Lack of Null Controllability of Hybrid PDE-ODE models.

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DyCon DYNAMIC CONTROL erc

Outline

Abstract Control System

Question : Can the solution trajectory of a dynamical system (PDE) be brought to a desired profile using a control? The control may act in an open subset of the domain, or on parts of the boundary.

- X (State space) and U (Control Space) are two Hilbert Spaces.
- Abstract Control system :

$$\dot{z}(t) = Az(t) + Bu(t),\ t \geqslant 0, \quad z(0) = z_0 \in X. \tag{1.1}$$

- $A : D(A) \to X$, generates a C^0-semigroup. $B \in \mathcal{L}(U, D(A^*)')$.
- $z \in C([0, T]; X)$.

Definition

Let $\tau > 0$. We say that the system (1.1) or the pair (A, B) is null controllable in time τ, if for every $z_0 \in X$ there exists a control $u \in L^2(0, \tau; U)$ such that

$$z(\tau) = 0.$$

Finite dimensional controlled system

- $X = \mathbb{R}^n$, $A \in M_{n\times n}$. $U = \mathbb{R}^m$, $B \in M_{n\times m}$ with $m \leqslant n$.
- Example 1 :

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix}$$

Then $z_1 = e^t z_{1,0}$, i.e., the control has no effect on the first component. The system is not controllable.
The system is controllable if and only if control acts on the both components. ($n = m = 2$.)

Examples

- Consider the following system

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix}$$

- Even though control acts on one component, the system is controllable.
- The system is equivalent to $z''(t) + z(t) = u(t)$.
- Consider any y such that $y(0) = z_0$ and $y(\tau) = 0$ and define $u(t) := y''(t) + y(t)$.
- **Kalman Rank Condition** The pair (A, B) is controllable if and only if

$$\mathrm{rank}\,[\mathrm{B\ AB\ \ldots A^{n-1}B}] = \mathrm{n}.$$

- Controllable in some time $\tau > 0 \Longrightarrow$ controllable in any time.

Examples : PDE

- Infinite-dimensional ODE.
  - Ω be a bounded domain and $\omega \subseteq \Omega$. $X = L^2(\Omega)$. We consider

$$z'(t,x) = z(t,x) + \chi_\omega u(t,x), \quad z(0,x) = z_0(x).$$

  - $z(t,x) = e^t z_0(x) + \int_0^t e^{(t-s)} \chi_\omega u(s,x)\, ds.$
  - If $\omega = \Omega$, then null controllable in any time $\tau > 0$.
- Heat equation.
  - We consider

$$z'(t) = \Delta z + \chi_\omega u, \quad z(0,x) = z_0(x).$$

  - Null contollable in any time $\tau > 0$ for any non empty open subset $\omega \subset \Omega$.
  - Carleman estimates (Fursikov-Imanuvilov), Spectral methods (Lebeau-Robianno) ....

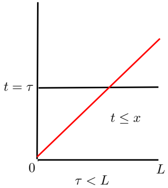
Example : 1D transport equation

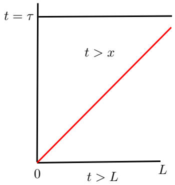
We consider

$$\begin{cases} z_t + z_x = 0 & t \geqslant 0, x \in (0, L), \\ z(t,0) = u(t), & z(0,x) = z_0(x). \end{cases}$$

Then

$$z(t,x) = \begin{cases} z_0(x-t) & t \leqslant x, \\ u(t-x) & t > x. \end{cases}$$





- Null controllable in time $\tau \geqslant L$.
- What happens to hybrid PDE + ODE system.

