Control and free boundaries

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About me

- BSc & MSc, Applied Mathematics, Université de Bordeaux
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- PhD Project: *Control and simulation of free boundary problems*
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Free boundary problems

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  where (T, s) are unknown, while $s_0 \geq 0$ and (f, T_0) are given
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  where (T, s) are unknown, while $s_0 \geq 0$ and (f, T_0) are given
- Model for the melting of a block of ice inside a container filled with water
- The *Stefan condition* describes the motion of the melting interface.

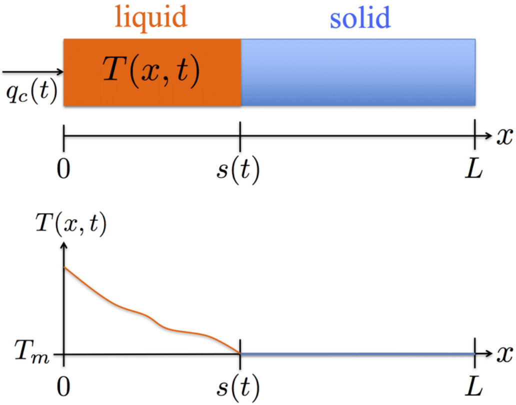

Figure: T is the temperature and s is the melting front. In our case, $q_c, T_m \equiv 0$.

Basics on parabolic equations

- The canonical example is the *heat equation*

$$\begin{cases} y_t - \Delta y = f \mathbb{1}_\omega & \text{in } (0,T) \times \Omega \\ y = 0 & \text{on } (0,T) \times \partial\Omega \\ y = y_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with C^2 boundary, $\omega \subset \Omega$, $\mathbb{1}$ is the indicator function, and (f, y_0) are given

- *Smoothing effect*: $f \mathbb{1}_\omega \equiv 0$ on $\Omega \setminus \omega \implies y(t,\cdot) \in C^\infty(\Omega \setminus \omega)$ for $t > 0$, even if $y_0 \in L^2(\Omega)$.

Controllability of the heat equation

There are multiple concepts of controllability, the "basic" one being

Definition (Exact controllability at time $T > 0$)

For any $y_0, y_1 \in L^2(\Omega)$, there exists $f \in L^2((0,T) \times \Omega)$ such that the solution y to (1) satisfies

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A few remarks are in order:

- Smoothing effect $\Rightarrow$ if $\omega \neq \Omega$, then (1) is not exactly controllable.
- Can we steer y to specific targets, such as $y_1 \equiv 0$? This is the problem of *null-controllability*.

Null-controllability of the heat equation

Hilbert Uniqueness Method: null-controllability at time $T > 0$ is equivalent to: $\exists C = C(T) > 0$ such that for all $\varphi^T \in L^2(\Omega)$,

$$\boxed{\int_\Omega |\varphi(x,0)|^2 dx \leq C \int_0^T \int_\Omega |\varphi(x,t)|^2 dxdt,}$$

where φ is the solution to the *adjoint problem*

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } (0,T) \times \Omega \\ \varphi = 0 & \text{on } (0,T) \times \partial\Omega \\ \varphi(T,\cdot) = \varphi^T & \text{in } \Omega. \end{cases}$$

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This is called an *observability inequality*, and we say that the adjoint problem is (final-state) *observable*.

Null-controllability of the heat equation

How to prove the observability inequality?

- Fourier techniques: if eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ known, roughly check if $\lambda_{k+1} - \lambda_k > 0$ for all $k \in \mathbb{N}$
- Biorthogonals: technical condition, again using eigenvalues
- Carleman inequalities

Remark: Distributed $\Longrightarrow$ boundary null-controllability (observability) for parabolic problems.

Null-control of the Stefan problem

E.Férnandez-Cara et al. (2016): null-controllability roughly by means of the scheme:

1. Fix $s \in C^1([0, T])$, and consider

$$\begin{cases} y_t - y_{xx} = f\mathbb{1}_\omega & \text{for } t \geq 0,\ 0 < x < s(t) \\ y(0,t) = y(s(t),t) = 0 & \text{for } t \geq 0 \\ y(x,0) = y_0(x) & \text{for } 0 < x < s_0. \end{cases} \tag{2}$$

   Notice that we have removed the Stefan condition.

