A time adaptive multirate Dirichlet-Neumann waveform relaxation method for heterogeneous coupled heat equations

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Outline

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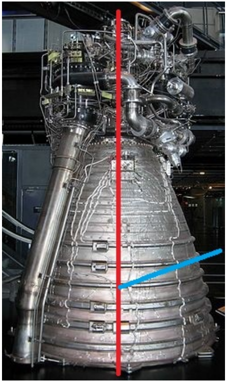
Applications of thermal FSI



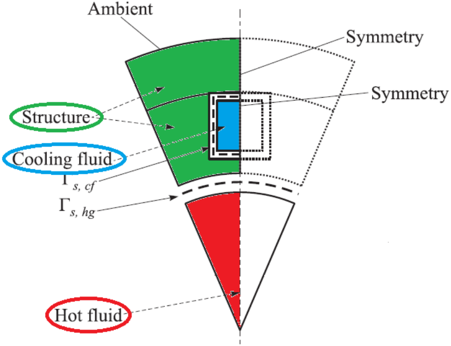
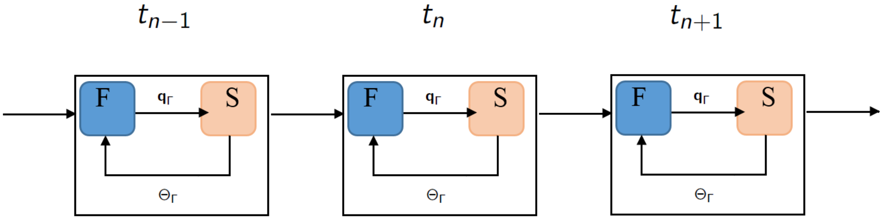


Figure: Left: Vulcain engine for Ariane 5. Right: Sketch of the cooling system; Pline, Wikimedia Commons

Thermal interaction between fluid and structure needs to be modelled

Dirichlet–Neumann coupling

- *Partitioned approach* for the solution of the coupled problem.
- *Fluid Model*: Compressible Navier–Stokes - FVM, DLR-TAU-Code
- *Structure Model*: Nonlinear heat equation - FEM, NATIVE inhouse code

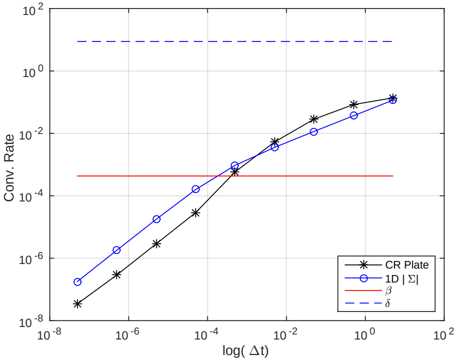


$$\Theta_\Gamma^{k+1} = \mathbf{S}(\mathbf{F}(\Theta_\Gamma^k)), \quad \text{Interface Temperature Iteration}$$

Outline

Limitations of the Dirichlet–Neumann method

Convergence rates FSI application



Limitations

- The subsolvers are sequential.
- Same time integration for both fields.

Use a different method!

More at: A. Monge, P. Birken, Computational Mechanics, 2017

Limitations of the Dirichlet–Neumann method

List of wishes

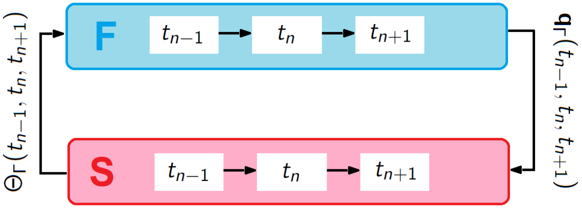
- Two independent time integration schemes.
- High order resolution (at least 2nd order).
- To be able to insert time adaptivity in the framework.

