The result follows from the following property.

Parabolic unique continuation

Given $s \in (0, 1)$ and $\varphi_0^T \in L^2(-1, 1)$, let φ be the unique solution to the adjoint equation. Let $\omega \subset (-1, 1)$ be an arbitrary open set. If $\varphi = 0$ on $\omega \times (0, T)$, then $\varphi = 0$ on $(-1, 1) \times (0, T)$.

This, in turn, is a consequence of the **unique continuation** property for the Fractional Laplacian.¹

¹ M.M. Fall and V. Felli, Comm. Partial Differential Equations, 2014..



Remarks

- The elliptic unique continuation property for the fractional Laplacian holds in **any space dimension**. In view of that, the approximate controllability for $(\mathcal{H})$ may be obtained also to the case $N > 1$. The same does not apply to the null-controllability, since our proof uses arguments that are designed specifically for one-dimensional problems.
- If one would like to analyze the multi-dimensional problem, other tools (**Carleman estimates**) are needed. These techniques are not available for problems involving the fractional Laplacian on a domain.

NUMERICAL IMPLEMENTATION

Penalized Hilbert Uniqueness Method

We have to solve the following minimization problem:

$$\min_{\varphi^T \in L^2(-1,1)} J_\varepsilon(\varphi^T) := \frac{1}{2} \int_{\omega \times (0,T)} |\varphi|^2 dx dt + \frac{\varepsilon}{2} \|\varphi^T\|_{L^2(-1,1)}^2 + \int_{-1}^1 z_0 \varphi(0)$$

where φ is the solution to the adjoint problem.

$$\begin{cases} -\varphi_t + (-d_x^2)^s \varphi = 0, & (x, t) \in (-1, 1) \times (0, T) \\ \varphi = 0, & (x, t) \in (-1, 1)^c \times (0, T) \\ \varphi(\cdot, T) = \varphi^T(x), & x \in (-1, 1) \end{cases}$$

J_ε is **continuous, coercive and strictly convex**, thus the existence and uniqueness of a minimizer φ_ε^T is guaranteed.

In fact, the control is chosen as

$$v = \varphi_\varepsilon|_\omega$$

where φ_ε is the solution to the adjoint with initial datum φ_ε^T .

Penalized Hilbert Uniqueness Method (cont.)

Analyzing the behavior of the penalized problem with respect to the parameter ε , we can deduce controllability properties for our system.

Theorem (F. Boyer, ESAIM Proc., 2013)

Let φ_ε^T be the unique minimizer of J_ε . System $(\mathcal{H})$ is

- *null controllable at time T* $\Leftrightarrow M_{z_0}^2 := 2 \sup_{\varepsilon > 0} (\inf J_\varepsilon) < +\infty$.

In this case:

$$\|g\|_{L^2(\omega \times (0, T))} \leq M_{z_0} \quad \text{and} \quad \|\varphi_\varepsilon^T\| \leq M_{z_0} \sqrt{\varepsilon}$$

- *approximately controllable at time T* $\Leftrightarrow \varphi_\varepsilon^T \rightarrow 0$ as $\varepsilon \rightarrow 0$.

DEVELOPMENT OF THE NUMERICAL SCHEME

Numerical implementation

For $M > 0$ and $\Delta t = T/M$, we write the fully-discrete system

$$\begin{cases} z^0 = z_0(x) \\ \mathcal{M}_h \frac{z^{n+1} - z^n}{\Delta t} + \mathcal{A}_h^s z^{n+1} = \chi_\omega v_h^{n+1}, \quad \forall n \in \{0, \dots, M-1\}, \end{cases}$$

where $\mathcal{A}_h^s$ and $\mathcal{M}_h$ are suitable approximation matrices.

We consider also the discrete version of the penalized HUM functional

$$J_{\varepsilon, h, \Delta t}(\varphi^T) = \frac{1}{2} \sum_{n=1}^M \Delta t \int_{\omega} |\varphi^n|^2 + \frac{\varepsilon}{2} |\varphi^T|_{L^2}^2 + (y_0, \varphi^1)_{L^2}$$

where $\varphi = (\varphi^n)_{1 \leq n \leq M+1}$ solution to the adjoint system

$$\begin{cases} \varphi^{M+1} = \varphi^T \\ \mathcal{M}_h \frac{\varphi^n - \varphi^{n+1}}{\Delta t} + \mathcal{A}_h^s \varphi^n = 0, \quad \forall n \in \{1, \dots, M\}. \end{cases}$$

