A Finite Element approximation of the one-dimensional fractional Poisson equation with applications to numerical control

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Outline of the talk

Introduction

We present a Finite Element (FE) scheme for the numerical approximation of the solution to the following non-local equation:

Fractional Poisson equation

$$(-d_x^2)^s u = f, \quad x \in (-L, L), \qquad\qquad u \equiv 0, \quad x \in \mathbb{R} \setminus (-L, L).$$

As a natural application, we analyze the numerical control problem for the following parabolic equation:

Fractional heat equation

$$\begin{cases} z_t + (-d_x^2)^s z = g\mathbf{1}_\omega, & (x,t) \in (-1,1) \times (0,T) \\ z = 0, & (x,t) \in [\mathbb{R} \setminus (-1,1)] \times (0,T) \\ z(x,0) = z_0(x), & x \in (-1,1) \end{cases}$$

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Fractional Laplacian

For any function u sufficiently regular and for any $s \in (0,1)$, the s-th power of the Laplace operator is given by

$$(-d_x^2)^s u(x) = C_s P.V. \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\, dy.$$

Functional setting: fractional Sobolev spaces

- $H^s(-L,L) := \left\{ u \in L^2(-L,L) \,:\, \frac{|u(x)-u(y)|}{|x-y|^{\frac{1}{2}+s}} \in L^2((-L,L)^2) \right\}.$
- $\|u\|_{H^s(-L,L)} := \left(\int_{-L}^{L} |u|^2 dx + \int_{-L}^{L}\int_{-L}^{L} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\, dxdy \right)^{\frac{1}{2}}.$
- $H_0^s(-L,L) := \left\{ u \in H^s(\mathbb{R}) \,:\, u = 0 \; in \; \mathbb{R} \setminus (-L,L) \right\}.$

Existing literature

- R.H. Nochetto, E. Otárola and A.J. Salgado, A. Bonito et al. :

  FE schemes for the discretization of elliptic and parabolic problems involving the **spectral** Fractional Laplacian (**DIFFERENT OPERATOR**).

- G. Acosta et al. :

  FE schemes for the discretization 2-D elliptic problems involving the Fractional Laplacian.

Variational formulation for the elliptic problem

Find $u \in H^s_0(-L, L)$ such that

$$\underbrace{\frac{c_{1,s}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+2s}} \, dxdy}_{a(u,v)} = \int_{-L}^{L} fv \, dx,$$

for all $v \in H^s_0(-L, L)$.

Well posedness

$a(\cdot, \cdot) : H^s_0(-L, L) \times H^s_0(-L, L) \to \mathbb{R}$ continuous and coercive $\Rightarrow$ If $f \in H^{-s}(-L, L)$, then there exists a unique weak solution $u \in H^s_0(-L, L)$.

Control results

Fractional heat equation

$$\begin{cases} z_t + (-d_x^2)^s z = g\mathbf{1}_\omega, & (x,t) \in (-1,1) \times (0,T) \\ z = 0, & (x,t) \in [\mathbb{R} \setminus (-1,1)] \times (0,T) \\ z(x,0) = z_0(x), & x \in (-1,1) \end{cases}$$

Proposition

For all $z_0 \in L^2(-1,1)$ and $T > 0$, the equation is null-controllable with a control function $g \in L^2(\omega \times (0,T))$ if and only if $s > 1/2$.

Proposition

Let $s \in (0,1)$ and $T > 0$. For all $z_0 \in L^2(-1,1)$, there exists a control function $g \in L^2(\omega \times (0,T))$ such that the unique solution z to the fractional Heat equation is approximately controllable.

