Remarks on Long Time Versus Steady State Optimal Control

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Abstract Control problems play a key role in many fields of Engineering, Economics and Sciences. This applies, in particular, to climate sciences where, often times, relevant problems are formulated in long time scales. The problem of the possible asymptotic simplification (as time tends to infinity) then emerges naturally. More precisely, assuming, for instance, that the free dynamics under consideration stabilizes towards a steady state solution, the following question arises: Do time averages of optimal controls and trajectories converge to the steady optimal controls and states as the time-horizon tends to infinity?

This question is very closely related to the so-called *turnpike* property stating that, often times, the optimal trajectory joining two points that are far apart, consists in, departing from the point of origin, rapidly getting close to the steady-state (the turnpike) to stay there most of the time, to quit it only very close to the final destination and time.

In this paper we focus on the semilinear heat equation. We prove some partial results and enumerate a number of interesting topics of future research, indicating also some connections with shape design and inverse problems theory.

Keywords Semilinear heat equations • Optimal control problems • Long time behavior • Steady states • Controllability • Observability • Turnpike property

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1 Introduction

In this paper, we address the question of the limiting behavior of optimal control problems as the time-horizon tends to infinity for semilinear heat equations. Although the question makes sense and is relevant for a much wider class of problems, we focus on this particular case to simplify the presentation, and to underline some of the main difficulties one encounters when addressing these problems and the fundamental tools needed in their analysis.

The motivation to consider this kind of problems is clear in many contexts but in particular in climate sciences where problems are naturally formulated in long time intervals. This is for instance the case in paleoclimatology (study of past climates) (see, for instance, [14]) where the problem of the inversion of past climates is addressed.

Note however that the models arising in climate sciences are extremely complex. Thus, rigorously speaking, although the topic addressed here can be of relevance in that field, the techniques we develop cannot be directly applied and will require significant further developments.

Sustainable economic development is another area in which these issues arises, playing a central role (see [7]).

Most often, the existing Partial Differential Equations (PDE) Control Theory, based on optimization and minimization of cost functionals, and the characterization of optimal controls through the corresponding optimality systems and adjoint methods, does not distinguish between short and long time horizons.

Here we are specifically interested in long-time horizon control problems and the possibility that optimal trajectories and controls simplify towards those of the corresponding steady state model.

In practice, in long time-horizons, the effective computation of the control can be very expensive since it requires iterative methods to solve the coupled optimality system combining the forward controlled state equation and the backward adjoint one.

It is then natural to look for some shortcuts. This makes sense, in particular, when, as it occurs often times in applications, the free dynamics associated to the state equation presents some property of asymptotic simplification: convergence towards a steady state solution, stabilization around a periodic trajectory or a self-similar solution, etc. When that occurs it is natural to investigate whether the optimal control and trajectories converge towards the corresponding simplified optimal control and states.

In other words, the question we are discussing consists in analyzing whether the processes of long time asymptotics and control commute.

This problem, as pointed out in [15], is related to the so-called *turnpike property*, mainly motivated by economic theories (see [22] and references therein). We also refer to the more recent paper on time-discrete finite-dimensional systems [10] and [9], the seminal continuous time paper [2] and the more recent and systematic one [20].

The question of whether the control process commutes with some qualitative aspect of PDE models has been analyzed in other contexts too. For instance, it is well known that the question is very subtle when dealing with numerical approximation methods. More precisely, convergent numerical algorithms for the free dynamics do not necessarily lead to convergent numerical methods for control problems, especially, when one is dealing with the more demanding problem of controllability (see [23]). This is so, in particular, when the numerical scheme is not stable enough to avoid the emergence of spurious numerical high frequency solutions. A certain amount of dissipativity of the numerical schemes is required and, as we shall see, the same can be said when dealing with the long-time horizon control problems.

The issue of long time versus steady state control is also relevant in shape design. In particular, in the field of aeronautics, most designs are computed based on steady state models and, although it is assumed or understood that these steady optimal shapes are close to the optimal time-evolving ones, there are not results justifying such a fact rigorously, especially for the relevant models in fluid mechanics such as Navier-Stokes or Euler equations (see [12]). Similar questions also arise in the context of inverse problems (see [11], section 9.4 and [8]).

In our earlier paper [15] we addressed the problem of long time horizon versus steady state control in the linear setting. There we analyzed in detail both the finite-dimensional case, and the paradigmatic PDE models, namely, the heat and the wave equations, and proved that, under suitable controllability conditions (see [24] for a general presentation of the theory of controllability for PDE), optimal controls and controlled trajectories converge, as the time horizon tends to infinity, towards the stationary optimal controls and states with an exponential rate induced by the stabilizing Riccati feedback operator.

The analysis in [15], however, is of purely linear nature, based on the properties of the optimality system characterizing the optimal controls and states through the coupling with the adjoint system.

But the problem makes sense in the nonlinear context too.

In this paper we briefly discuss the issue for the semilinear heat equation. We first present the main tools developed in [15] in the linear case, in order to later employ them to get results of local nature for the semilinear heat equation. We then consider the simpler problem of time-independent controls showing how simpler and more classical Γ-convergence arguments allow to handle it. This late result, although much simpler to be achieved, is also relevant from the point of view of applications, where the applied controls can be time-independent as well.

We close the paper formulating a number of open problems and directions of future research.

2 Preliminaries on the Linear Heat Equation

Let us briefly recall the main results obtained in [15] in the specific case of the following controlled heat equation.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the heat equation with Dirichlet boundary conditions and an applied control:

$$\begin{cases} y_t - \Delta y = u\chi_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0) = y_0 \in L^2(\Omega)\ . \end{cases} \tag{1}$$

Then, consider the following associated control problem

$$\min\ J^T(u) = \frac{1}{2}\int_0^T \left[|u(t)|^2_{L^2(\omega)} + |y(t)-z|^2_{L^2(\omega_0)}\right] dt \tag{2}$$

where $u \in L^2(0,T;L^2(\omega))$ and y solves (1), and $z \in L^2(\omega_0)$ is a given observation. Here ω and ω_0 are two open subsets of Ω and χ_ω stands for the characteristic function of the set ω where the control is being applied, while ω_0 denotes the subdomain where the tracking term of the cost functional is active.