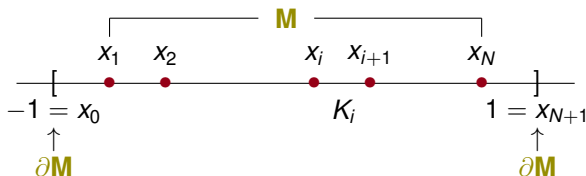
Finite element approximation of the elliptic problem

Partition of $(-1, 1)$

$$-1 = x_0 < x_1 < \dots < x_i < x_{i+1} < \dots < x_{N+1} = 1$$

$$x_{i+1} = x_i + h, \quad i = 0, \dots, N.$$

- $\mathbf{M} := \{x_i : i = 1, \dots, N\}$.
- $\partial\mathbf{M} := \{x_0, x_{N+1}\}$.
- $K_i := [x_i, x_{i+1}]$.



Consider the discrete space

$$V_h := \left\{ v \in H_0^s(-1, 1) \mid v|_{K_i} \in \mathcal{P}^1 \right\},$$

where $\mathcal{P}^1$ is the space of the continuous and piece-wise linear functions.

Given $\{\phi_i\}_{i=1}^N$ any basis of V_h , through a classical FE approach we are reduced to solve the linear system $\mathcal{A}_h u = F$

- $\mathcal{A}_h^s \in \mathbb{R}^{N \times N}$: stiffness matrix with components

$$a_{i,j} = \frac{c_{1,s}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi_i(x) - \phi_i(y))(\phi_j(x) - \phi_j(y))}{|x - y|^{1+2s}} dx dy,$$

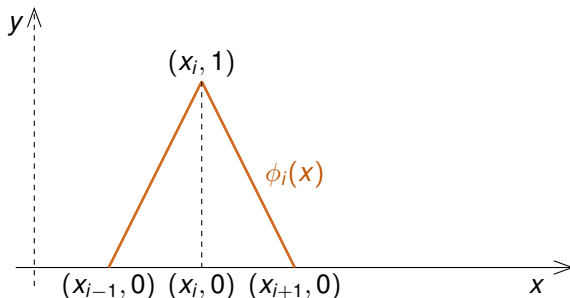
- $F \in \mathbb{R}^N$ given by $F = (F_1, \dots, F_N)$ with

$$F_i = \langle f, \phi_i \rangle = \int_{-1}^1 f \phi_i dx, \quad i = 1, \dots, N.$$

Basis functions

We employ the classical basis $\{\phi_i\}_{i=1}^N$ in which each ϕ_i is the tent function with $\text{supp}(\phi_i) = (x_{i-1}, x_{i+1})$ and verifying $\phi_i(x_j) = \delta_{i,j}$.

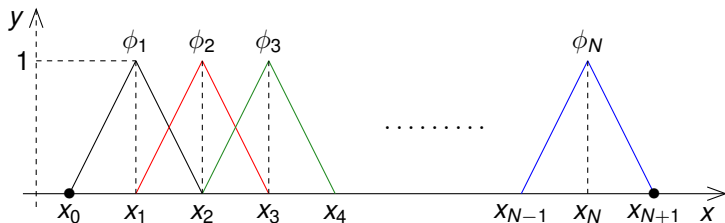
$$\phi_i(x) = 1 - \frac{|x - x_i|}{h}.$$



Construction of the stiffness matrix

Some remarks:

- $\mathcal{A}_h^s$ is **symmetric**. Therefore, in our algorithm we will only need to compute the values $a_{i,j}$ with $j \geq i$.
- Due to the non-local nature of the problem, the matrix $\mathcal{A}_h^s$ is **full**.
- The basis functions satisfy the zero Dirichlet B.C. This is important in the case $s > 1/2$.



Construction of $\mathcal{A}_h^s$ (cont.)

- $j \geq i + 2$: the easiest case, since $\text{Supp}(\phi_i) \cap \text{Supp}(\phi_j) = \emptyset$. Hence, the problem is reduced to compute

$$a_{i,j} = -2 \int_{x_{j-1}}^{x_{j+1}} \int_{x_{i-1}}^{x_{i+1}} \frac{\phi_i(x) \phi_j(y)}{|x - y|^{1+2s}} dx dy.$$

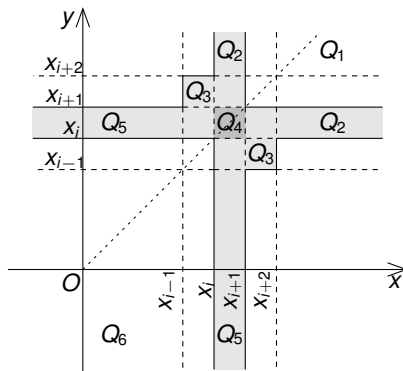
The contributions to the stiffness matrix in this case are

$$\mathcal{A}_h^s = \begin{pmatrix} & a_{1,3} & \cdots & \cdots & a_{1,N} \\ & & a_{2,4} & \cdots & a_{2,N} \\ a_{3,1} & & & \ddots & \vdots \\ \vdots & a_{4,2} & & & \\ \vdots & \vdots & \ddots & & \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N-2} & \end{pmatrix}$$

Construction of $\mathcal{A}_h^s$ (cont.)