Option 1

Exchange fixed point iteration with time recursion

Option 2

Use a different domain decomposition method



Limitations of the Dirichlet–Neumann method

Option 1

Dirichlet-Neumann Waveform Relaxation (DNWR) algorithm

+ Computationally cheap
– Sequential method

Option 2

Neumann-Neumann Waveform Relaxation (NNWR) algorithm

+ Parallel method
– Computationally expensive

- *DNWR and NNWR introduced by Gander and Kwok 2016: Constant coefficients and one single time integration scheme.*
- *More about Option 2*:
  1. A. Monge, P. Birken, arXiv:1805.04336, submitted to SISC 18.
  2. A. Monge, P. Birken, Proceedings of 25th Domain Decomposition Conference, submitted 18.

Outline

Model Problem: Coupled heat equations

$$\alpha_m \frac{\partial u_m(\mathbf{x}, t)}{\partial t} - \nabla \cdot (\lambda_m \nabla u_m(\mathbf{x}, t)) = 0,$$

$$t \in [t_0, t_f], \quad \mathbf{x} \in \Omega_m \subset \mathbb{R}^d, \ \ m = 1, 2$$

$$u_m(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega_m \backslash \Gamma$$

$$u_1(\mathbf{x}, t) = u_2(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma$$

$$\lambda_2 \frac{\partial u_2(\mathbf{x}, t)}{\partial \mathbf{n}_2} = -\lambda_1 \frac{\partial u_1(\mathbf{x}, t)}{\partial \mathbf{n}_1}, \quad \mathbf{x} \in \Gamma$$

$$u_m(\mathbf{x}, 0) = g_m(\mathbf{x}) \quad \mathbf{x} \in \Omega_m$$

where $\alpha_m = \lambda_m / D_m$.

Dirichlet-Neumann waveform relaxation (DNWR)

$$
(D)\begin{cases}
\alpha_1 \frac{\partial u_1^{k+1}(\mathbf{x},t)}{\partial t} - \nabla \cdot (\lambda_1 \nabla u_1^{k+1}(\mathbf{x},t)) = 0, & \mathbf{x} \in \Omega_1, \\
u_1^{k+1}(\mathbf{x},t) = 0, & \mathbf{x} \in \partial\Omega_1 \backslash \Gamma, \\
u_1^{k+1}(\mathbf{x},t) = g^k(\mathbf{x},t), & \mathbf{x} \in \Gamma, \\
u_1^{k+1}(\mathbf{x},0) = u_1^0(\mathbf{x}), & \mathbf{x} \in \Omega_1.
\end{cases}
$$

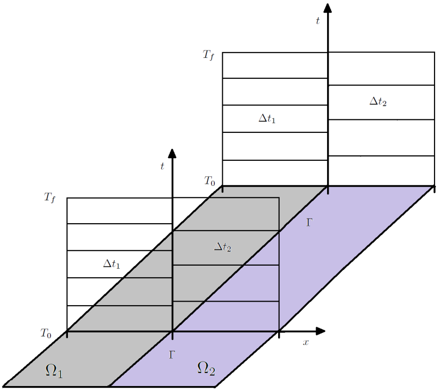
$$
(N)\begin{cases}
\alpha_2 \frac{\partial u_2^{k+1}(\mathbf{x},t)}{\partial t} - \nabla \cdot (\lambda_2 \nabla u_2^{k+1}(\mathbf{x},t)) = 0, & \mathbf{x} \in \Omega_2, \\
u_2^{k+1}(\mathbf{x},t) = 0, & \mathbf{x} \in \partial\Omega_2 \backslash \Gamma, \\
\lambda_2 \frac{\partial u_2^{k+1}(\mathbf{x},t)}{\partial \mathbf{n}_2} = -\lambda_1 \frac{\partial u_1^{k+1}(\mathbf{x},t)}{\partial \mathbf{n}_1}, & \mathbf{x} \in \Gamma, \\
u_1^{k+1}(\mathbf{x},0) = u_1^0(\mathbf{x}), & \mathbf{x} \in \Omega_1.
\end{cases}
$$

$$
(U) \quad g^{k+1}(\mathbf{x},t) = \Theta u_2^{k+1}(\mathbf{x},t) + (1-\Theta) g^k(\mathbf{x},t), \quad \mathbf{x} \in \Gamma.
$$

How to choose the relaxation parameter Θ properly?