2. Prove that (2) is null-controllable: HUM + Carleman inequality.
3. Transfer this knowledge to the free boundary problem by means of a Schauder fixed-point theorem applied to the map

$$\Lambda : s(t) \mapsto s_0 - \int_0^t y_x(s(\tau),\tau)d\tau.$$

A different strategy

Liu, Takahashi and Tucsnak (2013): null-controllability for

$$\begin{cases} v_t - v_{xx} + vv_x = u\mathbb{1}_\omega & \text{for } t \geq 0,\ x \in (-1,1) \setminus \{h(t)\} \\ v(\pm 1, t) = 0 & \text{for } t \geq 0 \\ \dot{h}(t) = v(h(t), t) & \text{for } t \geq 0 \\ \ddot{h}(t) = [v_x](h(t), t) & \text{for } t \geq 0 \\ h(0) = h_0, \quad \dot{h}(0) = h_1, & \\ v(x, 0) = v_0(x), & \text{for } x \in (-1,1) \setminus \{h_0\}. \end{cases}$$

- Model for the motion of a single particle in a viscous fluid occupying the pipe $(-1, 1)$
- v represents the fluid velocity and h the position of the particle
- Null-controllability result includes $h(T) = 0$, $\dot{h}(T) = 0$.

1. For $t \geq 0$, change of variable to fix the domain

$$\eta(\cdot, t) : (-1,1) \setminus \{h(t)\} \to (-1,1) \setminus \{0\}$$

yielding a nonlinear problem written in Cauchy-form

$$\begin{cases} \dot{z}(t) = Az(t) + B\hat{u}(t) + N \begin{bmatrix} z \\ h \end{bmatrix} \\ \dot{h}(t) = Cz(t) \\ z(0) = z_0 \\ h(0) = h_0. \end{cases}$$

2. Consider the linear problem: replace $N \begin{bmatrix} z \\ h \end{bmatrix}$ by f
3. Prove null-control. of the linear problem with $f \equiv 0$ using parabolic techniques
4. Transfer null-control. result to problem with $f \not\equiv 0$ if f has decay properties (called *source term method*)
5. N is a contraction $\Longrightarrow$ Banach's fixed-point.

The *porous medium equation*:

$$u_t = \Delta u^m \quad \text{on } (0,\infty)\times\mathbb{R}^d,$$

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- Can be written as a scalar conservation law

$$u_t + \nabla\cdot\Big(-u\nabla\frac{m}{m-1}u^{m-1}\Big) = 0;$$

density is advected by negative gradient of the *pressure* $\frac{m}{m-1}u^{m-1}$.

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- If u_0 has compact support $\implies$ free boundaries appear:

$$\Gamma = \partial\overline{\text{supp} u}.$$

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  The "empty" region $\{u=0\}$ and where there is gas $\{u>0\}$ are separated by these interfaces.
- In $d=1$, Γ consists of two Lipschitz curves $s_1(t)$, $s_2(t)$.

Barenblatt profile

We look to linearize around the explicit, source-type solution:

$$u_*(x,t) = \frac{1}{t^{d\alpha}}\left(1 - \frac{\alpha(m-1)}{2m}\frac{|x|^2}{t^{2\alpha}}\right)_+^{\frac{1}{m-1}}, \quad \alpha = \frac{1}{d(m-1)+2}.$$

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- We rescale: $x = \beta t^{\alpha}\hat{x}$ and $t = \exp(\alpha^{-1}\hat{t})$,

$$\hat{u} = (\alpha\beta^2)^{1/(m-1)} t^{-\alpha d}\hat{u},$$

  with $\beta = \sqrt{\frac{2m}{\alpha(m-1)}}$.