- $j = i + 1$: the most cumbersome case.

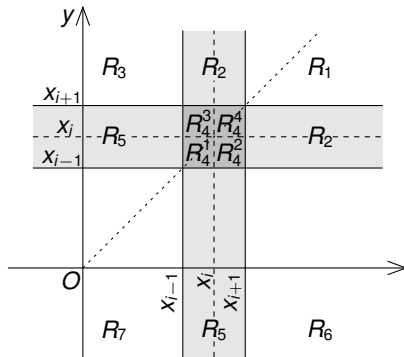
$$a_{i,i+1} = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi_i(x) - \phi_i(y))(\phi_{i+1}(x) - \phi_{i+1}(y))}{|x - y|^{1+2s}} dx dy := \sum_{k=1}^6 Q_k$$



Construction of $\mathcal{A}_h^s$ (cont.)

- $i = j$: we fill the diagonal of the matrix $\mathcal{A}_h^s$, which collects the values corresponding to the case $\phi_i(x) = \phi_j(x)$. We have

$$a_{i,i} = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi_i(x) - \phi_i(y))^2}{|x - y|^{1+2s}} dx dy := \sum_{k=1}^7 R_k.$$



Entries of the stiffness matrix $\mathcal{A}_h^s$

$$s \in (0, 1), s \neq 1/2$$

$$a_{i,j} = -h^{1-2s} \begin{cases} \frac{4(k+1)^{3-2s} + 4(k-1)^{3-2s}}{2s(1-2s)(1-s)(3-2s)} \\ - \frac{6k^{3-2s} + (k+2)^{3-2s} + (k-2)^{3-2s}}{2s(1-2s)(1-s)(3-2s)}, & k = j - i, k \geq 2 \\ \frac{3^{3-2s} - 2^{5-2s} + 7}{2s(1-2s)(1-s)(3-2s)}, & j = i + 1 \\ \frac{2^{3-2s} - 4}{s(1-2s)(1-s)(3-2s)}, & j = i. \end{cases}$$

Entries of the stiffness matrix $\mathcal{A}_h^s$

$$s = 1/2$$

$$a_{i,j} = \begin{cases} -4(k+1)^2 \log(k+1) - 4(k-1)^2 \log(k-1) \\ \quad + 6k^2 \log(k) + (k+2)^2 \log(k+2) \\ \quad + (k-2)^2 \log(k-2), & k = j - i, k > 2 \\ \\ 56 \ln(2) - 36 \ln(3), & j = i + 2. \\ \\ 9 \ln 3 - 16 \ln 2, & j = i + 1 \\ \\ 8 \ln 2, & j = i. \end{cases}$$

Some final remarks on the approximation $\mathcal{A}_h^s$

Some remarks

- Each entry $a_{i,j}$ of the matrix only depend on i, j, s and h .
- The matrix $\mathcal{A}_h^s$ has the structure of a N -diagonal matrix. This is analogous to the tridiagonal matrix approximating the classical Laplace operator

$$\mathcal{A}_h = \frac{1}{h} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

- $\mathcal{A}_h^s \rightarrow \mathcal{A}_h$ as $s \rightarrow 1$, which is in accordance to the behavior of the continuous operator. ¹

¹ E. Di Nezza, G. Palatucci, and E. Valdinoci, Bull. Sci. Math. 2012

NUMERICAL RESULTS

In order to test numerically the accuracy of our method, we use the following problem

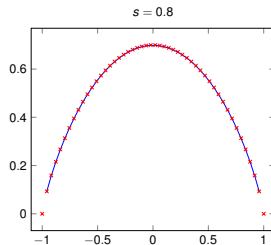
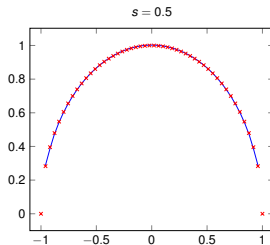
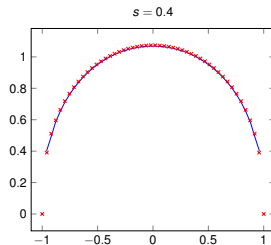
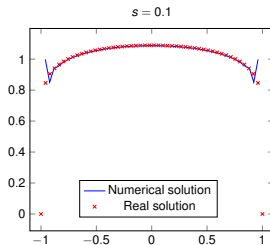
$$\begin{cases} (-d_x^2)^s u = 1, & x \in (-1, 1) \\ u \equiv 0, & x \in (-1, 1)^c. \end{cases}$$

