CONTROLLABILITY OF A 1-D FRACTIONAL HEAT EQUATION

THEORETICAL AND NUMERICAL ASPECTS

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INTRODUCTION AND MAIN THE- ORETICAL RESULTS

Introduction

Consider the following non-local parabolic equation

Fractional heat equation

$$\begin{cases} z_t + (-d_x^2)^s z = g\mathbf{1}_\omega, & (x,t) \in (-1,1) \times (0,T) \\ z = 0, & (x,t) \in (-1,1)^c \times (0,T) \\ z(x,0) = z_0(x), & x \in (-1,1) \end{cases} \tag{$\mathcal{H}$}$$

- $\omega \subset (-1,1)$.
- $(-d_x^2)^s$: fractional Laplacian.

We are interested in analyzing controllability properties, both from a theoretical and a numerical viewpoint.

Fractional Laplacian

For any function u sufficiently regular and for any $s \in (0,1)$, the s-th power of the Laplace operator is given by

$$(-d_x^2)^s u(x) = c_s P.V. \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\, dy.$$

$c_s = \frac{s 2^{2s}\Gamma\left(\frac{1+2s}{2}\right)}{\sqrt{\pi}\Gamma(1-s)}$ normalization constant ensuring that $\lim_{s\to 1^-}(-d_x^2)^s = -d_x^2$.

NULL CONTROLLABILITY at time $T > 0$:
given any initial datum $z_0 \in L^2(-1,1)$ there exists $g \in L^2(\omega \times (0,T))$ such that the corresponding solution z satisfies $z(x,T) = 0$.

APPROXIMATE CONTROLLABILITY at time $T > 0$:
given any $z_0, z_T \in L^2(-1,1)$ and any $\varepsilon > 0$ there exists $g \in L^2(\omega \times (0,T))$ such that the corresponding solution z with initial datum $z(x,0) = z_0$ satisfies $\|z(x,T) - z_T\|_{L^2(-1,1)} \leq \varepsilon$.

Theorem (U.B. and V. Hernández-Santamaría, IMA J. Math. Control Inf., 2018)

The fractional heat equation $(\mathcal{H})$ *is*

- *null-controllable at time* $T > 0$ *if and only if* $s > 1/2$.
- *approximately controllable at time* $T > 0$ *for all* $s \in (0,1)$.

Existence uniqueness and regularity of solutions

Definition

We say that $z \in L^2(0,T;H_0^s(-1,1)) \cap C([0,T],L^2(-1,1))$ *with* $z_t \in L^2(0,T;H^{-s}(-1,1))$ *is a weak solution for the parabolic problem* $(\mathcal{H})$ *with* $g \in L^2(0,T;H^{-s}(-1,1))$ *and* $z_0 \in L^2(-1,1)$ *if it satisfies*

$$\int_0^T \int_{-1}^1 z_t w \, dxdt + \int_0^T a(z,w)\, dt = \int_0^T \langle f, w\rangle_{-s,s}\, dt,$$

for any $w \in L^2(0,T;H_0^s(-1,1))$.

The bilinear form $a(\cdot,\cdot) : H_0^s(-1,1) \times H_0^s(-1,1) \to \mathbb{R}$ *is defined as*

$$a(u,v) = \frac{c_{1,s}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+2s}}\, dxdy.$$

Theorem (T. Leonori, I. Peral, A. Primo, and F. Soria, Discrete Contin. Dyn. Syst., 2015.)

Assume $f \in L^2(0,T;H^{-s}(-1,1))$. Then for any $z_0 \in L^2(-1,1)$, problem $(\mathcal{H})$ has a unique weak solution.

Theorem (U.B., M. Warma, and E. Zuazua, SEMA SIMAI Springer Series, 2018.)