- This renders Barenblatt stationary:

$$\hat{u}_*(\hat{x}) = \frac{m-1}{2m}(1 - |\hat{x}|^2)_+^{\frac{1}{m-1}}.$$

- The equation in the new variables reads

$$\hat{u}_{\hat{t}} - \Delta\hat{u}^m - \nabla\cdot(\hat{x}\hat{u}) = 0, \qquad \hat{x}\in\mathbb{R}^d.$$

We remove the hats. We recall the rescaled equation

$$u_t - \Delta u^m - \nabla \cdot (xu) = 0, \qquad x \in \mathbb{R}^d.$$

- It is convenient to work with the pressure: $v = \frac{m}{m-1} u^{m-1}$. We set $\sigma = \frac{2-m}{m-1}$ (recall that $m > 1$).
- The equation for v reads

$$v_t - v\Delta v - (\sigma + 1)(|\nabla v|^2 + x \cdot \nabla v) - dv = 0, \qquad x \in \mathbb{R}^d.$$

- the Barenblatt pressure reads

$$\rho(x) = \frac{1}{2}(1 - |x|^2)_+.$$

Fixing the domain (Koch's thesis, 1999)

For any $t \geq 0$, we make a change of spatial variable (*von Mises transformation*)

$$z = \frac{x}{\sqrt{2v(t,x) + |x|^2}}.$$

- Now $z \in B_1$ since $v(t,x) = 0$ for $|x| \to \infty$ ($B_1 =$ unit disk)
- Also, if $v = \rho$, then $z = x$.

Set

$$1 + w(t,z) = \sqrt{2v(t,x) + |x|^2}.$$

The equation for w reads

$$w_t - \rho^{-\sigma}\nabla \cdot (\rho^{\sigma+1}\nabla w) = N[w] \quad \text{on } B_1 \times (0,\infty),$$

where $N[w] = N[w, \nabla w]$ is the nonlinearity.

The linearized problem

The preceding computations lead us to analyze/control the linear problem

$$\begin{cases} w_t - \rho^{-\sigma}\nabla\cdot(\rho^{\sigma+1}\nabla w) = f & \text{in } B_1\times(0,\infty) \\ w(0,\cdot) = w_0 & \text{in } B_1. \end{cases}$$

- As $\rho(z) = \frac{1}{2}(1-|z|^2)$, the linear operator

$$\boxed{\mathcal{L} = \rho^{-\sigma}\nabla\cdot(\rho^{\sigma+1}\nabla) = -\rho\Delta + (\sigma+1)z\cdot\nabla}$$

  is not elliptic as it degenerates at the boundary ∂B_1.
- Well-posedness in which space? What about boundary conditions?

Functional setting

Let $d\mu_\sigma = \rho^\sigma dx$ with $\sigma > 0$. We consider the weighed Lebesgue space

$$L^2_\sigma = \left\{ u : \int_{B_1} |u|^2 d\mu_\sigma < \infty \right\},$$

and the weighed homogeneous Sobolev space

$$\dot{H}^1_{\sigma+1} = \left\{ u \in L^1_{loc}(B_1) : \int_{B_1} |\nabla u|^2 d\mu_{\sigma+1} < \infty \right\}.$$

We will consider $\mathcal{L} = -\rho\Delta + (\sigma+1)x \cdot \nabla$ as an *unbounded operator* on both spaces.

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Lemma (Hardy-Poincaré-Wirtinger inequality, Seis 2016)

There exists $C > 0$ such that

$$\inf_{c\in\mathbb{R}} \int_{B_1} |u-c|^2 d\mu_\sigma \leq C \int_{B_1} |\nabla u|^2 d\mu_{\sigma+1} \quad \forall u \in \dot{H}^1_{\sigma+1}.$$

The operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to L^2_\sigma$

Integration by parts shows that

$$\int_{B_1} \mathcal{L} uv d\mu_\sigma = \int_{B_1} \nabla u \cdot \nabla v d\mu_{\sigma+1} \quad \forall u, v \in C^\infty(\overline{B_1}).$$