We also consider the stationary version of the state equation:

$$\begin{cases} -\Delta y = u\chi_\omega & \text{in } \Omega \\ y = 0 & \text{on } \Omega, \end{cases} \tag{3}$$

and the corresponding problem of minimizing the functional

$$\min\ J(u) = \frac{1}{2}\left[|u|^2_{L^2(\omega)} + |y-z|^2_{L^2(\omega_0)}\right]\ . \tag{4}$$

Let us now consider the control problems (2) and (4) and the corresponding optimal solutions (u^T, y^T) and $(\bar{u}, \bar{y})$, respectively. Then, according to the results in [15], there exists $\mu > 0$ such that

$$\|y^T(t) - \bar{y}\|_{L^2(\Omega)} + \|u^T(t) - \bar{u}\|_{L^2(\Omega)} \le K(e^{-\mu t} + e^{-\mu(T-t)}) \tag{5}$$

for every $t \in [0,T]$. Let us now sketch the main steps of the proof that will give us a precise idea of the constants K, μ involved in this estimate.

The optimality systems for the time evolution and steady state problems read as follows, respectively:

$$\begin{cases} y^T_t - \Delta y^T = -q^T\chi_\omega & \text{in } \Omega\times(0,T) \\ y^T = 0 & \text{on } \partial\Omega\times(0,T) \\ y^T(0) = y_0 & \\ -q^T_t - \Delta q^T = (y^T - z)\chi_{\omega_0} & \text{in } \Omega\times(0,T) \\ q^T = 0 & \text{on } \partial\Omega\times(0,T) \\ q^T(T) = 0, & \end{cases} \tag{6}$$

and

$$\begin{cases} -\Delta \bar{y} = -\bar{q}\chi_\omega & \text{in } \Omega \\ \bar{y} = 0 & \text{on } \partial\Omega \\ -\Delta \bar{q} = (\bar{y} - z)\chi_{\omega_0} & \text{in } \Omega \\ \bar{q} = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

In the reference case where $z = 0$, we define a linear bounded operator in $L^2(\Omega)$ as

$$\mathscr{E}(T)y_0 := q^T(0)\ .$$

It turns out that $\mathscr{E}(t)$ is a positive operator which is increasing and uniformly bounded with respect to t, and we have

$$\|\mathscr{E}(t) - \hat{E}\|_{\mathscr{L}(L^2(\Omega),L^2(\Omega))} \le Ce^{-\mu t}, \tag{8}$$

for some $C > 0$ and $\mu > 0$, where $\hat{E}$ is the corresponding operator for the infinite horizon control problem with $z = 0$. Namely,

$$\hat{E}y_0 := \hat{q}(0)$$

where, in this case, the pair $(\hat{y}, \hat{q})$ solves the optimality system in infinite time associated with $z = 0$:

$$\begin{cases} \hat{y}_t - \Delta \hat{y} = -\hat{p}\chi_\omega & \text{in } \Omega \times (0,\infty) \\ \hat{y} = 0 & \text{on } \partial\Omega \times (0,\infty) \\ \hat{y}(0) = y_0 & \\ -\hat{q}_t - \Delta \hat{q} = \hat{y}\chi_{\omega_0} & \text{in } \Omega \times (0,\infty) \\ \hat{q} = 0 & \text{on } \partial\Omega \times (0,\infty) \\ ||\hat{q}(t)||_{L^2(\Omega)} \to 0 & \text{as } t \to \infty. \end{cases} \tag{9}$$

Notice that, by time invariance, we have $\hat{q}(t) = \hat{E}\hat{y}(t)$, and the first equation in (9) defines an operator $M := -\Delta + \hat{E}\,\chi_\omega$ which is exponentially stable, providing the rate μ which appears in (8), as well as in (5).

Once the operators $\mathscr{E}(t)$ and $\hat{E}$ are defined as above in terms of the reference problem where $z = 0$, the adjoint state of the general system (6) can be represented by the affine feedback law

$$q^T(t) - \bar{q} = \mathscr{E}(T - t)(y^T(t) - \bar{y}) + h^T(t) \tag{10}$$

where h^T solves

$$\begin{cases} -h_t^T + (-\Delta + \mathscr{E}(T-t)\chi_\omega)h^T = 0 & \text{in } \Omega \times (0,T) \\ h^T = 0 & \text{on } \partial\Omega \times (0,T) \\ h^T(T) = -\bar{q}. & . \end{cases} \tag{11}$$

In some sense, h^T is a kind of corrector taking care of the final cost at time T; and the equation satisfied by h^T can be deduced from equality (10) using the Riccati equation satisfied by $\mathscr{E}(t)$ and the optimality system (6). Alternatively, instead of using the Riccati equation, one can define h^T as a solution of (11) and verify a posteriori that equality (10) holds in a weak sense

$$\int_\Omega (q^T(t) - \bar{q})\varphi\, dx = \int_\Omega (y^T(t) - \bar{y})[\mathscr{E}(T-t)\varphi]\, dx + \int_\Omega h^T(t)\, \varphi\, dx$$

using properly the definition of $\mathscr{E}(t)$.

The corrector h^T can be estimated from (8) and the exponential stability of $-\Delta + \hat{E}\,\chi_\omega$; in fact, one can prove that

$$||h^T(t)||_{L^2(\Omega)} \le C^* \|\bar{q}\|_{L^2(\Omega)} e^{-\mu(T-t)}. \tag{12}$$

Once this structure is observed, the system (6) can be uncoupled by writing that the optimal trajectory y^T solves

$$y_t^T - \Delta y^T = -q^T \chi_\omega = -\bar{q}\chi_\omega - \mathscr{E}(T-t)\chi_\omega(y^T(t) - \bar{y}) - h^T\chi_\omega$$

which implies

$$(y^T(t) - \bar{y})_t - \Delta(y^T(t) - \bar{y}) = -\mathscr{E}(T-t)\chi_\omega(y^T(t) - \bar{y}) - h^T\chi_\omega\ .$$

The exponential proximity property (5) is now a straightforward consequence of (8), (12) and the decay of the stabilized dynamics. In fact, we have

$$(y^T(t) - \bar{y})_t + [-\Delta + \hat{E}\chi_\omega](y^T(t) - \bar{y}) = (\hat{E} - \mathscr{E}(T-t))\chi_\omega(y^T(t) - \bar{y}) - h^T\chi_\omega$$

which implies estimate (5) in the more precise form

$$\|y^T(t) - \bar{y}\|_{L^2(\Omega)} + \|u^T(t) - \bar{u}\|_{L^2(\Omega)} \le \tilde{K}(||y_0 - \bar{y}||_{L^2(\Omega)} e^{-\mu t} + \|\bar{q}\| e^{-\mu(T-t)})\ , \tag{13}$$

for every $t \in [0, T]$. Here the constant $\tilde{K}$ is independent of the choice of the initial data and of the target z.