Proof of the null controllability (sketch)

The result is equivalent to the existence of a constant $C > 0$ such that the following observability inequality holds

$$\|\varphi(x,0)\|^2_{L^2(-1,1)} \leq C \int_0^T \left| \int_\omega \varphi(x,t) g(x,t)\, dx \right|^2 dt,$$

where $\varphi(x,t)$ is the unique solution to the adjoint system

$$\begin{cases} -\varphi_t + (-d_x^2)^s \varphi = 0, & (x,t) \in (-1,1) \times (0,T) \\ \varphi = 0, & (x,t) \in [\,\mathbb{R} \setminus (-1,1)\,] \times (0,T) \\ \varphi(x,T) = \varphi^T(x), & x \in (-1,1). \end{cases}$$

Spectral expansion

$$\varphi(x,t) = \sum_{k\geq 1} \varphi_k e^{-\lambda_k (T-t)} \varrho_k(x), \quad \varphi_k = \langle \varphi^T, \varrho_k \rangle.$$

The observability inequality becomes

$$\sum_{k\geq 1} |\varphi_k|^2 e^{-2\lambda_k T} \leq C \int_0^T \left| \sum_{k\geq 1} \varphi_k g_k(t) e^{-\lambda_k t} \right|^2 dt, \quad g_k = \langle g \mathbf{1}_\omega, \varrho_k \rangle.$$

Müntz Theorem: the last inequality is true if and only if

$$\sum_{k\geq 1} \frac{1}{\lambda_k} < +\infty.$$

Eigenvalues of $(-d_x^2)^s$ on $(-1,1)$ with DBC[1]

$$\lambda_k = \left(\frac{k\pi}{2} - \frac{(1-s)\pi}{4}\right)^{2s} + O\left(\frac{1}{k}\right).$$

The series is convergent if and only if $s > 1/2$. Therefore, the observability inequality holds when $s > 1/2$, but it is false when $s \leq 1/2$.

[1] M. Kwaśnichi, J. Funct. Anal., 2012

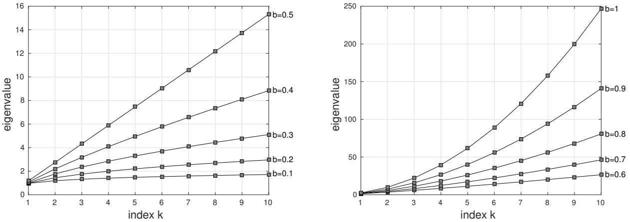


Figure: *

First ten eigenvalues of $(-d_x^2)^s$ on $(-1,1)$ with DBC for $s \leq 1/2$ (left) and $s > 1/2$ (right).

- $s \leq 1/2$: the equation is not null-controllable.
- $s > 1/2$: the equation is null-controllable.

Proof of the approximate controllability (sketch)

The result follows from the following property.

Parabolic unique continuation

Given $s \in (0,1)$ and $\varphi_0^T \in L^2(-1,1)$, let φ be the unique solution to the adjoint equation. Let $\omega \subset (-1,1)$ be an arbitrary open set. If $\varphi = 0$ on $\omega \times (0,T)$, then $\varphi = 0$ on $(-1,1) \times (0,T)$.

This, in turn, is a consequence of the Unique Continuation property for the Fractional Laplacian, obtained by Fall and Felli [2].

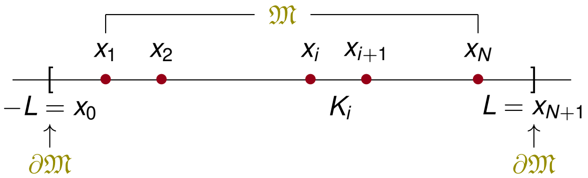
[2] M.M. Fall and V. Felli, Comm. Partial Differential Equations, 2014.. □

Finite element approximation of the elliptic problem

Partition of $(-L, L)$

$$-L = x_0 < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_{N+1} = L,$$
$$x_{i+1} = x_i + h, \; i = 0, \ldots N.$$

- $\mathfrak{M} := \{x_i \; : \; i = 1, \ldots, N\}$.
- $\partial\mathfrak{M} := \{x_0, x_{N+1}\}$.
- $K_i := [x_i, x_{i+1}]$.



Consider the discrete space

$$V_h := \left\{ v \in H_0^s(-L, L) \,\middle|\, v\,|_{K_i} \in \mathcal{P}^1 \right\},$$

where $\mathcal{P}^1$ is the space of the continuous and piece-wise linear functions.

