DYNAMICS AND CONTROL FOR MULTI-AGENT NETWORKED SYSTEMS

A FINITE DIFFERENCE APPROACH

Umberto Biccari

DeustoTech, Universidad de Deusto, Bilbao, Spain

Joint work with Dongnam Ko and Enrique Zuazua

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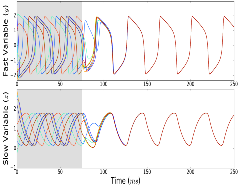
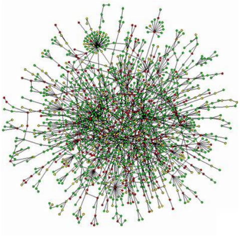
INTRODUCTION

Collective behavior models

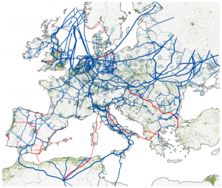
- They describe the dynamics of a system of interacting individuals.
- They are applied in a large spectrum of subjects such as synchronization of coupled oscillators, random networks, multi-area power grid, opinion propagation,...



Fitz-Hugh-Nagumo oscillators [Davison et al., Allerton 2016]



Yeast's protein interactions [Jeong et al., Nature, 2001]



European natural gas pipeline network [www.offiziere.ch]

Complex behavior by simple interaction rules

Systems of Ordinary Differential Equations (ODEs) in which each agent's dynamics follows a prescribed law of interactions.

First-order consensus model

$$\dot{x}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a_{i,j}(x_j(t) - x_i(t)), \quad i = 1, \ldots, N$$

- It describes the opinion formation in a group of N individuals.
- $x_i \in \mathbb{R}^d$, $d \geq 1$, represents the opinion of the i-th agent.
- It applies in several fields including information spreading of social networks, distributed decision-making systems or synchronizing sensor networks, ...

Linear versus Nonlinear

Linear networked multi-agent models[1]: $a_{i,j}$ are the elements of the adjacency matrix of a graph with nodes x_i

$$\begin{aligned} a_{i,j} > 0, &\quad \textit{if } i \neq j \textit{ and } x_i \textit{ is connected to } x_j \\ a_{i,j} = 0, &\quad \textit{otherwise}. \end{aligned}$$

Nonlinear alignment models[2]:

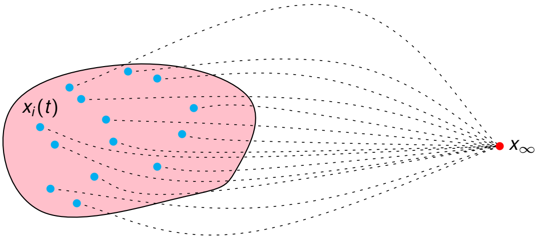
$$a_{i,j} := a(|x_j - x_i|), \quad \textit{where} \quad a : \mathbb{R}_+ \to \mathbb{R}_+,$$

$a \geq 0$ is the influence function. The connectivity depends on the contrast of opinions between individuals.

[1] Olfati-Saber, Fax, and Murray, IEEE Proc., 2007
[2] Motsch and Tadmor, SIAM Rev., 2014

Consensus

Depending on the nature of the interactions, the system may converge to a particular configuration, called consensus which is characterized by the property $x_1 = x_2 = \ldots = x_N := x_\infty$.



Caponigro, Carrillo, Fornasier, Piccoli, Tadmor,Trélat,...

Convergence to consensus happens naturally whenever the system is sufficiently close to this configuration.

Control strategies (N fixed)

Controlled model

$$\begin{cases} \dot{x}(t) = \frac{1}{N}\sum_{j=1}^{N} a_{i,j}(x_j(t) - x_i(t)) + \sum_{j=1}^{M} b_{i,j}u_j(t), \quad i = 1, \dots, N, \\ x(0) = x_0, \end{cases}$$

- **Linear**: $\dot{x} + Lx = Bu \Rightarrow$ Classical Kalman rank condition.
- **Nonlinear**: controllability and stabilizability are much more challenging[1].
- The linear model can be viewed as the linearization of the nonlinear one around the consensus configuration.

Different control strategies

- To act on all the components of the system (certainly effective but not always optimal).
- To focus on a small number of agents at each time (sparse control).
- To look for a single leader who acts on the whole crowd and steers it to the desired configuration (control through leadership).