Remark 1 Let us stress that (13) would take a more symmetric form if the adjoint state p^T had a different prescribed data at time $t = T$. This is the case if we consider a cost functional with an additional final pay-off such as

$$J^T(u) = \frac{1}{2}\int_0^T \left[|u(t)|^2_{L^2(\omega)} + |y(t) - z|^2_{L^2(\omega_0)}\right] dt + q_0 \cdot y(T)$$

for some $q_0 \in L^2(\Omega)$. In this case, the adjoint state q^T must satisfy $q^T(T) = q_0$; the above proof applies without changes except that now the corrector term h^T will take a different final condition (equal to $q_0 - \bar{q}$) and the estimate (13) would become

$$\|y^T(t) - \bar{y}\|_{L^2(\Omega)} + \|u^T(t) - \bar{u}\|_{L^2(\Omega)} \leq \tilde{K}(||y_0 - \bar{y}||_{L^2(\Omega)}e^{-\mu t} + \|q_0 - \bar{q}\|e^{-\mu(T-t)}) ,$$

for every $t \in [0, T]$.

The above remark points out that the exponential turnpike property is somehow symmetric with respect to what happens at $t = 0$ and $t = T$, see also a more general discussion of this fact in [20] for the finite-d case.

3 Local Results for the Semilinear Heat Equation

This section is divided in two parts. In the first one we formulate the problem under consideration and prove a turnpike property for the system of optimality under the condition that the initial and final states are close enough to the stationary primal and dual state, respectively. In the second subsection, we show, as an example, that this result applies at least in the case that the target and the initial datum are small enough. We stress, however, that this solution of the system of optimality is not guaranteed to be a minimizer for the time-dependent optimal control problem, unless one proves that the functional under consideration is (locally) convex. A similar strategy was used in [4] to analyze the optima of a nonlinear control problem arising in mean field games theory; unfortunately, for the semilinear heat equation considered below, the convexity of the functional is not clear and the equivalence between minima and solutions of the optimality system requires further investigation.

3.1 The System of Optimality

It is natural to consider the same issues of the previous section for the following semilinear heat equation:

$$\begin{cases} y_t - \Delta y + f(y) = u\chi_\omega & \text{in } \Omega \times (0,T) \\ y = 0 & \text{on } \partial\Omega \times (0,T) \\ y(0) = y_0 \in L^2(\Omega)\ , \end{cases} \tag{14}$$

f being a C^1 nondecreasing function.

The semilinear problem (14) is well-posed. More precisely, given $y_0 \in L^2(\Omega)$ and $u \in L^2(\omega \times (0,T))$, there exists a unique solution

$$y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).$$

We consider the optimal control problem:

$$\min\ \{J^T(u) := \frac{1}{2}\int_0^T |y(t) - z|^2_{L^2(\omega_0)} dt + \frac{1}{2}\int_0^T |u|^2_{L^2(\omega)} dt + q_0 \cdot y(T)\}\ , \tag{15}$$

where $z \in L^2(\omega_0)$ and $q_0 \in L^2(\Omega)$. In the stationary version the state equation is

$$\begin{cases} -\Delta y + f(y) = u\chi_\omega & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega\ , \end{cases} \tag{16}$$

together with the corresponding functional

$$\min\ \{J(u) := \frac{1}{2}\left[|y - z|^2_{L^2(\omega_0)} + |u|^2_{L^2(\omega)}\right]\}\ . \tag{17}$$

In both cases it is easy to see that the optima are achieved and we can easily write the corresponding optimality systems. They read as

$$\begin{cases} y_t^T - \Delta y^T + f(y^T) = -q^T\chi_\omega & \text{in } \Omega \times (0,T) \\ y^T = 0 & \text{on } \partial\Omega \times (0,T) \\ y^T(0) = y_0 & \\ -q_t^T - \Delta q^T + f'(y^T)q^T = (y^T - z)\chi_{\omega_0} & \text{in } \Omega \times (0,T) \\ q^T = 0 & \text{on } \partial\Omega \times (0,T) \\ q^T(T) = q_0, & \end{cases} \tag{18}$$

and

$$\begin{cases} -\Delta\bar{y} + f(\bar{y}) = -\bar{q}\chi_{\omega} & \text{in } \Omega \\ \bar{y} = 0 & \text{on } \partial\Omega \\ -\Delta\bar{q} + f'(\bar{y})\bar{q} = (\bar{y} - z)\chi_{\omega_0} & \text{in } \Omega \\ \bar{q} = 0 & \text{on } \partial\Omega. \end{cases} \tag{19}$$

But, due to the nonlinearity of the problems under consideration, the methods of the previous section cannot be applied directly. Hence we develop a local analysis around a given steady state optimal control.

Thus, let $(\bar{y}, \bar{u})$ be an optimal pair for the steady-state problem and introduce the change of variables:

$$\eta = y^T - \bar{y}; \quad \varphi = q^T - \bar{q}.$$

Then, (η, φ) satisfy:

$$\begin{cases} \eta_t - \Delta\eta + F(\eta) = -\varphi\chi_{\omega} & \text{in } (0,T) \times \Omega \\ \eta = 0 & \text{on } (0,T) \times \partial\Omega \\ \eta(0) = \eta_0 & \text{in } \Omega \\ -\varphi_t - \Delta\varphi + \Phi(\eta, \varphi) = \eta\chi_{\omega_0} & \text{in } (0,T) \times \Omega \\ \bar{q} = 0 & \text{on } (0,T) \times \partial\Omega \\ \varphi(T) = \varphi_0 & \text{in } \Omega \end{cases} \tag{20}$$

where

$$\eta_0 = y_0 - \bar{y}\,, \qquad \varphi_0 = q_0 - \bar{q}$$

and

$$F(\bar{y}, \eta) = f(\bar{y} + \eta) - f(\bar{y})\,,$$
$$\Phi(\eta, \varphi) = f'(\bar{y} + \eta)(\bar{q} + \varphi) - f'(\bar{y})\bar{q}$$

