Control of free boundary problems

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About me

- BSc & MSc, Applied Mathematics, Université de Bordeaux.
- PhD, *Control and simulation of free boundary problems*, Universidad Autónoma de Madrid, since July 2018.
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Free boundary problems

- Also known as *moving interface problems*
- **Unknowns:** the **state** and a **part of the boundary of the domain**

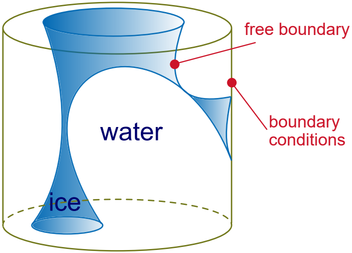

Figure: *The Stefan problem.*

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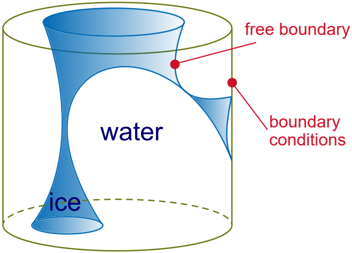

$0 < x < s(t)$	is the water
$s(t) < x$	is the ice
$y_t - y_{xx} = 0$	in the water
$\dot{s}(t) = -y_x$	at melting $x = s(t)$

Figure: *The Stefan problem.*

- Null-controllability (Fernandez-Cara et al. '16), feedback stabilization (Krstic et al. '16).

Fluid-structure interaction

Free boundary (moving interface) problems appear naturally in FSI:

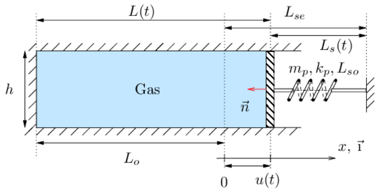

Figure: *The piston problem.*

Null-controllability (Liu, Takahashi, Tucsnak '13), feedback stabilization (Takahashi et al. '14); related studies in dimension 2.

Porous medium equation

For $m > 1$,

$$\rho_t = \Delta\rho^m \quad \text{on } \mathbb{R}^d \times (0, \infty),$$

$\rho \geq 0$ is a *density*.

- Can be written as a scalar conservation law

$$\rho_t + \nabla \cdot \left(-\rho\nabla \frac{m}{m-1}\rho^{m-1}\right) = 0;$$

  ρ is advected by a velocity $=$ to negative gradient of the *pressure* (Darcy's law).
- *Nonlinear, degenerate* diffusion: $\Delta\rho^m = \nabla \cdot (m\rho^{m-1}\nabla\rho)$.
- ρ_0 *compact support* $\implies$ *finite speed of propagation* and *free boundary* $\partial\{\rho > 0\}$.
- In $d = 1$, $\{\rho(t) > 0\} = (s_1(t), s_2(t))$ for all $t > 0$, with s_i Lipschitz; Darcy's law holds for the interface velocity.

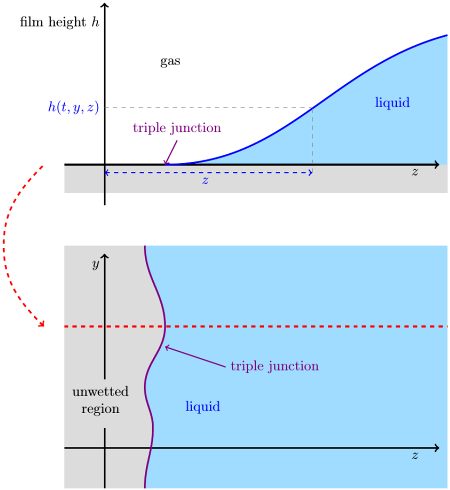

Figure: A thin film spreading along a solid surface.

Transformation and linearization

Goal: Linearize around the Barenblatt solution (Koch '99):

$$\rho_*(x,t) = \frac{1}{t^{d\alpha}}\left(1 - \frac{\alpha(m-1)}{2m}\frac{|x|^2}{t^{2\alpha}}\right)_+^{\frac{1}{m-1}}, \quad \alpha = \frac{1}{d(m-1)+2}.$$

1. **Rescale**: Barenblatt becomes stationary, $a(x) = \frac{1}{2}(1-|x|^2)_+$.
2. **Change of variable**: fix the domain $\{\rho(t) > 0\}$ to B_1 (*von Mises transform*).
3. Deduce a **transformed nonlinear equation**:

$$\begin{cases} y_t - a^{-\sigma}\nabla\cdot(a^{\sigma+1}\nabla y) = \mathcal{N}[y] & \text{in } B_1 \\ a^{\sigma+1}\nabla y \cdot \nu = 0 & \text{on } \partial B_1 \end{cases}$$

where $\sigma = \sigma(m) > 0$.

Problem: We need a trace theory (not justified in Seis JDE '15).

Null-controllability at time $T > 0$

We act by means of a distributed control localized on $\omega \subset B_1$:

$$\begin{cases} y_t = \mathcal{L}y + \mathcal{N}[y] + f\mathbb{1}_\omega(x) & \text{in } B_1 \\ a^{\sigma+1}\nabla y \cdot \nu = 0 & \text{on } \partial B_1, \end{cases}$$

where $\mathcal{L} = a^{-\sigma}\nabla \cdot (a^{\sigma+1}\nabla y)$.

1. **First**: null-controllability of the linear homogeneous problem ($\mathcal{N}[y] = g = 0$)
2. **Second**: null-controllability of the linear problem with source term ($\mathcal{N}[y] = g$), with g decaying sufficiently fast w.r.t. control cost as $t \to T$
3. **Third**: Contraction mapping principle ...

Other questions: bounded controls (to trajectories) to preserve positivity, extension and control to more sophisticated systems.

Thank you for your attention.