[1] Caponigro, Fornasier, Piccoli, and Trélat, Math. Models Methods Appl. Sci., 2015

TWO LIMIT MODELS

Mean-field limit equations

When the number of agents N tends to infinity, the ODE consensus model is replaced with a suitable PDE.

Nonlinear alignment models:

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} a(|x_j - x_i|)(x_j - x_i), \quad i = 1, \dots, N, \quad a : \mathbb{R}_+ \to \mathbb{R}_+.$$

Classical mean-field theory suggests to consider the N-particle distribution function

$$\mu^N = \mu^N(x,t) := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}.$$

and to look for the equation it satisfies as $N \to +\infty$.

Particle method

Analogies with the particle method (P. A. Raviart, J. Comp. Math., 1986) which refers to numerical schemes for time-dependent problems in PDE where, for each time t, the exact solution is approximated by a linear combination of Dirac measures.

The limit $\mu = \lim_{N \to +\infty} \mu^N$ solves the non-local transport equation [1,2]

$$\partial_t \mu = \partial_x \left(\mu(x,t) \int_{\mathbb{R}^d} a(|y-x|)(x-y)\mu(y,t)\, dy \right).$$

The convolution kernel describes the mixing of opinions due to the interactions among the agents during the time evolution of the dynamics.

The system of ODEs describing the agents dynamics defines the characteristics of the underlying transport equation. The coupling of the agents dynamics introduces the non-local effects on transport.

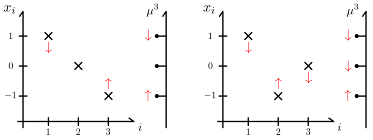
[1] Ha and Tadmor, Kinetic Relat. Methods, 2008
[2] Motsch and Tadmor, SIAM Rev., 2014

Mean-field limit for linear models?

- The mean-field equation involves the density μ, which does not contain the full information of the state since it does not keep track of the identities of agents (label i).

Linear networked model with three agents

$$\dot{x} + Lx = 0 \quad \text{and} \quad L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

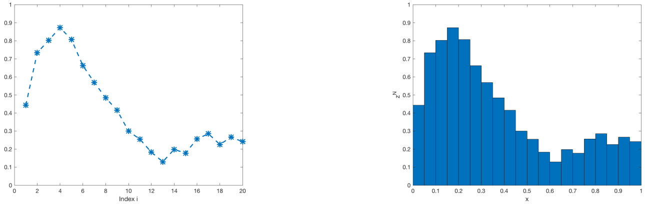


Two different initial data $x^1(0) = (-1, 0, 1)$ (left) and $x^2(0) = (-2, 3, -1)$ (right) whose dynamics are different though they have the same distribution μ^3.

Graph limit method

- Based on the theory of graph limits.
- Considers the phase-value function $x^N(s,t)$ defined as

$$x^N(s,t) = \sum_{i=1}^{N} x_i(t)\chi_{I_i}(s,t), \quad s \in (0,1),\ t > 0,\ \bigcup_{i=1}^{N} I_i = [0,1].$$



An opinion datum for $N = 20$ and its function z^{20} on $[0,1]$

Formal procedure

Let $(x_i^N)_{i=1}^N$ be the solution of the following consensus model

$$\dot{x}_i^N = \frac{1}{N}\sum_{j=1}^{N} a_{i,j}^N \psi(x_j^N - x_i^N),$$

where $a_{i,j}^N$ are constant and ψ represents nonlinearity.

The graph limit theory says that if

$$W^N(s,s_*) = \sum_{i,j=1}^{N} a_{i,j}^N \chi_{\left[\frac{i}{N},\frac{(i+1)}{N}\right)}(s)\chi_{\left[\frac{j}{N},\frac{(j+1)}{N}\right)}(s_*)$$

is uniformly bounded and converges to W, then the phase-value function $x^N(s,t)$ converges to the solution of the non-local diffusive equation

$$\partial_t x(s,t) = \int_0^1 W(s,s_*)\psi(x(s_*,t) - x(s,t))ds_*{}^1 .$$

[1] Medvedev, SIAM J. Math. Anal., 2014

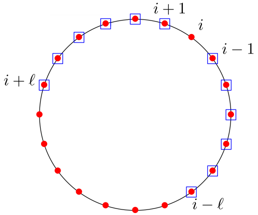
Example of the graph limit

Model with a periodic dense network

$$\dot{x} + L_r x = 0, \quad L_r = \frac{1}{N}(l_{i,j})_{i,j=1}^N, \quad r \in (0, 1/2]$$

$$l_{i,j} = \begin{cases} \ell, & \text{if } i = j \\ -1, & \text{if } j - i \in [-\ell, \ell] \setminus \{0\} \ (mod\ N) \\ 0, & \text{otherwise} \end{cases}$$

$\ell = [rN]$, the closest integer to rN.