Our aim is to build a pair (u^T, y^T) fulfilling the turnpike property, i.e. such that $(u^T, y^T) \sim (\bar{u}, \bar{y})$ in the sense of the previous section. In the (η, φ) variables, this is equivalent to finding $(\eta, \varphi) \sim (0, 0)$. It is therefore natural to look at the linearization of the above functions F, Φ near $(\eta = 0, \varphi = 0)$ and at the corresponding linearised optimality system:

$$\begin{cases} \eta_t - \Delta\eta + f'(\bar{y})\eta = -\varphi\chi_\omega & \text{in } (0,T)\times\Omega \\ \eta = 0 & \text{on } (0,T)\times\partial\Omega \\ \eta(0) = \eta_0 & \text{in } \Omega \\ -\varphi_t - \Delta\varphi + f'(\bar{y})\varphi = \eta\chi_{\omega_0} - f''(\bar{y})\bar{q}\,\eta & \text{in } (0,T)\times\Omega \\ \bar{q} = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T) = 0 & \text{in } \Omega. \end{cases} \tag{21}$$

It defines a bounded (linear) feedback in $L^2(\Omega)$ as

$$\mathscr{E}(T)\eta_0 = \varphi(0).$$

Let us assume that for some $C > 0$ and $\mu > 0$, we have

$$\begin{aligned} &\|\mathscr{E}(t) - \hat{E}\|_{\mathscr{L}(L^2(\Omega),L^2(\Omega))} \le Ce^{-\mu t}, \\ &\|e^{-tM}\|_{\mathscr{L}(L^2(\Omega),L^2(\Omega))} \le e^{-\mu t}, \quad M := -\Delta + f'(\bar{y}) + \hat{E}\,\chi_\omega \end{aligned} \tag{22}$$

where $\hat{E}$ is the corresponding operator for the problem in $(0,\infty)$.

Using this exponential stability property of the linearized system, we will be able to prove the following statement. For simplicity, we assume that $f \in C^3$ and the dimension $n \le 3$.

Theorem 1 *Assume that system (21) satisfies the exponential turnpike property. Then, there exists some $\varepsilon > 0$ such that for every y_0, q_0 with*

$$||y_0 - \bar{y}||_{L^\infty(\Omega)} + \|q_0 - \bar{q}\|_{L^\infty(\Omega)} \le \varepsilon,$$

there exists a solution of the optimality system

$$\begin{cases} y_t^T - \Delta y^T + f(y^T) = -q^T\chi_\omega & \textit{in } \Omega\times(0,T) \\ y^T = 0 & \textit{on } \partial\Omega\times(0,T) \\ y^T(0) = y_0 \\ -q_t^T - \Delta q^T + f'(y^T)q^T = (y^T - z)\chi_{\omega_0} & \textit{in } \Omega\times(0,T) \\ q^T = 0 & \textit{on } \partial\Omega\times(0,T) \\ q^T(T) = q_0, \end{cases} \tag{23}$$

which satisfies

$$||y^T(t) - \bar{y}||_{L^\infty(\Omega)} + ||q^T(t) - \bar{q}||_{L^\infty(\Omega)} \le K(e^{-\mu t} + e^{-\mu(T-t)}),\ \forall 0 < t < T\,. \tag{24}$$

As mentioned above, the turnpike property is established for solutions of the optimality system. Thus, the result does not have the nature we expect, in other words, it does not really apply to the minimizers of the functional under consideration. Dealing with minimizers requires further analysis as the one developed in [20] in the finite-dimensional case.

As we shall see in the following section, this theorem applies at least when the target z is small enough, in which case the steady state problem has a unique minimum which is also small and system (21) satisfies the exponential turnpike property. In this special case, one could expect the solution of the parabolic optimality system to be also unique and to coincide with the optimal state and control.

Proof We look at (20) as a perturbation of the linear system (21) and we aim at finding a solution through a fixed point argument. Namely, for $\hat{\eta}, \hat{\varphi}$ given, we set

$$R_1(\hat{\eta}) := -\left\{f(\bar{y}+\hat{\eta}) - f(\bar{y}) - f'(\bar{y})\hat{\eta}\right\} ,$$
$$R_2(\hat{\eta},\hat{\varphi}) := -\bar{q}\left\{f'(\bar{y}+\hat{\eta}) - f'(\bar{y}) - f''(\bar{y})\hat{\eta}\right\} + [f'(\bar{y}) - f'(\bar{y}+\hat{\eta})]\hat{\varphi}$$

and we define the operator

$$(\eta,\varphi) := \mathscr{K}(\hat{\eta},\hat{\varphi})$$

where (η,φ) solve

$$\begin{cases} \eta_t - \Delta\eta + f'(\bar{y})\eta = -\varphi\chi_\omega + R_1(\hat{\eta}) & \text{in } (0,T)\times\Omega \\ \eta = 0 & \text{on } (0,T)\times\partial\Omega \\ \eta(0) = \eta_0 & \text{in } \Omega \\ -\varphi_t - \Delta\varphi + f'(\bar{y})\varphi = \eta\chi_{\omega_0} - f''(\bar{y})\bar{q}\,\eta + R_2(\hat{\eta},\hat{\varphi}) & \text{in } (0,T)\times\Omega \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T) = \varphi_0 & \text{in } \Omega. \end{cases} \tag{25}$$

with $\eta_0 = y_0 - \bar{y}$, $\varphi_0 = q_0 - \bar{q}$.

Notice that a fixed point would solve the system (20), hence $y^T = \bar{y} + \eta$ and $q^T = \bar{q} + \varphi$ provide a solution of (23).