Assume $f \in L^2((-1,1)\times(0,T))$, $z_0 = 0$, and let $z \in L^2((0,T);H_0^s(-1,1)) \cap C([0,T];L^2(-1,1))$ with $z_t \in L^2((0,T);H^{-s}(-1,1))$ be the unique weak solution of system $(\mathcal{H})$. Then

$$z \in L^2((0,T);H^{2s}_{loc}(-1,1)) \cap L^\infty((0,T);H_0^s(-1,1))$$

and

$$z_t \in L^2((-1,1)\times(0,T)).$$

Known controllability results

- S. Micu and E. Zuazua, 2006: in the 1-d setting the authors consider spectral fractional Laplacian which is defined by

$$(-d_x^2)_S^s u(x) = \sum_{k\geq 1} \langle u, \psi_k \rangle \lambda_k^s \psi_k(x),$$

  They prove that for $s > 1/2$ null controllability holds. But

  spectral $\neq$ integral[1]

- L. Miller, 2006: the same result holds in multi-d setting using Lebeau-Robbiano strategy.

[1] R. Servadei and E. Valdinoci, Proc. R. Soc. Edinb. A, 2014

Proof of the null controllability (sketch)

By employing the classical moment method [1], the equation is null-controllable if and only if

- $\sum_{k\geq 1} \lambda_k^{-1} < +\infty$.
- $\lambda_{k+1} - \lambda_k \geq \gamma > 0, \quad \forall\, k \in \mathbb{N}$.

Eigenvalues of the fractional Laplacian on $(-1,1)$ with Dirichlet B.C.[2]

$$\begin{cases} (-d_x^2)^s \varrho_k = \lambda_k \varrho_k, & x \in (-1,1), \;\; k \in \mathbb{N} \\ \varrho_k = 0, & x \in \mathbb{R} \setminus (-1,1), \end{cases}$$

$$\lambda_k = \left(\frac{k\pi}{2} - \frac{(1-s)\pi}{4} \right)^{2s} + O\left(\frac{1}{k}\right).$$

Therefore, the two conditions above are both satisfied if and only if $s > 1/2$.

[1] Fattorini and Russell, Quart. Appl. Math., 1974. [2] Kwaśnicki, J. Funct. Anal., 2012.

Proof of the approximate controllability (sketch)

The result follows from the following property.

Parabolic unique continuation

Given $s \in (0,1)$ and $\varphi_0^T \in L^2(-1,1)$, let φ be the unique solution to the adjoint equation. Let $\omega \subset (-1,1)$ be an arbitrary open set. If $\varphi = 0$ on $\omega \times (0,T)$, then $\varphi = 0$ on $(-1,1) \times (0,T)$.

This, in turn, is a consequence of the unique continuation property for the Fractional Laplacian.[1]

[1] M.M. Fall and V. Felli, Comm. Partial Differential Equations, 2014.. $\square$

Remarks

- The elliptic unique continuation property for the fractional Laplacian holds in any space dimension. In view of that, the approximate controllability for $(\mathcal{H})$ may be obtained also to the case $N > 1$. The same does not applies to the null-controllability, since our proof uses arguments that are designed specifically for one-dimensional problems.
- If one would like to analyze the multi-dimensional problem, other tools (Carleman estimates) are needed. These techniques are not available for problems involving the fractional Laplacian on a domain.

NUMERICAL IMPLEMENTATION

Penalized Hilbert Uniqueness Method

We have to solve the following minimization problem:

$$\min_{\varphi^T \in L^2(-1,1)} J_\varepsilon(\varphi^T) := \frac{1}{2}\int_{\omega\times(0,T)} |\varphi|^2 dxdt + \frac{\varepsilon}{2}\left\|\varphi^T\right\|^2_{L^2(-1,1)} + \int_{-1}^{1} z_0\varphi(0)$$

where φ is the solution to the adjoint problem.

$$\begin{cases} -\varphi_t + (-d_x^2)^s\varphi = 0, & (x,t)\in(-1,1)\times(0,T) \\ \varphi = 0, & (x,t)\in(-1,1)^c\times(0,T) \\ \varphi(\cdot,T) = \varphi^T(x), & x\in(-1,1) \end{cases}$$

J_ε is continuous, coercive and strictly convex , thus the existence and uniqueness of a minimizer φ_ε^T is guaranteed.

In fact, the control is chosen as

$$v = \varphi_\varepsilon|_\omega$$

where φ_ε is the solution to the adjoint with initial datum φ_ε^T.

Penalized Hilbert Uniqueness Method (cont.)

Analyzing the behavior of the penalized problem with respect to the parameter ε, we can deduce controllability properties for our system.