In this particular case, the solution can be computed exactly and it reads as follows,

Solution

$$u(x) = \frac{2^{-2s} \sqrt{\pi}}{\Gamma\left(\frac{1+2s}{2}\right) \Gamma(1+s)} \left(1^2 - x^2\right)^s \chi_{(-1,1)}.$$

Comparison for different values of s

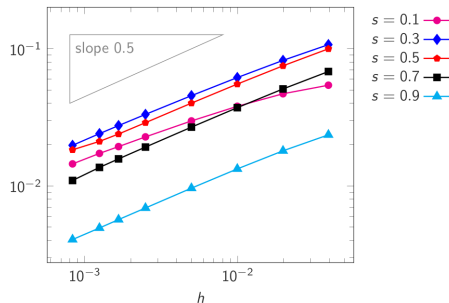


Convergence of the error

Theorem (G. Acosta and J. P. Borthagaray, SIAM J. Numer. Anal., 2017)

For h sufficiently small and f regular enough, the following estimate holds

$$\|u - u_h\|_{H_0^s(-1,1)} \lesssim Ch^{1/2} \quad \text{for } s \in (0, 1)$$



Control experiments: practical considerations

$$J_{\varepsilon, h, \Delta t}(\varphi^T) = \frac{1}{2} \sum_{n=1}^M \Delta t \int_{\omega} |\varphi^n|^2 + \frac{\varepsilon}{2} |\varphi^T|_{L^2}^2 + (y_0, \varphi^1)_{L^2}$$

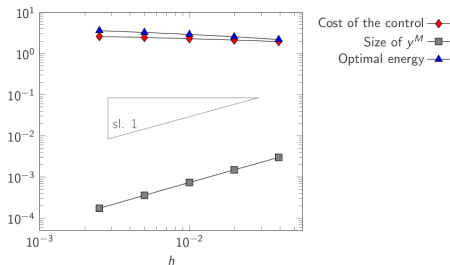
- We use conjugate gradient (CG).
- Choose the penalization parameter $\varepsilon = \phi(h)$
- **Practical rule:** choose $\varepsilon \sim h^{2p}$ where p is the order of accuracy in space of the discretization method.

Controlled solution

- $T = 0.3s$,
- $s = 0.8$,
- $\omega = (-0.4, 0.8)$.

Behavior of the penalized HUM

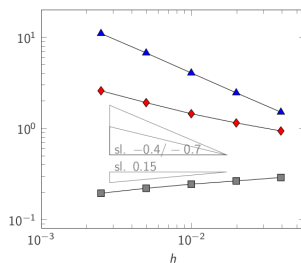
$s = 0.8$



- Cost of control and opt. energy are bounded as $h \rightarrow 0$.
- $|y(T)|_{L^2} \sim \sqrt{\varepsilon}$

null controllable.

$s = 0.4$



- Cost of control and opt. energy are not bounded as $h \rightarrow 0$.
- $|y(T)|_{L^2} \sim Ch^{0.15}$

approximately controllable.

FINAL REMARK

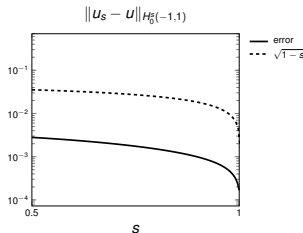
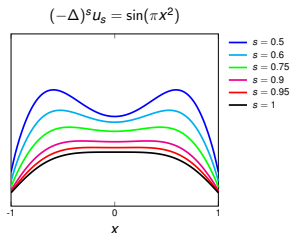
Convergence of u_s to u

- $(-\Delta)^s \xrightarrow{s \rightarrow 1} -\Delta$, (Di Nezza et al., '12)
- $\mathcal{A}_h^s \xrightarrow{s \rightarrow 1} \mathcal{A}_h$.

Theorem (U. B. and V. Hernández-Santamaría., Elec. J. Diff. Eq., 2018)

Given a u.b. sequence of $f_s \in H^{-s}(\Omega)$, $u_s \rightarrow u$ in $H_0^{1-\delta}(\Omega)$ strongly for all $0 < \delta \leq 1$. Moreover, $u \in H_0^1(\Omega)$ and satisfies

$$\int_{-1}^1 \nabla u \cdot \nabla v \, dx = \int_{-1}^1 f v \, dx, \quad \forall v \in \mathcal{D}(\Omega)$$



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