Assume that $(\hat{\eta},\hat{\varphi}) \in X$, where

$$X = \left\{(\eta,\varphi)\ :\ \|\eta(t)\|_\infty + \|\varphi(t)\|_\infty \le M(e^{-\mu t} + e^{-\mu(T-t)}) \quad \forall\, t\in[0,T]\right\}$$

for some $M \le 1$. Notice that

$$\begin{aligned} \|R_1(\hat{\eta})(t)\|_2 &\le C\|R_1(\hat{\eta})(t)\|_\infty \le c_0\, M^2(e^{-2\mu t} + e^{-2\mu(T-t)}) , \\ \|R_2(\hat{\eta},\hat{\varphi})(t)\|_2 &\le C\|R_2(\hat{\eta},\hat{\varphi})(t)\|_\infty \le c_1\, M^2(e^{-2\mu t} + e^{-2\mu(T-t)}) \end{aligned} \tag{26}$$

where c_0, c_1 depend on $\|f\|_{C^3[-\|\bar{y}\|_\infty - 1, \|\bar{y}\|_\infty + 1]}$ and on $\|\bar{q}\|_\infty$.

We first remark the following: by defining

$$h^T := \varphi - \mathscr{E}(T-t)\eta$$

then h^T solves the problem

$$\begin{cases} -h^T_t - \Delta h^T + f'(\bar{y})h^T + \mathscr{E}(T-t)\chi_\omega\, h^T = \mathscr{E}(T-t)R_1(\hat{\eta}) + R_2(\hat{\eta},\hat{\varphi}) & \text{in } (0,T)\times\Omega \\ h^T = 0 & \text{on } (0,T)\times\partial\Omega \\ h^T(T) = \varphi_0 & \text{in } \Omega. \end{cases}$$

We estimate h^T as in Sect. 2. In particular, we have

$$h^T(t) = e^{-M(T-t)}(\varphi_0) + \int_t^T e^{M(t-s)}[(\hat{E} - \mathscr{E}(T-s))\chi_\omega h^T(s)ds$$
$$+ \int_t^T e^{M(t-s)}[\mathscr{E}(T-s)R_1(\hat{\eta}) + R_2(\hat{\eta},\hat{\varphi})]ds$$

where $M := -\Delta + f'(\bar{y}) + \hat{E}\chi_\omega$. Since M is exponentially stable (with rate μ) and (22) holds, and by means of (26), we get

$$\|h^T(t)\|_2 \le e^{-\mu(T-t)}\|\varphi_0\|_2 + \int_t^T e^{\mu(t-s)}e^{-\mu(T-s)}\|h^T(s)\|_2 ds$$
$$+ c\,M^2 \int_t^T e^{\mu(t-s)}[e^{-2\mu(T-s)} + e^{-2\mu s}]ds\ ,$$

hence

$$\|h^T(t)\|_2 \le e^{-\mu(T-t)}[\|\varphi_0\|_2 + cM^2] + c\,M^2 e^{-2\mu t} + e^{-\mu(T-t)} \int_t^T \|h^T(s)\|_2 ds\ .$$

This implies that

$$\|h^T(t)\|_2 \le c\,[\|\varphi_0\|_2 + c\,M^2]e^{-\mu(T-t)} + c\,M^2\, e^{-2\mu t}\ .$$

Coming back to system (25), the first equation now reads as

$$\eta_t - \Delta\eta + f'(\bar{y})\eta + \hat{E}\chi_\omega\eta = -(\mathscr{E}(T-t) - \hat{E})\chi_\omega\eta + R_1(\hat{\eta}) - h^T\chi_\omega$$

so that

$$\eta(t) = e^{-Mt}\eta_0 - \int_0^t e^{-M(t-s)}(\mathscr{E}(T-s) - \hat{E})\chi_\omega \eta(s)ds + \int_0^t e^{-M(t-s)}[R_1(\hat{\eta})(s) - h^T(s)\chi_\omega]ds \,.$$

On account of (22), (26) and the estimate on h we get

$$\|\eta(t)\|_2 \le e^{-\mu t}\|\eta_0\|_2 + \int_0^t e^{-\mu(t-s)}e^{-\mu(T-s)}\|\eta(s)\|_2 ds$$
$$+c\int_0^t e^{-\mu(t-s)}\left\{[cM^2(e^{-2\mu(T-s)} + e^{-2\mu s})] + [\|\varphi_0\|_2 + cM^2]e^{-\mu(T-s)}\right\} ds \,.$$

We apply Gronwall lemma to conclude that

$$\|\eta^T(t)\|_2 \le c\,[\|\eta_0\|_2 + \|\varphi_0\|_2 + c\,M^2](e^{-\mu(T-t)} + e^{-\mu t}) \,.$$

Now, from the equality $\varphi = \mathscr{E}(T-t)\eta + h^T$, we deduce that a similar estimate holds for φ, namely,

$$\|\varphi^T(t)\|_2 \le c\,[\|\eta_0\|_2 + \|\varphi_0\|_2 + c\,M^2](e^{-\mu(T-t)} + e^{-\mu t}) \,.$$

We go back again on the first equation, observing that

$$\eta_t - \varDelta\eta = \chi$$

where $\chi := -f'(\bar{y})\eta - \varphi\chi_\omega + R_1(\hat{\eta})$ satisfies

$$\|\chi(t)\|_2 \le c\left\{[\|\eta_0\|_2 + \|\varphi_0\|_2 + c\,M^2](e^{-\mu(T-t)} + e^{-\mu t})\right\} \tag{27}$$

Since in dimension $n \le 3$ we have $2 > \frac{n}{2}$, this implies an estimate for η in L^∞, because, as is well-known, the heat semigroup yields

$$\|\eta(t)\|_\infty \le c\left\{\|\chi\|_{L^\infty((t-1,t);L^r(\Omega))} + \|\eta\|_{L^2((t-1,t)\times\Omega)}\right\}$$

for $r > \frac{n}{2}$. Therefore, we conclude that

$$\|\eta(t)\|_\infty \le c\left\{[\|\eta_0\|_2 + [\|\varphi_0\|_2 + c\,M^2](e^{-\mu(T-t)} + e^{-\mu t})\right\} \,,$$

for $t \ge 1$ and, if $\eta_0 \in L^\infty(\Omega)$, the estimate extends to [0, 1] as well (with a constant now depending on $\|\eta_0\|_\infty$). Similarly we reason for φ; finally, we proved that

$$\|\eta(t)\|_\infty + \|\varphi(t)\|_\infty \le c[\|\eta_0\|_\infty + \|\varphi_0\|_\infty + cM^2](e^{-\mu(T-t)} + e^{-\mu t}) \,.$$

Choose now some $M \le 1$ such that $cM^2 \le \frac{M}{2}$; then, if $\|\eta_0\|_\infty + \|\varphi_0\|_\infty$ are suitably bounded, we have

$$c[\|\eta_0\|_\infty + \|\varphi_0\|_\infty + cM^2] \le M$$

so that X becomes an invariant convex subset of $L^2(0,T;L^2(\Omega))$. Continuity and compactness of the operator $\mathscr{K}$ are easy to prove, which allows us to conclude the existence of a fixed point (η,φ) which is therefore a solution to (18). □

3.2 Small Solutions

As mentioned above, in this section we consider the particular case where both the target z and the initial datum y_0 are small in $L^2(\Omega)$. In this case, one can prove that the optimal pair for the steady-state problem is unique and the linearized optimality system exponentially stable. The uniqueness of the optima could be expected for the evolution problem as well. This would yield the uniqueness of the solution of the optimality system and would allow to apply the previous result. But the arguments of the steady-state case only allow proving the smallness of the time-averages of the optimal pairs and this is not sufficient by now to prove the strict convexity of the functional and to conclude.