Theorem (F. Boyer, ESAIM Proc., 2013)

Let φ_ε^T be the unique minimizer of J_ε. System $(\mathcal{H})$ is

- *null controllable at time $T \Leftrightarrow M_{z_0}^2 := 2\sup_{\varepsilon>0}(\inf J_\varepsilon) < +\infty$.*
  In this case:
  $$\|g\|_{L^2(\omega\times(0,T))} \leq M_{z_0} \quad \text{and} \quad \|\varphi_\varepsilon^T\| \leq M_{z_0}\sqrt{\varepsilon}$$
- *approximately controllable at time $T \Leftrightarrow \varphi_\varepsilon^T \to 0$ as $\varepsilon \to 0$.*

DEVELOPMENT OF THE NUMERICAL SCHEME

Numerical implementation

For $M > 0$ and $\Delta t = T/M$, we write the fully-discrete system

$$\begin{cases} z^0 = z_0(x) \\ \mathcal{M}_h \dfrac{z^{n+1} - z^n}{\Delta t} + \mathcal{A}_h^s z^{n+1} = \chi_\omega v_h^{n+1}, \quad \forall n \in \{0, \dots, M-1\}, \end{cases}$$

where $\mathcal{A}_h^s$ and $\mathcal{M}_h$ are suitable approximation matrices.

We consider also the discrete version of the penalized HUM functional

$$J_{\varepsilon,h,\Delta t}(\varphi^T) = \frac{1}{2} \sum_{n=1}^{M} \Delta t \int_\omega \left|\varphi^n\right|^2 + \frac{\varepsilon}{2} |\varphi^T|_{L^2}^2 + \left(y_0, \varphi^1\right)_{L^2}$$

where $\varphi = (\varphi^n)_{1 \le n \le M+1}$ solution to the adjoint system

$$\begin{cases} \varphi^{M+1} = \varphi^T \\ \mathcal{M}_h \dfrac{\varphi^n - \varphi^{n+1}}{\Delta t} + \mathcal{A}_h^s \varphi^n = 0, \quad \forall n \in \{1, \dots, M\}. \end{cases}$$

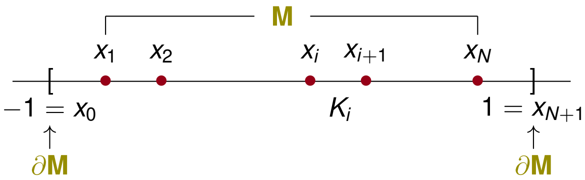
Finite element approximation of the elliptic problem

Partition of $(-1, 1)$

$$-1 = x_0 < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_{N+1} = 1$$

$$x_{i+1} = x_i + h, \; i = 0, \ldots N.$$

- $\mathbf{M} := \{x_i \; : \; i = 1, \ldots, N\}$.
- $\partial\mathbf{M} := \{x_0, x_{N+1}\}$.
- $K_i := [x_i, x_{i+1}]$.



Consider the discrete space

$$V_h := \left\{ v \in H_0^s(-1,1) \,\middle|\, v|_{K_i} \in \mathcal{P}^1 \right\},$$

where $\mathcal{P}^1$ is the space of the continuous and piece-wise linear functions.

Given $\{\phi_i\}_{i=1}^N$ any basis of V_h, through a classical FE approach we are reduced to solve the linear system $\mathcal{A}_h u = F$

- $\mathcal{A}_h^s \in \mathbb{R}^{N\times N}$: stiffness matrix with components

$$a_{i,j} = \frac{c_{1,s}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi_i(x) - \phi_i(y))(\phi_j(x) - \phi_j(y))}{|x-y|^{1+2s}}\, dxdy,$$

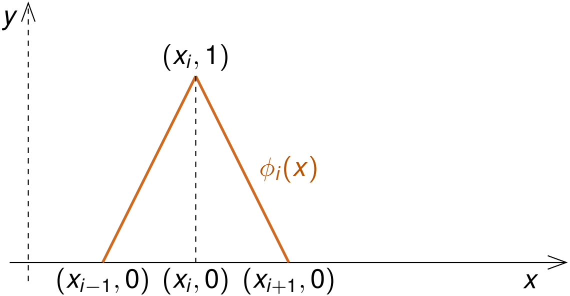
- $F \in \mathbb{R}^N$ given by $F = (F_1, \dots, F_N)$ with

$$F_i = \langle f, \phi_i \rangle = \int_{-1}^{1} f\phi_i\, dx, \quad i = 1, \dots, N.$$

Basis functions

We employ the classical basis $\{\phi_i\}_{i=1}^{N}$ in which each ϕ_i is the tent function with $supp(\phi_i) = (x_{i-1}, x_{i+1})$ and verifying $\phi_i(x_j) = \delta_{i,j}$.