Choices

- *Space discretization*: 1D and 2D finite elements
- *Time discretization*: Implicit Euler and SDIRK2
- *Matching space grid* at the interface, unknowns on interface
- *Nonmatching time grids* at the interface, linear interpolation



Relaxation parameter

Q: How to choose the relaxation parameter Θ?

A: Find Θ s.t. minimizes the spectral radius of Σ w.r.t $\mathbf{u}_\Gamma(T_f)$,

$$\mathbf{u}_\Gamma^{k+1,T_f} = \Sigma \mathbf{u}_\Gamma^{k,T_f} + \sum_{i=2}^{2} \left(\psi^{k+1,\tilde{\tau}_i} + \psi^{k,\tau_i} \right).$$

Procedure to find the iteration matrix

1. Isolate $\mathbf{u}_I^{(m),k+1,T_f}$ from the Dirichlet solver.
2. Isolate $\mathbf{u}_I^{(2),k+1,T_f}$ from the Neumann solver.
3. Use the update step to get Σ w.r.t $\mathbf{u}_\Gamma(T_f)$.

Iteration Matrix

After lengthly derivations one gets,

$$\Sigma = \mathbf{I} - \Theta \left(\mathbf{I} + {\mathbf{S}^{(2)}}^{-1} \mathbf{S}^{(1)} \right),$$

with

$$\mathbf{S}^{(m)} = (\mathbf{M}_{\Gamma\Gamma}^{(m)} + \Delta t \mathbf{A}_{\Gamma\Gamma}^{(m)}) - (\mathbf{M}_{\Gamma I}^{(m)} + \Delta t \mathbf{A}_{\Gamma I}^{(m)})(\mathbf{M}_m + \Delta t \mathbf{A}_m)^{-1}(\mathbf{M}_{I\Gamma}^{(m)} + \Delta t \mathbf{A}_{I\Gamma}^{(m)})$$

for $m = 1, 2$.

Iteration Matrix

A closer look at the iteration matrix Σ:

$$\mathbf{S}^{(m)} = \mathbf{M}_{\Gamma\Gamma}^{(m)} - \Delta t\, \mathbf{A}_{\Gamma\Gamma}^{(m)} - \left(\mathbf{M}_{\Gamma I}^{(m)} - \Delta t \mathbf{A}_{\Gamma I}^{(m)}\right) \left(\mathbf{M}_m - \Delta t \mathbf{A}_m\right)^{-1} \left(\mathbf{M}_{I\Gamma}^{(m)} - \Delta t \mathbf{A}_{I\Gamma}^{(m)}\right)$$

- Problem: Matrices $\mathbf{M}_m + \Delta t \mathbf{A}_m$ are sparse but $(\mathbf{M}_m + \Delta t \mathbf{A}_m)^{-1}$ are dense.
- Solution: Use their eigendecompositions to compute the desired entries of the Toeplitz matrices: $(\mathbf{M}_m + \Delta t \mathbf{A}_m)^{-1} = \mathbf{V} \Lambda_m^{-1} \mathbf{V}$.

Optimal Relaxation Parameter

- *Space discretization*: 1D equidistant FE/FE.
- *Time integration*: nonmultirate Implicit Euler.