We consider first the elliptic problem:

$$\min \left\{J(u) := \frac{1}{2}|y-z|^2_{L^2(\omega_0)} + \frac{1}{2}|u|^2_{L^2(\omega)}\right\}, \tag{28}$$

associated to

$$\begin{cases} -\Delta y + f(y) = u\chi_\omega & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{29}$$

Obviously

$$I = \min J \le J(0) = \frac{1}{2}|z|^2_{L^2(\omega_0)}.$$

Consequently, any minimizer $(\bar{u},\bar{y})$ satisfies

$$|\bar{y}-z|^2_{L^2(\omega_0)} + |\bar{u}|^2_{L^2(\omega)} \le |z|^2_{L^2(\omega_0)}.$$

Now, assuming that the target z is small enough, this ensures necessarily the smallness of the optimal control $\bar{u}$ and of the optimal state $\bar{y}$, that consequently live in a ball B in $L^2(\omega)\times L^2(\omega_0)$. The radius of this ball, centered at the origin, can be made small as the norm of z in $L^2(\Omega)$ tends to zero. Let us now explain why,

z being small, the functional J is strictly convex in the relevant ball B. In view of Proposition 2.3 in [5] (see also [6] and [21]) we have

$$J''(u)v_1v_2 = \int_{\omega_0} \eta_{v_1}\eta_{v_2}dx + \int_{\omega} v_1v_2dx - \int_{\Omega} f''(y)q\eta_{v_1}\eta_{v_2}, \tag{30}$$

where q is the adjoint state solution of

$$\begin{cases} -\Delta q + f'(y)q = (y-z)\chi_{\omega_0} & \text{in } \Omega \\ q = 0 & \text{on } \partial\Omega,\ j = 1,2. \end{cases} \tag{31}$$

and η_{v_1}, η_{v_2} are the linearized solutions in the direction of v_1 and v_2 respectively, i.e.

$$\begin{cases} -\Delta \eta_{v_j} + f'(y)\eta_{v_j} = v_j\chi_{\omega} & \text{in } \Omega \\ \eta_{v_j} = 0 & \text{on } \partial\Omega,\ j = 1,2. \end{cases} \tag{32}$$

Now, assuming that the target z is small enough in $L^2(\Omega)$, we can deduce that both y and q are small in $L^\infty(\Omega)$. Indeed, since $n \le 3$, the smallness for the control in $L^2(\omega)$ implies that the right-hand side of (29) is small in some $L^p(\Omega)$ with $p > \frac{n}{2}$. Since the nonlinearity is accretive (i.e. it has a good sign), the elliptic regularity implies the smallness of the state in $H_0^1(\Omega) \cap L^\infty(\Omega)$. In turn, a similar property holds for q from (31), in view of the fact that y is bounded and $f'(y) \ge 0$. Therefore, since

$$\|\eta_v\|_{H_0^1(\Omega)} \le C||v||_{L^2(\omega)}$$

the last term in (30) can be absorbed in the second one thanks to the bound of y and the smallness of q in $L^\infty(\Omega)$, namely

$$J''(u)vv \ge \int_{\omega_0} \eta_v^2 dx + (1 - c\,\|q\|_\infty) \int_{\omega} v^2 dx\,.$$

This guarantees the uniqueness of the minimizer of J, but also the uniqueness of a critical point of J in the ball B. Accordingly, one can guarantee that the unique solution of the stationary optimality system on that ball is the minimizer $\bar{u}$. In addition, as shown above, $J''(\bar{u})$ turns out to be coercive, which implies that the linearized optimality system (21) satisfies the exponential turnpike property in this case and Theorem 1 can be applied producing a turnpike solution of the optimality system. Whether the uniqueness of the minimizer is true in the parabolic case and, consequently, if the turnpike property actually holds for the optima under smallness conditions on the initial datum and target is an interesting open problem.

4 Time Independent Controls by Γ-Convergence

Let us consider again the semilinear heat equation

$$\begin{cases} y_t - \Delta y + f(y) = u(x)\chi_\omega & \text{in } \Omega \times (0,T) \\ y = 0 & \text{on } (0,T) \times \partial\Omega \\ y(0) = y_0 \in L^2(\Omega)\ , \end{cases} \tag{33}$$

but this time with controls $u = u(x)$ independent of time.

We focus in the particular case:

$$\begin{cases} y_t - \Delta y + |y|^{p-1}y = u(x)\chi_\omega & \text{in } \Omega \times (0,T) \\ y = 0 & \text{on } (0,T) \times \partial\Omega \\ y(0) = y_0 \in L^2(\Omega)\ , \end{cases} \tag{34}$$

with $p > 1$. We now consider the optimal control problem:

$$\min\ J^T(u) = \frac{1}{2}\int_0^T |y(t) - z|^2_{L^2(\omega_0)} dt + \frac{T}{2}|u|^2_{L^2(\omega)}\ , \tag{35}$$

and the steady state version

$$\begin{cases} -\Delta y + |y|^{p-1}y = u\chi_\omega & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega\ , \end{cases} \tag{36}$$

together with the corresponding functional

$$\min\ J(u) = \frac{1}{2}\left[|u|^2_{L^2(\omega)} + |y - z|^2_{L^2(\omega_0)}\right]\ . \tag{37}$$

Employing Γ-convergence arguments and taking advantage of the fact that the controls under consideration are independent of t the following can be proved.