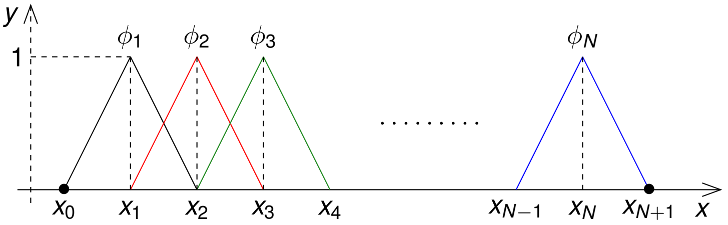
$$\phi_i(x) = 1 - \frac{|x - x_i|}{h}.$$



Construction of the stiffness matrix

Some remarks:

- $\mathcal{A}_h^s$ is symmetric. Therefore, in our algorithm we will only need to compute the values $a_{i,j}$ with $j \geq i$.
- Due to the non-local nature of the problem, the matrix $\mathcal{A}_h^s$ is full.
- The basis functions satisfy the zero Dirichlet B.C. This is important in the case $s > 1/2$.



Construction of $\mathcal{A}_h^s$ (cont.)

- $j \geq i+2$: the easiest case, since $\mathrm{Supp}(\phi_i) \cap \mathrm{Supp}(\phi_j) = \emptyset$. Hence, the problem is reduced to compute

$$a_{i,j} = -2 \int_{x_{j-1}}^{x_{j+1}} \int_{x_{i-1}}^{x_{i+1}} \frac{\phi_i(x)\phi_j(y)}{|x-y|^{1+2s}}\, dxdy.$$

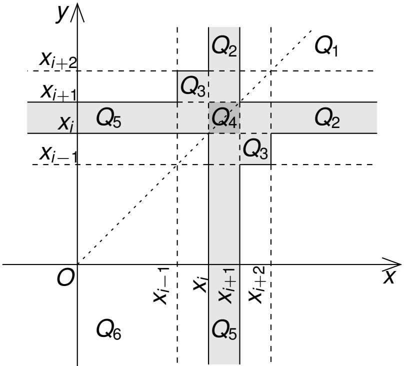
The contributions to the stiffness matrix in this case are

$$\mathcal{A}_h^s = \begin{pmatrix} & & a_{1,3} & \cdots & \cdots & a_{1,N} \\ & & & a_{24} & \cdots & a_{2,N} \\ & & & & \ddots & \vdots \\ a_{31} & & & & & \\ \vdots & a_{42} & & & & a_{N-2,N} \\ \vdots & \vdots & \ddots & & & \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N-2} & & \end{pmatrix}$$

Construction of $\mathcal{A}_h^s$ (cont.)

- $j = i + 1$: the most cumbersome case.

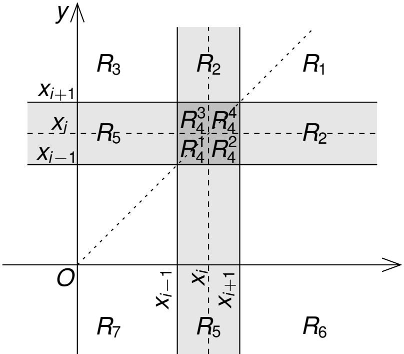
$$a_{i,i+1} = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi_i(x) - \phi_i(y))(\phi_{i+1}(x) - \phi_{i+1}(y))}{|x - y|^{1+2s}} \, dxdy := \sum_{k=1}^{6} Q_k$$



Construction of $\mathcal{A}_h^s$ (cont.)