This leads to the non-local diffusion equation with

$$W(\theta, \theta_*) = \chi_{[-2\pi r, 2\pi r]}(\theta_* - \theta), \quad \theta, \theta_* \in \mathbb{S}^1.$$

Non-local diffusion equation

$$\partial_t x(\theta, t) = \int_{\mathbb{S}^1} W(\theta_*, \theta)(x(\theta_*, t) - x(\theta, t))\, d\theta_*$$

$$W(\theta, \theta_*) = \chi_{[-2\pi r, 2\pi r]}(\theta_* - \theta), \quad \theta_*, \theta \in \mathbb{S}^1.$$

- $\int_{\mathbb{S}^1} W(\theta_*, \theta)(x(\theta_*, t) - x(\theta, t))\, d\theta_* = 0$ for weak interactions $r \to 0$, leading to the trivial dynamics:

$$\partial_t x(s, t) \equiv 0.$$

- A non trivial limit dynamics requires a large number of interactions among the agents

$$(\text{\# of nonzero } a_{ij}) \sim N^2 \quad \text{as} \quad N \to +\infty.$$

Nonlinear subordination

- Finite ODE collective dynamics:

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} a(|x_j - x_i|)(x_j - x_i).$$

- Graph limit model:

$$x_t(s,t) = \int_0^1 a(|x(s_*,t) - x(s,t)|)(x(s_*,t) - x(s,t))ds_*.$$

- Mean-field limit model:

$$\mu_t(x,t) + \nabla_x(V[\mu]\mu) = 0, \quad \text{where} \quad V[\mu] := \int_X a(x_* - x)\mu(x_*,t)dx_*.$$

Subordination transformation

From non-local "parabolic" to non-local "hyperbolic": $\mu(x,t) = \int_S \delta(x - x(s,t))ds$.

Similar to the link between kinetic equations and conservation laws.

P.-L. Lions, B. Perthame and E. Tadmor, J. Amer. Math. Soc., 1994.

CONTROLLABILITY OF LINEAR MODEL

Consider the control problem associated to the linear consensus model

$$\dot{x} + Lx = Bu.$$

One looks for $u = u(t)$ so to steer the system into the consensus at time T:

$$x(T) = (\bar{x}, \dots, \bar{x})^T.$$

Different types of control actions

On one single agent or a few ones:

$$B = (1, 0, \dots, 0)^T \quad \text{or} \quad B = (0, \dots, 0, I_k, 0, \dots, 0)^T,$$

for $k \times k$ identity matrix I_k.

For finite-dimensional linear systems, the Kalman rank condition provides a necessary and sufficient condition

$$\text{rank}[B, LB, \dots, L^{N-1}B] = N.$$

Challenge

Analyze the behavior as $N \to +\infty$.

We discuss four examples of linear networked consensus models, inspired in previous knowledge on:

- $1-d$ heat equations.
- $2-d$ heat equations.
- Non-local diffusive equations.
- Fractional heat equations.

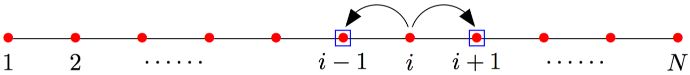
Rough answer

Control properties ARE NOT UNIFORM as $N \to +\infty$. The goal should rather be getting sharp estimates on how these properties diverge with N.

A model with sparse graph

Sparse graph

$$a_{i,j} = 1 \quad \textit{if } j = i \pm 1, \qquad a_{i,j} = 0 \quad \textit{otherwise}.$$



Each agent i communicates with its neighbors, $i-1$ and $i+1$.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_N \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & \dots & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & -1 & 1 \end{pmatrix}}_{L} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

Link with semi-discrete PDEs

Rescaled version of the finite difference semi-discretization of the one-dimensional heat equation with homogeneous Neumann boundary conditions on [0, 1].