- Thus $\mathcal{L}_{|C^\infty(\overline{B_1})}$ is nonnegative and symmetric w.r.t. $\langle \cdot, \cdot \rangle_{L^2_\sigma}$
- $C^\infty(\overline{B_1})$ dense in $\dot{H}^1_{\sigma+1}$, we don't know if $C^\infty(\overline{B_1})$ dense in $L^2_\sigma \dots$

Lemma (Seis 2016)

For all $f \in L^2_\sigma$ s.t. $\int_{B_1} f dx = 0$, there exists $u \in H^2_{loc}(B_1) \cap \dot{H}^1_{\sigma+1}$ s.t.

$$\begin{cases} -\nabla \cdot (\rho^{\sigma+1}) = f & \text{in } B_1 \\ \rho^{\sigma+1} \nabla u \cdot \nu = 0 & \text{on } \partial B_1. \end{cases}$$

The solution u is unique up to an additive constant.

The operator $\mathcal{H} : \mathcal{D}(\mathcal{H}) \to \dot{H}^1_{\sigma+1}$

Integration by parts shows that

$$\int_{B_1} \nabla(\mathcal{H}u)\cdot\nabla v d\mu_{\sigma+1} = c\int_{B_1} \nabla\cdot(\rho^{\sigma+1}\nabla u)\nabla\cdot(\rho^{\sigma+1}\nabla v)d\mu_\sigma \quad \forall u, v \in C^\infty(\overline{B}_1).$$

- Thus $\mathcal{H}_{|C^\infty(\overline{B_1})}$ is nonnegative and symmetric w.r.t. $\langle\cdot,\cdot\rangle_{H^1_{\sigma+1}}$
- By density $\Longrightarrow$ may be extended to a self-adjoint operator on $\dot{H}^1_{\sigma+1}$ with domain

$$\mathcal{D}(\mathcal{H}) = \{u \in H^3_{loc} \cap \dot{H}^1_{\sigma+1} : \mathcal{H}u \in \dot{H}^1_{\sigma+1},\ \rho^{\sigma+1}\nabla u \cdot \nu = 0 \text{ on } \partial B_1\}.$$

- The boundary condition $\rho^{\sigma+1}\nabla u \cdot \nu = 0$ on ∂B_1 interpreted in the sense

$$\int_{B_1} \nabla u \cdot \nabla v \rho^{\sigma+1} dx = -\int_{B_1} u\nabla\cdot(\rho^{\sigma+1}\nabla v)dx \quad \forall v \in \dot{H}^1_{\sigma+1}.$$

Null-controllability

Theorem (Seis 2016)

The spectrum of both $\mathcal{L}$ and $\mathcal{H}$ is discrete,

$$\Sigma(\mathcal{L}) = \Sigma(\mathcal{H}) \cup \{0\},$$

and the eigenvalues of $\mathcal{H}$ are

$$\lambda_{\ell,k} = (\sigma+1)(\ell+2k) + k(2k+2\ell+d+2)$$

where $(\ell, k) \in \mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}$ if $d \geq 2$ and $(\ell, k) \in \{0,1\} \times \mathbb{N} \setminus \{(0,0)\}$ if $d = 1$.

Here $\mathbb{N} = \{0, 1, \dots\}$. An indicator if null-controllability holds is to check for a gap between consecutive eigenvalues: $\exists \gamma > 0$ s.t.

$$\lambda_{j+1} - \lambda_j \geq \gamma \quad \text{for all } j \in \mathbb{N}$$

after relabeling the eigenvalues $\{\lambda_{\ell,k}\}_{\ell,k} = \{\lambda_j\}_j$.

Perspectives

- Clarify the well-posedness for the linear problem: what are the boundary conditions (if any), what is the adequate functional setting
- Check if null-controllability can hold, in $d = 1$ the spectrum seems easier to manipulate.
- Consider alternative control/optimization properties that may be studied
- A numerical study may hint possible results.

Thank you for your welcome and attention.