Discrete variational formulation

Find $u_h \in V_h$ such that

$$\frac{c_{1,s}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u_h(x) - u_h(y))(v_h(x) - v_h(y))}{|x-y|^{1+2s}}\, dxdy = \int_{-L}^{L} f v_h \, dx,$$

for all $v_h \in V_h$.

Given $\{\phi_i\}_{i=1}^N$ any basis of V_h, it is sufficient that the discrete variational formulation is satisfied for

$$u_h(x) = \sum_{j=1}^N u_j \phi_j(x), \quad v_h(x) = \phi_j(x)$$

In this way, we are reduced to solve the linear system $\mathcal{A}_h u = F$

- $\mathcal{A}_h \in \mathbb{R}^{N\times N}$: stiffness matrix with components

$$a_{i,j} = \frac{c_{1,s}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi_i(x) - \phi_i(y))(\phi_j(x) - \phi_j(y))}{|x-y|^{1+2s}} \, dxdy,$$

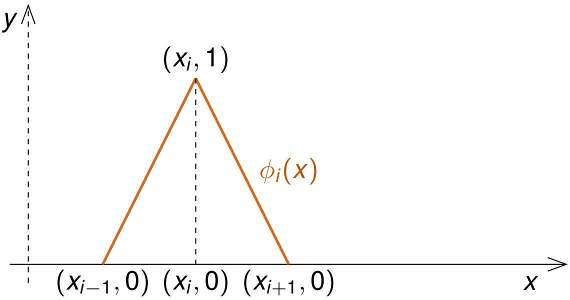
- $F \in \mathbb{R}^N$ given by $F = (F_1, \ldots, F_N)$ with

$$F_i = \langle f, \phi_i \rangle = \int_{-L}^{L} f\phi_i \, dx, \quad i = 1, \ldots, N.$$

Basis functions

We employ the classical basis $\{\phi_i\}_{i=1}^{N}$ in which each ϕ_i is the tent function with $supp(\phi_i) = (x_{i-1}, x_{i+1})$ and verifying $\phi_i(x_j) = \delta_{i,j}$.

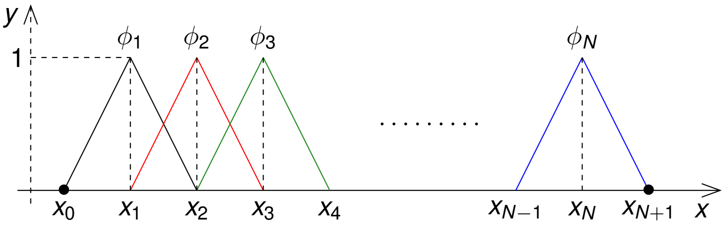
$$\phi_i(x) = 1 - \frac{|x - x_i|}{h}.$$



Construction of the stiffness matrix

Remark

- *$\mathcal{A}_h$ is symmetric. Therefore, in our algorithm we will only need to compute the values $a_{i,j}$ with $j \geq i$.*
- *Due to the non-local nature of the problem, the matrix $\mathcal{A}_h$ is full.*
- *While computing the values $a_{i,j}$, we will only work on the mesh $\mathfrak{M}$, not considering the points of the set $\partial\mathfrak{M}$. In this way, we will ensure that the basis functions ϕ_i satisfy the zero Dirichlet boundary conditions.*