Theorem 2 *Let u^T be a family of optimal controls for (34) and (35), with $T \to \infty$. Then, this family is relatively compact in $L^2(\omega)$ and any accumulation point $\bar{u}$ as $T \to \infty$ is an optimal control for the steady state problem (36) and (37).*

Remark 2 Note that the uniqueness of the optimal control is not guaranteed nor for the time-dependent problem nor for the steady state one. This is due to the lack of convexity of the functionals under minimization which is derived from the nonlinear character of the state equations. Thus, the statement above necessarily refers to the accumulation points of the family u^T as T tends to infinity and its inclusion within the set of steady state controls.

Proof We proceed in several steps.

Step 1. Let I^T and I be the values of the minimizers for the time-dependent problem in $[0, T]$ and the steady state one. We claim that

$$\frac{I^T}{T} \leq I + O(T^{-1}).$$

To prove this first estimate on the comparison between I and I^T we take a minimizer $\bar{u}$ for I and plug it into the functional J^T. We have

$$\frac{J^T(\bar{u})}{T} - J(\bar{u}) = \frac{\int_0^T |y^T(t) - z|^2_{L^2(\omega_0)} dt}{2T} - \frac{1}{2}|\bar{y} - z|^2_{L^2(\omega_0)}. \tag{38}$$

Here $\bar{y}$ stands for the steady state solution associated to the optimal control $\bar{u}$ and y^T is the corresponding solution of the evolution problem in the interval $[0, T]$. Obviously, because of the monotone character of the nonlinearity, standard energy estimates lead to the following exponential convergence property:

$$||y^T(t) - \bar{y}||_{L^2(\Omega)} \leq \exp(-\lambda_1 t)||y_0 - \bar{y}||_{L^2(\Omega)}, \tag{39}$$

λ_1 being the first eigenvalue of the Dirichlet Laplacian. Obviously, in view of this, the right hand side of (38) can be estimated as $O(T^{-1})$.

Step 2. Similarly, we may prove that

$$I \leq \frac{I^T}{T} + O(T^{-1}) \ .$$

To prove it we proceed all the way around. We plug the minimizer u^T of J^T into the functional J. We have

$$I \leq J(u^T) = \frac{J^T(u^T)}{T} - \frac{\int_0^T ||y^T(t) - z||^2_{L^2(\omega_0)}}{T} + ||\bar{y}^T - z||^2_{L^2(\omega_0)} \ . \tag{40}$$

This time, y^T stands for the solution of the evolution problem corresponding to the optimal control u^T while $\bar{y}^T$ is the corresponding steady state solution.

From previous estimates we know that

$$||y^T(t) - \bar{y}^T||_{L^2(\Omega)} \leq \exp(-\lambda_1 t)||y_0 - \bar{y}^T||_{L^2(\Omega)} \ , \tag{41}$$

so that, in order to guarantee uniform (with respect to T) decay rates, we need $\bar{y}^T$ to be uniformly bounded in $L^2(\Omega)$ and this requires uniform bounds on u^T in $L^2(\omega)$. But the uniform bound on u^T is easy to achieve.

Indeed, the solution of the evolution problem with $u = 0$ decays exponentially to zero. Thus, $\frac{1}{T}J^T(0)$ is uniformly bounded as T tends to infinity. Consequently

I^T/T is bounded above and this yields to the uniform bound of u^T in $L^2(\Omega)$. This automatically leads to uniform estimates for $\bar{y}^T$ in $H_0^1(\Omega)\cap L^{p+1}(\Omega)$. Therefore, (41) implies

$$||y^T(t) - \bar{y}^T||_{L^2(\Omega)} \leq c\ \exp(-\lambda_1 t)\ ,$$

and thanks to this estimate we have

$$\begin{aligned} &||\bar{y}^T - z||^2_{L^2(\omega_0)} - \tfrac{1}{T}\int_0^T ||y^T(t) - z||^2_{L^2(\omega_0)} \\ &\leq \tfrac{1}{T}\int_0^T [||\bar{y}^T - z||^2_{L^2(\omega_0)} - ||y^T(t) - z||^2_{L^2(\omega_0)}]dt = O(T^{-1})\ . \end{aligned}$$

Therefore from (40) we conclude that $I \leq \frac{I^T}{T} + O(T^{-1})$.

Step 3. Let us now show that the accumulation points of a sequence of minimizers u^T of J^T as T tends to infinity, are necessarily minimizers of J. The sequence u^T being bounded in $L^2(\omega)$, by extracting subsequences, we can get a weak limit in $L^2(\omega)$ that we denote as u^*. We claim that u^* is a minimizer for J. In other words, that $J(u^*) = I$. To prove it, we compute $J(u^*)$. We claim that, using lower semicontinuity properties, it follows that

$$J(u^*) \leq \liminf_{T\to\infty} \frac{J^T(u^T)}{T}\ . \tag{42}$$

Furthermore, according to the results in Steps 1 and 2, it follows that

$$\liminf_{T\to\infty} \frac{J^T(u^T)}{T} = \liminf_{T\to\infty} \frac{I^T}{T} = \lim_{T\to\infty} \frac{I^T}{T} = I\ .$$

This implies that $J(u^*) \leq I$ and, consequently, u^* is a minimizer for the steady state problem.

Let us now prove the claim (42). Obviously, by the weak convergence of the controls we have

$$||u^*||_{L^2(\omega)} \leq \liminf_{T\to\infty} ||u^T||_{L^2(\omega)}\ . \tag{43}$$

We need also to compare the solution $\bar{y}^*$ of the steady state problem associated to u^* and the solution y^T associated to u^T, the chosen optimal pairs for the time evolution problem in the time intervals $[0, T]$. We claim that $[\int_0^T y^T dt]/T$ converges to $\bar{y}^*$ in $L^2(\Omega)$.