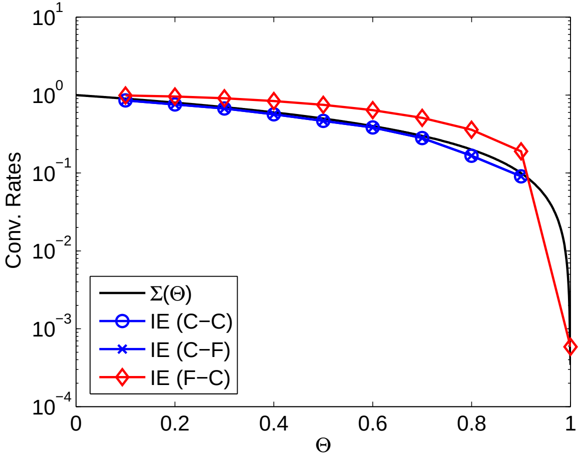
$$\Theta_{opt} = \left(1 + \frac{\mathbf{S}^{(1)}}{\mathbf{S}^{(2)}}\right)^{-1},$$

with

$$\mathbf{S}^{(m)} = (6\Delta x(\alpha_m \Delta x^2 + 3\lambda_m \Delta t) - (\alpha_m \Delta x^2 - 6\lambda_m \Delta t)^2 s_m).$$

Θ_{opt} gives the optimal parameter for any coupled materials!

Convergence Rates: Air-Water coupling



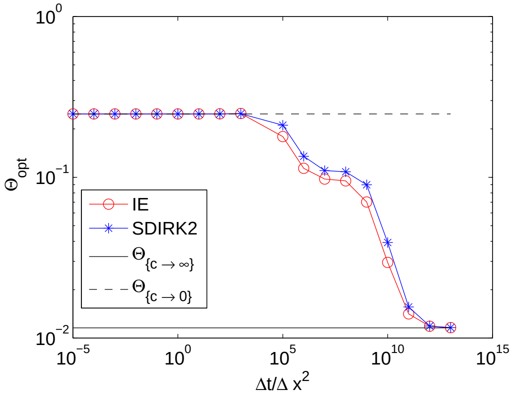
Θ_{opt} hits the optimal parameter.

Outline

1. Motivation
2. Limitations of the Dirichlet-Neumann method
3. A multirate approach
4. A time adaptive approach

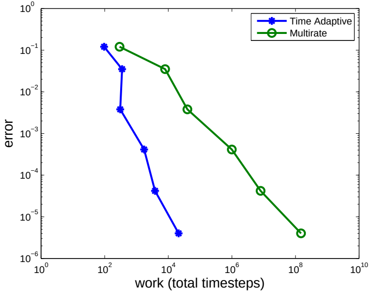
Choices for Time Adaptivity

- Add step size controllers on the Dirichlet and Neumann problems
- Problem: Θ_{opt} depends on α_m, λ_m, $m = 1, 2$, Δx and Δt
- Solution: Initial relaxation parameter $\Theta = (\Theta_{\{c \to 0\}} + \Theta_{\{c \to \infty\}})/2$
- Use Δt_1 and Δt_2 to update $\Theta = \Theta_{opt}(\Delta t_1, \Delta t_2)$.

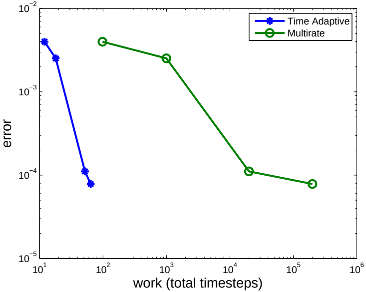


Time adaptive numerical results

Air-Steel 1D



Air-Water 2D



Time adaptive approach performs better than multirate approach.

Summary and Further Work

- Time adaptive multirate method for coupled parabolic problems (DNWR).
- We have performed a 1D analysis to find Θ_{opt}.
- Θ_{opt} is dependent on Δt, Δx, λ_m and α_m, $m = 1, 2$.
- Θ_{opt} works for 2D, multirate and **time adaptivity**.

- Apply to FSI test cases.
- Load balancing.

More at: P. Birken, S22 Scientific Computing, February 20th, 5:10-5:30.

Thank you!