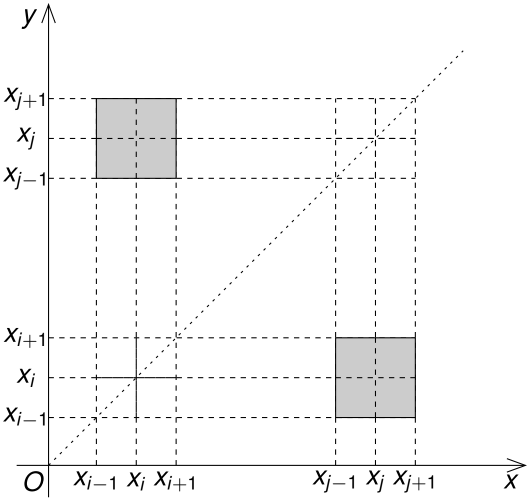


The building of the stiffness matrix $\mathcal{A}_h$ is done it in three steps

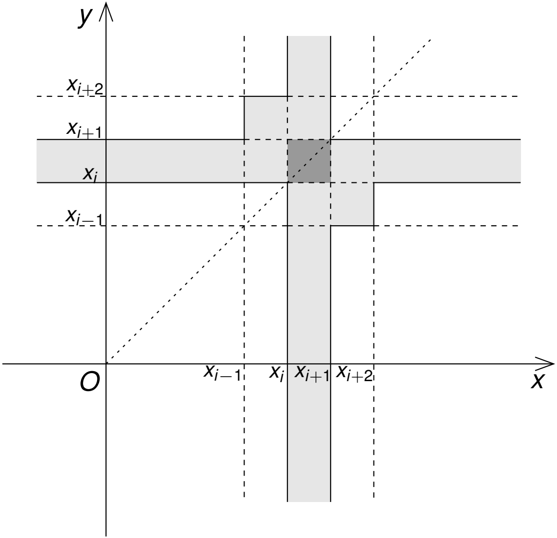
1. We fill the upper triangle, corresponding to $j \geq i + 2$.
2. We fill the upper diagonal corresponding to $j = i + 1$.
3. We fill the diagonal.

$$\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots\cdots & \cdots\cdots & a_{1,N} \\
 & a_{2,2} & a_{2,3} & a_{2,4} & \cdots\cdots & a_{2,N} \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & \ddots & \ddots & a_{N-2,N} \\
 & & & & a_{N-1,N-1} & a_{N-1,N} \\
 & & & & & a_{N,N}
\end{pmatrix}$$

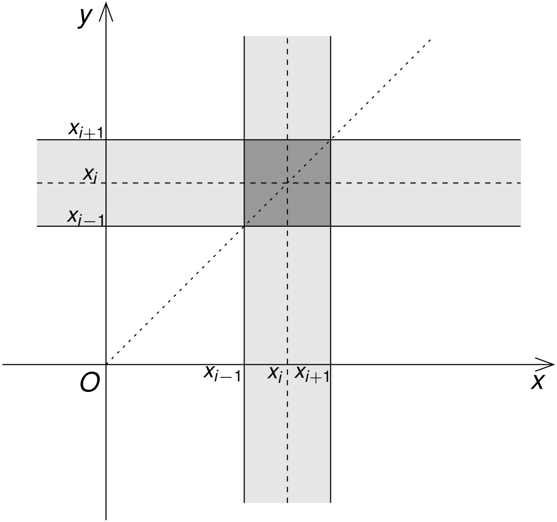
Values of $a_{i,j}$ for $j \geq i + 2$



Values of $a_{i,i+1}$



Values of $a_{i,i}$



Entries of the stiffness matrix $\mathcal{A}_h$

$s \neq 1/2$

$$a_{i,j} = -h^{1-2s} \begin{cases} \dfrac{4(k+1)^{3-2s} + 4(k-1)^{3-2s}}{2s(1-2s)(1-s)(3-2s)} \\ \qquad - \dfrac{6k^{3-2s} + (k+2)^{3-2s} + (k-2)^{3-2s}}{2s(1-2s)(1-s)(3-2s)}, & k = j-i,\ k \geq 2 \\ \dfrac{3^{3-2s} - 2^{5-2s} + 7}{2s(1-2s)(1-s)(3-2s)}, & j = i+1 \\ \dfrac{2^{3-2s} - 4}{s(1-2s)(1-s)(3-2s)}, & j = i. \end{cases}$$

Entries of the stiffness matrix $\mathcal{A}_h$

$s = 1/2$

$$a_{i,j} = \begin{cases} -4(k+1)^2\log(k+1) - 4(k-1)^2\log(k-1) \\ \quad +6k^2\log(k) + (k+2)^2\log(k+2) \\ \quad +(k-2)^2\log(k-2), & k = j-i,\ k > 2 \\ 56\ln(2) - 36\ln(3), & j = i+2. \\ 9\ln 3 - 16\ln 2, & j = i+1 \\ 8\ln 2, & j = i. \end{cases}$$