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_N \end{pmatrix} + N^2 \underbrace{\begin{pmatrix} 1 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & \dots & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & -1 & 1 \end{pmatrix}}_{D} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

Our system corresponds to the finite-difference discretization of

$$u_t - N^{-2} u_{xx} = 0, \quad t \in [0, T]$$

or, alternatively, to the finite-difference discretization of

$$u_t - u_{xx} = 0, \quad t \in [0, T/N^2].$$

Control cost

Applying known results of semi-discretized heat equations we deduce that the cost of controlling the sparse network of N agents to consensus in finite time T is of the order of

$$\mathcal{C} \sim \exp(cN^2/T).$$

Control of sparse networked system

- If $T \sim N^2$, $\|u^N(t)\|_{L^2(0,T)} \leq C$.
- If T is independent of N, then $\|u^N(t)\|_{L^2(0,T)} \sim C_1 \exp(C_2 N^2)$.

The $1-d$ sparse network exhibits a slow diffusion rate. Controlling the system through the action on a % of individuals of the network requires a time of the order of $T \sim N^2$ or controls of exponential size, so to steer the system to consensus.

Spectral analysis

The control properties of an infinite-dimensional symmetric system rely heavily on these two properties of its spectrum $\{\lambda_k\}_{k\ \geq 1}$:

- $\lambda_{k+1} - \lambda_k \geq \gamma > 0$, for all $k \geq 1$.
- $\sum_{k\geq 1} \lambda_k^{-1} < +\infty$.

These conditions are satisfied uniformly in N by the eigenvalues of the semi-discrete heat equation, which essentially behave like those of the actual heat equation ($\lambda_k \sim k^2$), but not by the ones of the consensus model

Link with Müntz theorem

We are considering dynamical systems generated by real exponential $\exp(-\lambda_k t)$ that, under the change of variables $z = \exp(t)$ take the polynomial form z^{λ_k}.

2D sparse networked model

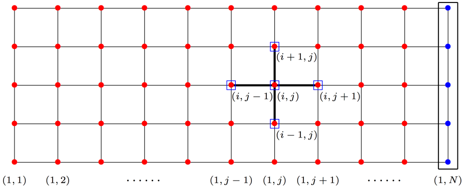
Similar results can be achieved for more general networks related to the finite-difference semi-discretization of the heat equation in $\mathbb{R}^2$.

Model on a 2D graph

$$\dot{x}_{i,j}(t) = \sum_{k,l=1}^{N} a_{(i,j),(k,l)}(x_{k,l}(t) - x_{i,j}(t)), \quad i,j = 1,\ldots,N,$$

$$a_{(i,j),(k,l)} = 1, \quad \text{if } (k,l) = (i \pm 1,\ j) \text{ or } (i,\ j \pm 1)$$

$$a_{(i,j),(k,l)} = 0, \quad \text{otherwise}.$$



This model corresponds to the control problem on the semi-discretized two-dimensional heat equation with scaling N^{-2}:

$$\dot{x} + N^{-2}Qx = Bu$$
$$B = [I, 0, \ldots, 0]^T.$$

Analogously to the one-dimensional case, we have a control cost exponentially large in N.

Control of the 2D sparse networked system

The control cost behaves $C \sim \exp(N^2/T)$ as in the one-dimensional case.

Example of the graph limit

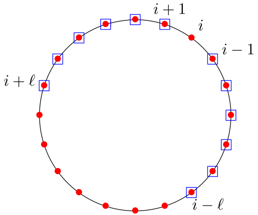
For examples with dense interactions, we may scale the model with a periodic dense network presented before:

Model with a periodic dense network

$$\dot{x} + L_r x = 0, \quad L_r = \frac{1}{N}(l_{i,j})_{i,j=1}^N, \quad r \in (0, 1/2]$$

$$l_{i,j} = \begin{cases} 2, & \text{if } i = j \\ -1/\ell, & \text{if } j - i \in [-\ell, \ell] \setminus \{0\} \ (modN) \\ 0, & \text{otherwise} \end{cases}$$

$\ell = [rN]$, the closest integer to rN.



This leads to the non-local diffusion equation with

$$W(\theta, \theta_*) = \frac{1}{2\pi r}\chi_{[-2\pi r, 2\pi r]}(\theta_* - \theta), \quad \theta, \theta_* \in \mathbb{S}^1.$$

We may expect that a dense graph with many interactions among the agents improves the control properties. Spectral analysis shows that this is not the case.