Indeed, from previous estimates we know the uniform boundedness of $\bar{y}^T$ in $H_0^1(\Omega) \cap L^{p+1}(\Omega)$, where $\bar{y}^T$ is the steady state solution associated to u^T. Passing to the limit in the steady state problem it can be shown easily that $\bar{y}^T$ weakly converges in $H_0^1(\Omega) \cap L^{p+1}(\Omega)$ to $\bar{y}^*$. Since we have

$$||\frac{\int_0^T y^T(t)dt}{T} - \bar{y}^*||_{L^2(\Omega)} \leq ||\frac{\int_0^T y^T(t)dt}{T} - \bar{y}^T||_{L^2(\Omega)} + ||\bar{y}^T - \bar{y}^*||_{L^2(\Omega)} \ ,$$

taking into account the fact that

$$||\bar{y}^T - \bar{y}^*||_{L^2(\Omega)} \to 0 \ ,$$

as T tends to ∞, and that, due to (41),

$$||\frac{\int_0^T y^T(t)dt}{T} - \bar{y}^T||_{L^2(\Omega)} = O(T^{-1}) \ ,$$

we conclude that

$$\frac{\int_0^T y^T(t)dt}{T} \to \bar{y}^* \quad \text{in} \quad L^2(\Omega) \ \text{as} \ T \to \infty \ .$$

Finally, we have

$$\frac{J^T(u^T)}{T} \geq \frac{1}{2}\left\{\|\frac{1}{T}\int_0^T y^T dt - z\|^2_{L^2(\Omega)} + \|u^T\|^2_{L^2(\Omega)}\right\}$$

and with the convergence established above we complete the proof of the claim (42).

So far we have proved the weak convergence in L^2 of the controls u^T towards a limit control u^*. But the arguments of Step 1 and 2, showing that $\frac{1}{T}I^T \to I$, imply that the norms also converge. This leads to strong convergence as stated in the Theorem. □

Remark 3 Several remarks are in order:

- The result above, employing Γ-convergence, does not use the controllability properties of the system but only its exponential stability as time tends to infinity.
- The proof above does not yield any convergence rates.

5 Further Comments and Open Problems

1. The Γ-convergence proof above uses in an essential way the fact that the controls under consideration are independent of time.

   It would be very interesting to analyze whether these techniques can be applied for time-evolving controls.

   In this argument we use standard stability properties of the semilinear heat equation with nonlinearities satisfying the good sign condition. Of course this proof can be generalized for a larger class of semilinear problems enjoying properties similar to (39). These techniques can also be employed, for instance, for damped semilinear wave equations.

2. It would be interesting to investigate similar questions for more general nonlinearities, leading possibly to more complex dynamics, such as:

$$y_t - \Delta y + y^3 - Ly = u\chi_\omega \quad \text{in } \Omega \times (0, \infty) \tag{44}$$

for $L > 0$. Note that, indeed, when $L > \lambda_1$, the first eigenvalue of the Dirichlet laplacian, the existence of a steady state solution is guaranteed but not its uniqueness. Also, the trajectories of the parabolic problem are bounded and the Lyapunov function

$$\frac{1}{2}\int_\Omega [|\nabla y|^2 - Ly^2]dx + \frac{1}{4}\int_\Omega y^4 dx - \int_\omega uy dx$$

allows proving that all the elements of the ω-limit set are steady state solutions.

In the present case, according to the results in [13] and the analyticity of the nonlinearity one can prove that every trajectory converges as $t \to \infty$ to a steady state solution. Whether this kind of results and the techniques in [13] can be further developed to obtain results of turnpike nature is an interesting open problem.

3. The results we have obtained in Sect. 3 apply to a particular class of solutions of the optimality system. But, as mentioned above, even in the case of small initial data and target, further work is needed to show its applicability to the optimal pairs.

In any case, the obtained results are of local nature. The obtention of global results would require a more complete understanding of the controlled dynamics. This has been done successfully in a number of examples (see [3] and [4]). But a systematic approach to these problems is still to be developed.

Note in particular that the optimal control and states are not unique for the steady state problem under consideration. Actually, the multiplicity of its solutions, its stability properties and the impact this might have on the problem under consideration could be worth investigating.

4. It would be natural to consider the analogues of the optimal control problems above in the context of optimal shape design. Often in applications, in particular in aeronautics (see [12]), optimal shapes are computed on the basis of the steady state modeling but they are then employed in time evolving ones. In the particular context of the semilinear heat equation of the previous section, the following problem makes sense. Does the optimal shape in the time interval $[0, T]$ converge, as T tends to infinity, to the optimal steady state shape for the elliptic equation? Of course this problem can be formulated in a variety of contexts, depending mainly on the admissible class of shapes considered.

There is very little in the subject except for the paper [1] where the issue is addressed in the context of the two-phase optimal design of the coefficients. In this setting it is indeed proved that, as time tends to infinity, optimal designs of the parabolic dynamics converge to those of the elliptic steady-state problem.

5. Note that the problems of optimal shape design and its possible stabilization in long time horizons could be much more complex if one would consider, for the evolution problem, shapes allowed to evolve in time as well.
6. At this stage it is important to observe that, according to earlier results in [16] and [17] for the conservative wave equations, in the absence of damping, optimal shape design problems for the collocation of actuators and sensors in long time intervals lead to spectral problems, and not really to steady state ones. In other words, for conservative dynamics one does not expect the simplification of optimal design problems to occur towards the steady state optimal design problem, but rather towards a spectral version of it. The results in [19] for the heat equation show that the optimal designs are determined by a finite number of eigenfunctions that diminishes as the time horizon increases. Thus even in this parabolic context, the relevant optimal design problem is not a steady state one but of spectral nature even if it involves only a finite number of eigenfunctions. Thus, the overall role of the turnpike property for shape design problems, in particular in the context of optimal placement of sensors and actuators, is to be clarified.
7. At this point it is worth to underline that, while in [1] the authors work with given initial data, in the problem considered in [19], the optimal placement of sensors and actuators is determined within the whole class of solutions. The questions are then of different nature and, accordingly, the expected results are not necessarily the same. This issue is important for a correct formulation of the optimal design problem to be addressed and the comparison of the corresponding results.

   Note also that the reduction in [19] requires of a randomization procedure so that optimal shapes are defined to be optimal in some probabilistic sense. But, roughly, it can be said that the results above apply in the context where the optimal shape of the actuator or sensor is computed so to be optimal within the whole class of solutions of the PDE under consideration. Of course, all these problems are expected to be easier to handle when one considers given fixed initial data as pointed out in [18], which corresponds, somehow, to the situation considered in [1].
8. The same can be said about inverse problems. The problem of the connection between the inversion process in long time intervals and in the steady state regime makes fully sense both in the context of linear and nonlinear problems and in a variety of inverse problems. Very little is known in this subject (see [11], section 9.4).

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