Penalized Hilbert Uniqueness Method

We have to solve the following minimization problem: find

$$\varphi_\varepsilon^T = \min_{\varphi \in L^2(-1,1)} J_\varepsilon(\varphi^T)$$

where

$$J_\varepsilon(\varphi^T) := \frac{1}{2}\int_0^T \int_\omega |\varphi|^2\, dxdt + \frac{\varepsilon}{2}\|\varphi^T\|^2_{L^2(-1,1)} + \int_\Omega z_0\varphi(0)\, dx$$

and where φ is the solution to the adjoint problem

$$\begin{cases} -\varphi_t + (-d_x^2)^s\varphi = 0, & (x,t) \in (-1,1)\times(0,T) \\ \varphi = 0, & (x,t) \in \big[\mathbb{R}\setminus(-1,1)\big]\times(0,T) \\ \varphi(x,T) = \varphi^T(x), & x \in (-1,1). \end{cases}$$

The approximate and null controllability properties of the system, for a given initial datum z_0, can be expressed in terms of the behavior of the penalized HUM approach. In particular, we have:

Theorem (F. Boyer, ESAIM: PROCEEDINGS, 2013)

- *The equation is approximately controllable at time T from the initial datum z_0 if and only if*

$$\varphi_\varepsilon^T \to 0, \quad as \;\; \varepsilon \to 0.$$

- *The equation is null-controllable at time T from the initial datum z_0 if and only if*

$$M_{z_0}^2 := 2 \sup_{\varepsilon > 0} \left(\inf_{L^2(0,T;L^2(\omega))} J_\varepsilon \right) < +\infty.$$

In this case, we have

$$\|g\|_{L^2(0,T;L^2(\omega))} \leq M_{z_0}, \quad \|\varphi_\varepsilon^T\|_{L^2(-L,L)} \leq M_{z_0}\sqrt{\varepsilon}.$$

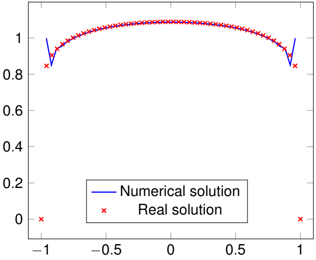
In order to test numerically the accuracy of our method, we use the following problem

$$\begin{cases} (-d_x^2)^s u = 1, & x \in (-L, L) \\ u \equiv 0, & x \in \mathbb{R} \setminus (-L, L). \end{cases}$$

In this particular case, the solution can be computed exactly and it reads as follows,

Solution

$$u(x) = \frac{2^{-2s}\sqrt{\pi}}{\Gamma\left(\frac{1+2s}{2}\right)\Gamma(1+s)}\left(L^2 - x^2\right)^s.$$



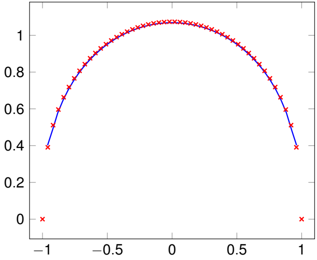


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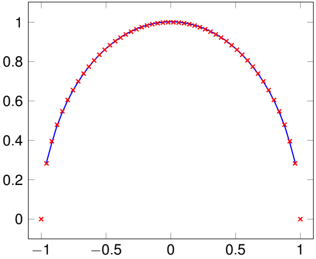
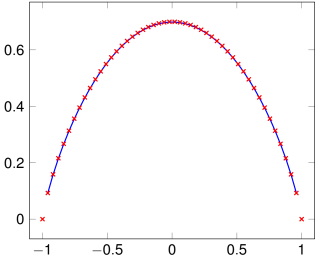
Figure: *