The eigenvalues λ_k^ℓ and eigenvectors ψ_k^ℓ can be calculated explicitly since the matrix L_r is Toeplitz:

Spectrum

$$\lambda_k^\ell = \frac{4}{\ell} \sum_{j=1}^{\ell} \sin^2\left(\frac{k\pi j}{N}\right),$$

$$\psi_k^\ell = \left(\sin\left(\frac{2k\pi j}{N}\right) + \cos\left(\frac{2k\pi j}{N}\right)\right)_{j=1}^{N}, \quad k = 1, \dots, N.$$

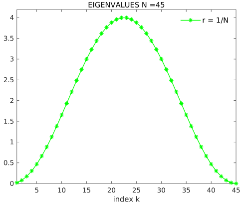
Also for this model we have a bad spectral behavior from the controllability point of view.

The case $r = 1/N$

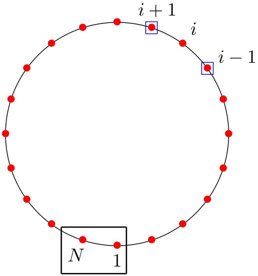
- For $r = 1/N$, corresponding to $\ell = 1$, we can quantify explicitly the spectral properties.
- Notice that, in this case, the graph is not really dense, since each agent is communicating only with the left and right neighbors.

$$\lambda_k^1 = 4\sin^2\left(\frac{k\pi}{N}\right), \quad k = 1, \ldots, N$$
$$\lambda_{N-k}^1 = \lambda_k^1.$$

We have eigenvalues with multiplicity 2. This is consequence of the periodicity of the network.



Distribution of the eigenvalues for $r = 1/N \Rightarrow \ell = 1$.



- This case corresponds to a rescaled semi-discrete heat equation with periodic boundary conditions.
- It is enough to take

$$B = (1, 0, \ldots, 0, 1)^T$$

that is, controlling only two agents (black box in the figure).

Then, as for our first example, the controllability cost is of the order of

$$\mathcal{C} \sim \exp(N^2/T).$$

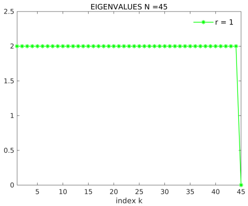
- When the time of control is $\boldsymbol{T \sim N^2}$, controllability to consensus is achievable with a control of size uniformly bounded on N.
- When we need a control time **T independent of N**, it requires controls exponentially large.

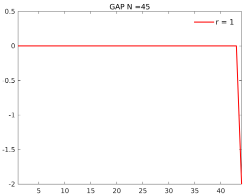
The case $r = 1/2$

- Also for $r = 1/2$, corresponding to $\ell = N/2$, we can easily analyze the spectrum.
- In this case, all the agents are in communication with each other.

$$\lambda_k^N = 2, \quad k = 1, \ldots, N-1, \quad \lambda_N^N = 0.$$

We have eigenvalues with multiplicity $N - 1$.



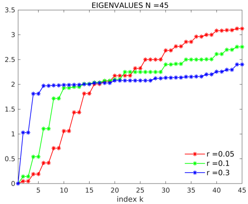


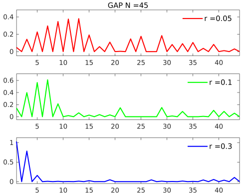
Eigenvalues (left) and spectral gap (right) for $r = 1/2 \Rightarrow \ell = N/2$.

The intermediate cases

- For $r \in (1/N, 1/2)$, it is difficult to study the spectral properties analytically.

$$\lambda_k^\ell = \frac{1}{\ell}\left[2\ell + 1 - \csc\left(\frac{k\pi}{N}\right)\sin\left(\frac{k\pi(2\ell+1)}{N}\right)\right], \quad k = 1, \ldots, N$$





Eigenvalues (left) and spectral gap (right) with $N = 45$ and various r.

- Simulations show that the spectral properties deteriorate as r increases.
- We have many repeated eigenvalues, but it is not easy to explicitly track their distribution.

Conclusions

Our analysis shows that dense graphs have worse controllability properties than the sparse ones (first example).