- $i = j$: we fill the diagonal of the matrix $\mathcal{A}_h^s$, which collects the values corresponding to the case $\phi_i(x) = \phi_j(x)$. We have

$$a_{i,i} = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi_i(x) - \phi_i(y))^2}{|x-y|^{1+2s}} \, dxdy := \sum_{k=1}^{7} R_k.$$



Entries of the stiffness matrix $\mathcal{A}_h^s$

$s \in (0,1)$, $s \neq 1/2$

$$a_{i,j} = -h^{1-2s} \begin{cases} \dfrac{4(k+1)^{3-2s} + 4(k-1)^{3-2s}}{2s(1-2s)(1-s)(3-2s)} \\[2ex] -\dfrac{6k^{3-2s} + (k+2)^{3-2s} + (k-2)^{3-2s}}{2s(1-2s)(1-s)(3-2s)}, & k = j-i,\ k \geq 2 \\[2ex] \dfrac{3^{3-2s} - 2^{5-2s} + 7}{2s(1-2s)(1-s)(3-2s)}, & j = i+1 \\[2ex] \dfrac{2^{3-2s} - 4}{s(1-2s)(1-s)(3-2s)}, & j = i. \end{cases}$$

Entries of the stiffness matrix $\mathcal{A}_h^s$

$s = 1/2$

$$a_{i,j} = \begin{cases} -4(k+1)^2\log(k+1) - 4(k-1)^2\log(k-1) & \\ \quad +6k^2\log(k) + (k+2)^2\log(k+2) & \\ \quad +(k-2)^2\log(k-2), & k = j-i,\ k > 2 \\ 56\ln(2) - 36\ln(3), & j = i+2. \\ 9\ln 3 - 16\ln 2, & j = i+1 \\ 8\ln 2, & j = i. \end{cases}$$

Some final remarks on the approximation $\mathcal{A}_h^s$

Some remarks

- Each entry $a_{i,j}$ of the matrix only depend on i, j, s and h.
- The matrix $\mathcal{A}_h^s$ has the structure of a N-diagonal matrix. This is analogous to the tridiagonal matrix approximating the classical Laplace operator

$$\mathcal{A}_h = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

- $\mathcal{A}_h^s \to \mathcal{A}_h$ as $s \to 1$, which is in accordance to the behavior of the continuous operator. [1]

[1] E. Di Nezza, G. Palatucci, and E. Valdinoci, Bull. Sci. Math. 2012

NUMERICAL RESULTS

In order to test numerically the accuracy of our method, we use the following problem

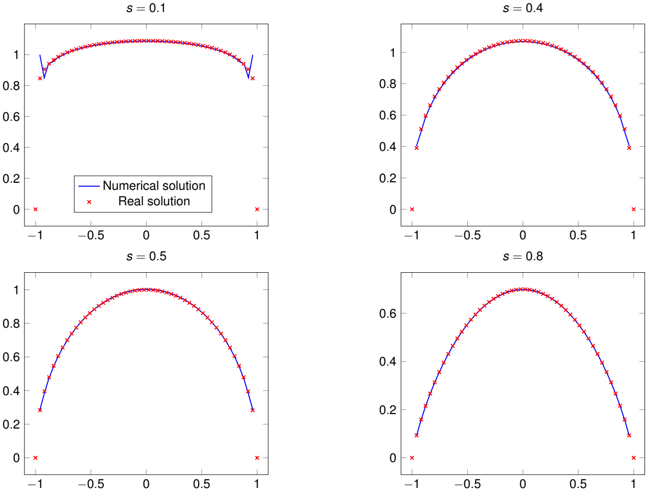
$$\begin{cases} (-d_x^2)^s u = 1, & x \in (-1,1) \\ u \equiv 0, & x \in (-1,1)^c. \end{cases}$$

In this particular case, the solution can be computed exactly and it reads as follows,

Solution

$$u(x) = \frac{2^{-2s}\sqrt{\pi}}{\Gamma\left(\frac{1+2s}{2}\right)\Gamma(1+s)}\left(1^2 - x^2\right)^s \chi_{(-1,1)}.$$

Comparison for different values of s

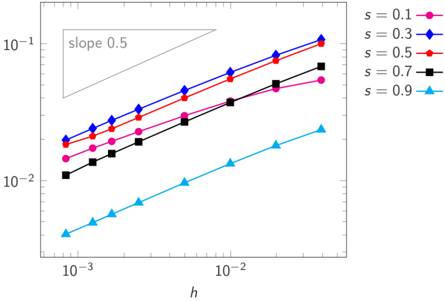


Convergence of the error

Theorem (G. Acosta and J. P. Borthagaray, SIAM J. Numer. Anal., 2017)