$s = 0.1$

$s = 0.4$





Error analysis

The computation of the error in the space $H_0^s(-L,L)$ can be readily done by using the definition of the bilinear form, namely

$$\|u-u_h\|^2_{H_0^s(-L,L)} = a(u-u_h, u-u_h) = \int_{-L}^{L} f(x)\,(u(x)-u_h(x))\,dx.$$

$$f \equiv 1 \;\Rightarrow\; \|u-u_h\|_{H_0^s(-L,L)} = \left(\int_{-L}^{L} (u(x)-u_h(x))\;dx\right)^{1/2}.$$

The right-hand side can be easily computed, since we have the closed formula

$$\int_{-L}^{L} u\,dx = \frac{\pi L^{2s+1}}{2^{2s}\Gamma(s+\frac{1}{2})\Gamma(s+\frac{3}{2})}$$

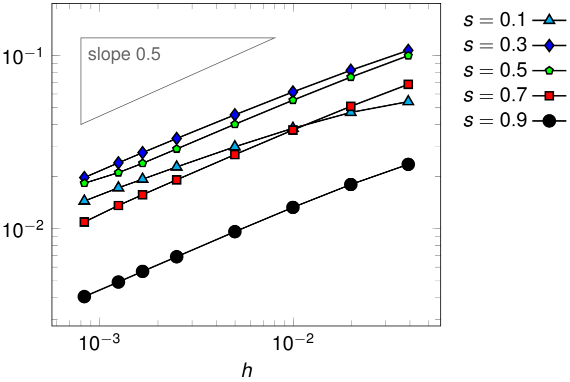
and the term corresponding to $\int_{-L}^{L} u_h$ can be carried out numerically.

Theorem (G. Acosta and J.P. Borthagaray, SIAM J. Numer. Anal., 2017)

For the solution u of the elliptic problem and its FE approximation u_h, if h is sufficiently small, the following estimates hold

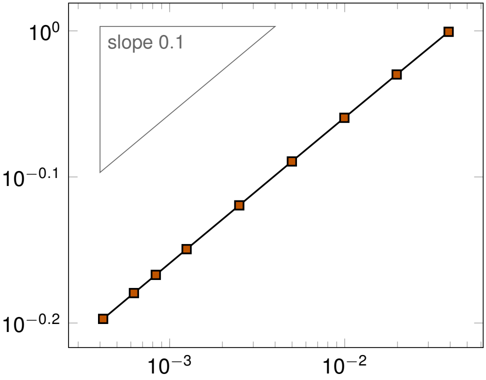
$$
\begin{aligned}
\|u-u_h\|_{H_0^s(-L,L)} &\leq Ch^{1/2}|\ln h|\,\|f\|_{C^{\frac{1}{2}-s}(-L,L)}, && \text{if } s<1/2,\\
\|u-u_h\|_{H_0^s(-L,L)} &\leq Ch^{1/2}|\ln h|\,\|f\|_{L^\infty(-L,L)}, && \text{if } s=1/2,\\
\|u-u_h\|_{H_0^s(-L,L)} &\leq \frac{C}{2s-1}h^{1/2}\sqrt{|\ln h|}\,\|f\|_{C^\beta(-L,L)}, && \text{if } s>1/2,
\end{aligned}
$$

where C is a positive constant not depending on h.



The convergence rate is maintained also for small values of s. This confirms that the behavior obtained for $s = 0.1$ is not in contrast with the known theoretical results. Indeed, since it is well-known that the notion of trace is not defined for the spaces $H^s(-L, L)$ with $s \leq 1/2$, it is somehow natural that we cannot expect a point-wise convergence in this case.

As a further validation of this fact, we plot the behavior of the L^∞-norm of the difference between the real and the numerical solution to the fractional Poisson equation.



Increasing the number of point of discretization, the L^∞-norm is decreasing with a rate (in h) of 0.1.

Controllability of the fractional heat equation

- We use the finite-element approximation of $(-d_x^2)^s$ for the space discretization and the implicit Euler scheme in the time variable.