Outline

Properties of Null Controllable System

- Recall: $z' = Az + Bu(t), \quad z(0) = z_0.$
- Then $z(\tau) = e^{\tau A} z_0 + \int_0^\tau e^{(\tau - s)A} Bu(s)\, ds.$
- We define $\mathbb{T}_t z_0 = e^{tA} z_0$ and $\Phi_\tau u = \int_0^\tau e^{(\tau - s)A} Bu(s)\, ds.$
- Null controllability equivalent to $\mathrm{Ran}\mathbb{T}_\tau \subset \mathrm{Ran}\Phi_\tau.$
- $\|\mathbb{T}_\tau^* z\| \leqslant C\|\Phi_\tau^* z\|$ for all $z \in D(A^*)$.
- Conseder the adjoint $\phi'(t) = A^*\phi(t)\ t > 0, \quad \phi(0) = \phi_0$. Null controllability is equivalent to final time observability inequality

$$\|\phi(\tau)\|_X^2 \leqslant C \int_0^\tau \|B^*\phi(s)\|_U^2.$$

- If the system is null controllable in time τ meaning that $u(\tau)$ is "smooth".

Outline

Compressible NS in 1D

We consider CNS linearized around $(1, 0)$

$$\begin{cases} \partial_t \rho + \partial_x u = \chi_{\omega_1} f & t > 0, x \in (0, L), \\ \partial_t u - \partial_{xx} u + \partial_x \rho = \chi_{\omega_2} f_2 & t > 0, x \in (0, L), \\ u(t, 0) = u(t, L) = 0 & t > 0. \end{cases} \tag{3.1}$$

Theorem (Chowdhury, Ramaswamy, Raymond, 13)

$\omega_1 = \emptyset$, $\omega_2 = (0, L)$ $\rho_0 \in H^1_m$ and $v_0 \in L^2$. Then the system is null controllable in any time $\tau > 0$.

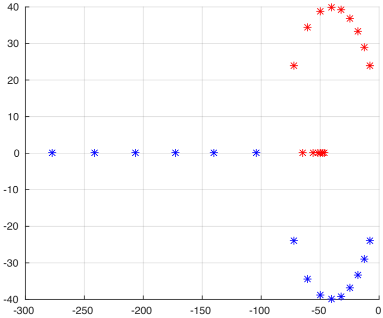
Theorem (DM, 15)

Let $\omega_1 \subset \Omega$. Then the system is not null controllable in any time in $L^2 \times L^2$.

- The system behaves like infinite-dimensional ODE.

What is the connection

- Let us look at the Spectrum.
  - Infinte-dimensional ODE $\lambda = 1$.
  - CNS :



  - They both have accumulation point in the spectrum.

- **Observability inequality**

$$\begin{cases} \partial_t \sigma - \partial_x v = 0 & t > 0, x \in (0, L), \\ \partial_t v - \partial_{xx} v - \partial_x \sigma = 0 & t > 0, x \in (0, L), \\ v(t, 0) = v(t, L) = 0 & t > 0, \\ \sigma(0, x) = \sigma_0, \quad v(0, x) = v_0. & \end{cases}$$

$$\exists C > 0, \|(\sigma(\tau), v(\tau))\|_{L^2}^2 \leqslant C \int_0^\tau \int_{\omega_1} \sigma^2 + \int_0^\tau \int_{\omega_2} v^2, \quad \forall (\sigma_0, v_0) \in L^2$$

•**Idea :** Existence of Gaussian beam solutions, i.e, solutions whose energy is localized near certain curves $(t, x(t))$ in space-time. (Cf. Ralston 1975).

Gaussian Beam Solutions

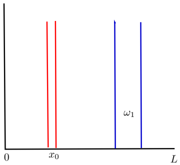
Theorem

Let $\epsilon > 0$. *There exists* $(\sigma_0, v_0) \in (L^2(\mathbb{R}))^3$ *such that, the solution of the adjoint system in* $\mathbb{R} \times (0, T)$, *satisfies*

$$\|\sigma(\cdot, t)\|^2_{L^2(0,T;L^2(|x-x(t)|>\eta))} \leq C\sqrt{\epsilon}, \|\sigma(\tau)\|_{L^2(\mathbb{R})} \geq \frac{1}{4\pi}$$

$$\|v\|^2_{L^2(0,T;L^2(\mathbb{R}))} \leq C\epsilon^2.$$

where $x(t) = x_0$, *for any* $x_0 \in \mathbb{R}$ *and constants* $\eta > 0$.