For h sufficiently small and f regular enough, the following estimate holds

$$\|u - u_h\|_{H_0^s(-1,1)} \lesssim C h^{1/2} \quad \textit{for } s \in (0,1)$$



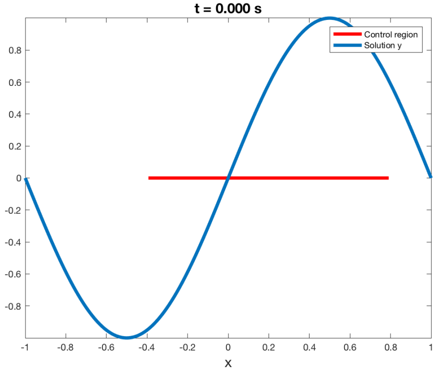
Control experiments: practical considerations

$$J_{\varepsilon,h,\Delta t}(\varphi^T) = \frac{1}{2}\sum_{n=1}^{M} \Delta t \int_{\omega} |\varphi^n|^2 + \frac{\varepsilon}{2}|\varphi^T|_{L^2}^2 + \left(y_0, \varphi^1\right)_{L^2}$$

- We use conjugate gradient (CG).
- Choose the penalization parameter $\varepsilon = \phi(h)$
- Practical rule: choose $\varepsilon \sim h^{2p}$ where p is the order of accuracy in space of the discretization method.

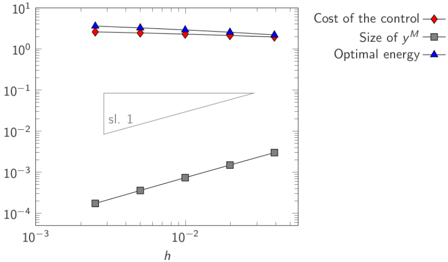
Controlled solution

- $T = 0.3s$,
- $s = 0.8$,
- $\omega = (-0.4, 0.8)$.



Behavior of the penalized HUM

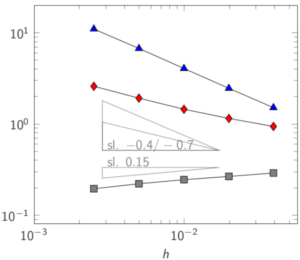
$s = 0.8$



- Cost of control and opt. energy are bounded as $h \to 0$.
- $|y(T)|_{L^2} \sim \sqrt{\varepsilon}$

null controllable.

$s = 0.4$



- Cost of control and opt. energy are not bounded as $h \to 0$.
- $|y(T)|_{L^2} \sim Ch^{0.15}$

approximately controllable.

Final remark

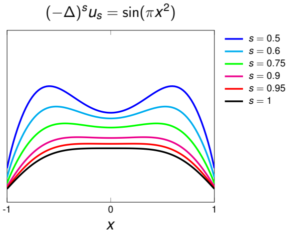
Convergence of u_s to u

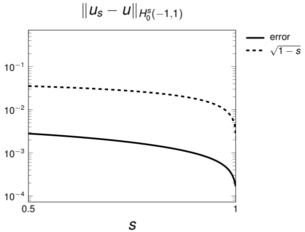
- $(-\Delta)^s \underset{s\to 1}{\to} -\Delta$, (Di Nezza et al., '12)
- $\mathcal{A}_h^s \underset{s\to 1}{\to} \mathcal{A}_h$.

Theorem (U. B. and V. Hernández-Santamaría., Elec. J. Diff. Eq., 2018)

Given a u.b. sequence of $f_s \in H^{-s}(\Omega)$, $u_s \to u$ in $H_0^{1-\delta}(\Omega)$ strongly for all $0 < \delta \leq 1$. Moreover, $u \in H_0^1(\Omega)$ and satisfies

$$\int_{-1}^{1} \nabla u \cdot \nabla v \, dx = \int_{-1}^{1} fv \, dx, \quad \forall v \in \mathcal{D}(\Omega)$$





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