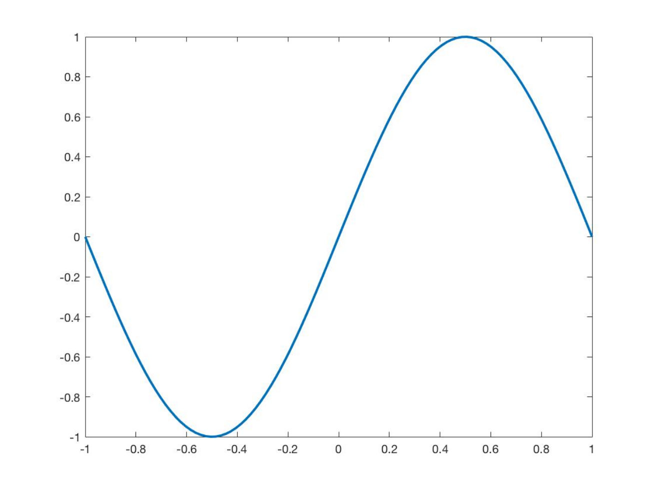
$$\begin{cases} \mathcal{M}_h \dfrac{z^{n+1} - z^n}{\delta t} + \mathcal{A}_h z^{n+1} = \mathbf{1}_\omega v_h^{n+1}, \quad \forall n \in \{1, \ldots, M-1\} \\ z^0 = z_0 \end{cases}$$

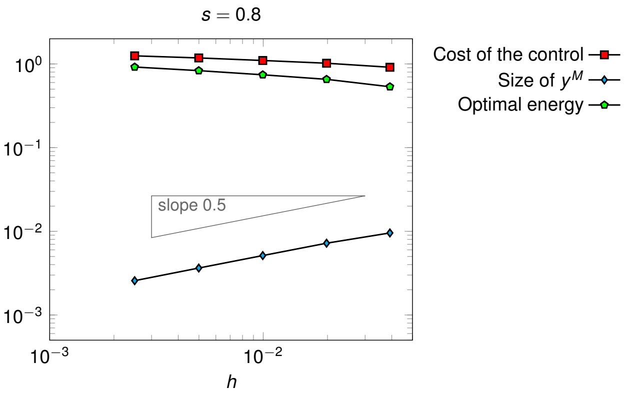
- We choose the penalization term ε as a function of h.
- PRACTICAL RULE: choose $\varepsilon \sim h^{2p}$ where p is the order of accuracy in space of the numerical method used for the discretization of the spatial operator involved. (in this case, we take $p = 1/2$).

Control experiments

Uncontrolled solution - $s = 0.8$, $T = 0.2\,s$

Controlled solution - $s = 0.8$, $T = 0.2\,s$, $\omega = (-0.3, 0.8)$

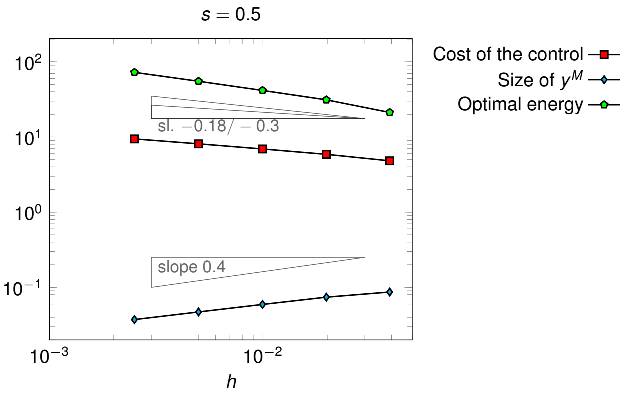




For $s = 0.8$ we observe that:

- The control cost and the optimal energy remain bounded as $h \to 0$.
- $|y^M|_{L^2(\mathbb{R}^{\mathfrak{M}})} \sim C\sqrt{\phi(h)} = Ch^{1/2}$.

This confirms that the system is null controllable.



For $s = 0.5$ we observe that:

- The control cost and the optimal energy do not remain bounded as $h \to 0$.
- $|y^M|_{L^2(\mathbb{R}^{\mathfrak{M}})} \sim Ch^{0.4}$.

This confirms that the system is only approximately controllable.

THANK YOU FOR YOUR ATTENTION!