- Construct solutions along the eigenfunction of the convergent e.v in the Fourier variable.

Viscoelastic flows

Conservation of mass and momentum:

$$\nabla . u = 0, \quad \rho u_t \;=\; \nabla . \tau - \nabla p.$$

Constitutive Law: Relates the stress tensor to the motion.

Newtonian Fluid : $\quad \tau_{New} = \eta(\nabla u + (\nabla u)^T), \quad \eta > 0,$

Linear viscoelastic fluid : Stress depends on the current motion as well as the history.
Then stress tensor satisfies : (Maxwell model)

$$\tau_t + \lambda \tau = \kappa \Big(\nabla u + (\nabla u)^T \Big),$$

$\lambda > 0, \; \kappa > 0, \; 1/\lambda =$ Stress relaxation time.
Jeffreys model:

$$\text{Total Stress} = \tau_{New} + \tau.$$

Jeffreys and Maxwell Models

System in $\Omega \subset \mathbb{R}^d$**,** $d = 2, 3$

$$
\begin{aligned}
\rho u_t \;&=\; \eta \triangle u + \nabla.\tau - \nabla p + f\chi_{\mathcal{O}} \quad \text{in } \Omega \times (0,T),\\
&\nabla.u = 0 \quad \text{in } \Omega \times (0,T), \quad u = 0 \quad \text{in } \partial\Omega \times (0,T),\\
&\tau_t + \lambda\tau = \kappa(\nabla u + (\nabla u)^T) \quad \text{in } \Omega \times (0,T),\\
&u(\cdot,0) = u_0, \quad \tau(\cdot,0) = \tau_0 \quad \text{in } \Omega,
\end{aligned}
$$

f is a control localized in $\mathcal{O}$, an open subset of Ω.

- For $\eta > 0$, Jeffreys system,
- For $\eta = 0$, Maxwell system.
- Several relaxation mode: $\tau = \sum_{i=1}^{N} \tau_i$,, for $i = 1, \cdots, N$,

$$
(\tau_i)_t + \lambda_i \tau_i = \kappa_i \Big(\nabla u + (\nabla u)^T\Big), \quad \lambda_i > 0, \quad \kappa_i > 0,
$$

- Contollability of the both component.

An equivalent system

The stress tensor is

$$\tau(x,t) = \tau_0 + \int_0^t e^{-\lambda(t-s)}\Big(\nabla u + (\nabla u)^T\Big)ds.$$

Necessarily, $\tau \in R_0 := \left\{\nabla u + \nabla u^\top | v \in H_0^1, \mathrm{div} v = 0\right\}.$
So we define

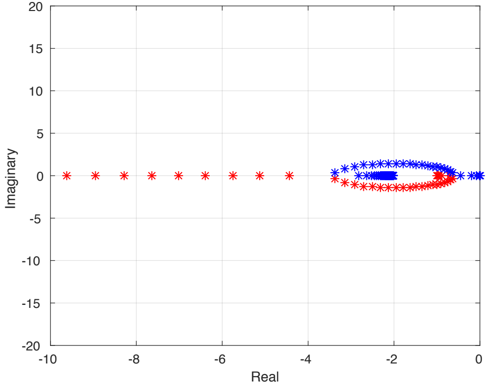
$$\tau = (\nabla v + (\nabla v)^T).$$

Thus the new system for divergence free (u, v) vanishing on boundary,

$$\begin{aligned}
\rho u_t \ &= \ \eta \triangle u + \sum_{i=1}^N \triangle v_i - \nabla p + f\chi_{\mathcal{O}} \quad \text{in } \Omega \times (0,T), \quad i = 1, \cdots, N, \\
v_{it} + \lambda_i v_i &= \kappa_i u \quad \text{in } \Omega \times (0,T), \quad i = 1, \cdots, N, \\
u(\cdot, 0) &= u_0, \quad v_i(\cdot, 0) = v_{i,0} \quad \text{in } \Omega, \quad i = 1, \cdots, N.
\end{aligned}$$

Spectrum of single mode Jeffreys system

$\eta > 0$.