- $\ell = 1$: with $B = (1, 0, \ldots, 0, 1)^T$ we recover the same controllability time and cost as our first example.
- $\ell > 2$: the controllability properties of the system deteriorate as r increases.
  - ▷ Our previous discussion suggests that we may recover better controllability properties by increasing the number of controlled agents

$$B = [I_\ell, 0, \ldots, 0, I_\ell]^T.$$

WORK IN PROGRESS

The fractional heat equation

Model on a weighted graph

$$\dot{x} + L_{frac}x = Bu, \quad L_{frac} = (a_{i,j})_{i,j=1}^{N}, \quad B = [0, \dots, 0, I_N, 0, \dots, 0]^T.$$

$$a_{i,j} = \begin{cases} -\dfrac{c(\alpha)}{|i-j|^{1+2\alpha}}, & \text{if } j \neq i, \\ \displaystyle\sum_{j \neq i} a_{i,j}, & \text{if } i = j. \end{cases}, \quad \alpha \in (0,1)$$

In contrast with the previous example, the communication rate among different agents is weighted as a function of the distance $|i-j|$. Hence, the interactions among close agents have a higher impact on the dynamics.

L_{frac} describes a dense network inspired on the fractional Laplacian.

$$D_{frac} := N^{2\alpha} L_{frac}$$

is the finite difference discretization of the fractional Laplace operator.

Fractional Laplacian

$$(-d_x^2)^\alpha u(x) := c_\alpha P.V. \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2\alpha}} dy.$$

$$\dot{x} + D_{frac} x = Bu$$

is the semi-discretized control problem for the fractional heat equation

$$\partial_t u + (-d_x^2)^\alpha u = f \chi_\omega, \quad t \geq 0.$$

It corresponds the graph limit non-local diffusive model with

$$W(x, y) = |x - y|^{-1-2\alpha},$$

The fractional heat equation is null-controllable in time $T > 0 \Leftrightarrow \alpha > 1/2$.[3,4]

The eigenvalues of D_{frac} behave as $\lambda_k^D \sim k^{2\alpha}, \quad k \geq 1$.

Spectral behavior

$$\alpha \leq 1/2 \Rightarrow \sum_{k=1}^{N} \left(\lambda_k^D\right)^{-1} \geq N, \quad \alpha > 1/2 \Rightarrow \sum_{k=1}^{N} \left(\lambda_k^D\right)^{-1} \leq C < +\infty$$

$$\inf_{k=1,\ldots,N-1} \left(\lambda_{k+1}^D - \lambda_k^D\right) = \begin{cases} \lambda_N^D - \lambda_{N-1}^D = \mathcal{O}(N^{2\alpha-1}), & \alpha < 1/2 \\ \lambda_2^D - \lambda_1^D = \mathcal{O}(1), & \alpha \geq 1/2. \end{cases}$$

For $\alpha \leq 1/2$, the control cost is not bounded in N. In particular, for $\alpha < 1/2$ it blows-up exponentially as $\exp(N^{1-2\alpha})$.

[3] Micu and Zuazua SIAM J. Cont. Optim., 2006

[4] Biccari and Hernández-Santamaría, IMA J. Math. Control. Inf., 2018

What about $\dot{x} + L_{frac}x = Bu$?

- This time, even in the case $\alpha > 1/2$, the controllability properties are not uniform in N due to the scaling of the matrix L_{frac}.
- The eigenvalues of L_{frac} behave as

$$\lambda_k^L = N^{-2\alpha}\lambda_k^D \sim \left(\frac{k}{N}\right)^{2\alpha}.$$

Consequently, the spectral gap is very small even for $\alpha > 1/2$.

- The two systems are equivalent up to time-scaling $t \mapsto N^{-2\alpha}t$:

$$\dot{x} + D_{frac}x = 0, \quad t \in [0, T/N^{2\alpha}].$$

Hence, the cost of controlling $\dot{x} + L_{frac}x = Bu$ is of the order of $\exp(CN^{2\alpha}/T)$.

Conclusions

- We considered finite-dimensional collective behavior models and we discussed their infinite-agents limits.
- The nature of the interactions among the individuals determines the limit approach one should use. Networked systems require the employment of a graph limit, while for aligned ones it is possible to rely on the classical mean-field theory.
- These two limit approaches lead to substantially different kinds of equations, a diffusion and a transport one, respectively. We showed that the diffusion equation is subordinated to the transport one through an averaging process.
- We analyzed controllability properties of linear networked models by linking them to the finite difference semi-discretization of heat-like equations.
- This allows to get to some conclusions learning from the existing theory of control of parabolic PDEs and their numerical counterparts, and to get some estimates on the cost of controlling systems as $N \to +\infty$.

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