Theorem (M, Mitra, Renardy, 18)

$\mathcal{O} \subset \Omega$. *Jeffreys system is not null controllable in any time* $\tau > 0$.

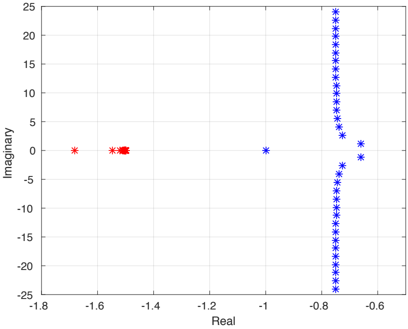
- $\mathrm{Ran}\Phi_\tau$ contains smooth functions outside $\mathcal{O}$.
- Consider the free dynamics in $\mathbb{R}^d \times (0, \tau)$.
- We can find initial data such that

$$u(t,x) = c_0(t)u_0(x) + \text{ regular terms }, \qquad c_0 \in C^\infty[0,\tau].$$

- Possible because of accumulating eigenvalue in the Fourier variable.
- Similar result holds for for several mode Jeffreys system.

Spectrum of multimode Maxwell system

$N = 2$, $\eta = 0$.



• In general $(N-1)$ convergent real eigenvalue and one pair of complex eigenvalue (like wave). • $N = 1 \sim$ Wave.

Theorem (M,Mitra , Renardy, 18)

$\mathcal{O} \subset \Omega$. *Multimode Maxwell system is not null controllable in any time* $\tau > 0$.

- The solution can not be smooth at any time τ. (Propagation of singularities, uniqueness theorem ...)
- For simplicity $d = 2$, $N = 2$. Set $a = \operatorname{curl} u$, and $b_i = \operatorname{curl} v_i$.
- (a, b_i) solves

$$\begin{cases} \rho a_t \ = \ \displaystyle\sum_{i=1}^{N} \triangle b_i + \operatorname{curl} f & x \in \Omega \\ (b_i)_t + \lambda_i b_i = \kappa_i a & x \in \Omega \\ a(\cdot, 0) = a_0 = \operatorname{curl} u_0, \quad b_i(\cdot, 0) = b_{i,0} = \operatorname{curl} v_{i,0}. \end{cases}$$

- a can not be smooth in any time τ.

Sketch of the proof

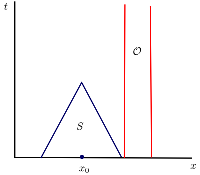
Step 1. Propagation of Singularities in $\mathbb{R}^d$.

$$\begin{cases} \rho \widetilde{a}_t \ = \ \displaystyle\sum_{i=1}^{N} \triangle \widetilde{b}_i & \text{in } (0,\tau)\times\mathbb{R}^d, \\ (\widetilde{b}_i)_t + \lambda_i \widetilde{b}_i = \kappa_i \widetilde{a} & \text{in } (0,\tau)\times\mathbb{R}^d, \\ \widetilde{a}(0) = \widetilde{a}_0, \quad \widetilde{b}_i(\cdot,0) = \widetilde{b}_{i,0}. \end{cases}$$

$x_0 \in \Omega \setminus \mathcal{O}$. Construct $(\widetilde{a}_0, \widetilde{b}_{i,0})$ such that $\widetilde{a}(t,\cdot)$ has singularity at x_0 for any $t > 0$.

Step 2. Singularity of a from singularity of $\widetilde{a}$. Holmgreen's uniqueness theorem.

If a_0 and $\widetilde{a}_0$ agrees in a small neighbourhood around x_0, then there exists a region S such that $a = \widetilde{a}$ in S.



Sketch of the proof

Step 3. From Step 2, $a(t,\cdot)$ has singularity for $t \leqslant T_0$. Singularity of a propagates for all the time. Due to Hörmander.
The line segment (t, x_0) does not meet the observation region.

Step 4. Construct $u_0, v_{i,0}$ such that $a_0 = \mathrm{curl}\, u_0, ...$

How to obtain Null controllability : Using moving control for the velocity field. Possible with control on both velocity and the tensor.

Thank You