Control of linearized porous medium flow

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Homogenization, Spectral problems and other topics in PDEs
ICMAT, 6-7 May, 2019

Scope of the talk

Given $T > 0$, our interest is the *controllability problem*: for any y_0, find $u = u(t,x)$ such that the solution y to

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = u\mathbb{1}_\omega + f[y] & \text{in } (0,T)\times(-1,1) \\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,\cdot) = y_0(\cdot) & \text{in } (-1,1) \end{cases} \tag{1}$$

satisfies

$$y(T,\cdot) = 0 \quad \text{in } (-1,1).$$

- Here $\sigma > 0, \rho(x) = \frac{1}{2}(1-x^2)$, $\omega = (a,b) \subset (-1,1)$ and f is an explicit, nonlinear term (given below).
- In this talk, we will only consider the linearized problem, replacing $f[y]$ by a source term f.

Motivation

For $m > 1$,

$$\partial_t h = \partial_z^2(h^m) \quad \text{in } \{h > 0\},$$

$h \geq 0$ is a gas density or height of thin film.

- *Nonlinear, degenerate* diffusion:

$$h = 0 \quad \Longrightarrow \quad \partial_z(h^{m-1}\partial_z h) = 0.$$

- *Free boundary* $\partial\{h(t) > 0\}$ is liquid-solid interface. Darcy's law $V = -\partial_z h^{m-1}$ gives the normal velocity V.
- *Controllability*: may we steer the film's height to a parabolic shape in finite time $T > 0$?

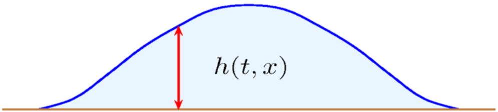

Figure: A two-dimensional thin fluid film.

Target is the Barenblatt self-similar solution

$$h_*(t,z) = \frac{1}{t^\alpha}\left(1 - \frac{\alpha(m-1)}{2m}\frac{|z|^2}{t^{2\alpha}}\right)_+^{\frac{1}{m-1}} \quad z \in \mathbb{R}.$$

It is convenient to consider the problem in

- Self-similar coordinates

$$z = \beta t^\alpha \hat{z}, \quad t = \exp(\alpha^{-1}\hat{t})$$

and

$$h(t,z) = \frac{1}{t^\alpha}(\alpha\beta^2)^{\frac{1}{m-1}}\hat{h}(\hat{t},\hat{z})$$

where $\beta = \sqrt{\frac{2m}{\alpha(m-1)}}$.

- Pressure variable: $v = \frac{m}{m-1}\hat{h}^{m-1}$.
- Barenblatt is now the parabola $\rho(\hat{z}) = \frac{1}{2}(1-\hat{z}^2)_+$.

Pressure equation in self-similar coordinates:

$$\partial_t v - v\partial_z^2 v - (\sigma+1)((\partial_z v)^2 + z\partial_z v) - v = 0 \quad \text{in } \{v > 0\}$$

where $\sigma(m) > 0$. We seek to linearize around $\rho(z) = \frac{1}{2}(1-z^2)_+$.

- Lagrangian-like change of variable (H. Koch '99, C. Seis '15):

$$x = \frac{z}{\sqrt{2v(t,z)+z^2}}, \quad y(t,x) = \sqrt{2v(t,z)+z^2} - 1.$$

  In the new variable x, Barenblatt is the constant 1.
- Yields perturbation equation (recall Problem (1))

$$\partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = \rho F[y] - \rho^{-\sigma}\partial_x(\rho^{\sigma+1} x F[y]) \quad \text{in } (-1,1)$$

  where $F[y] = \frac{(\partial_x y)^2}{1+y+x\partial_x y}$.

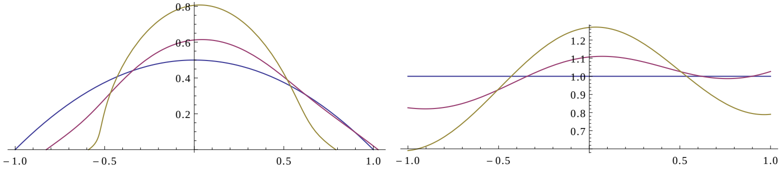

Figure: Barenblatt pressure (blue) and two perturbations in (z, v) coordinates versus corresponding graphs in $(x, 1+y)$ coordinates.

A null-controllability result ($y(T, \cdot) = 0$) of perturbation equation (1) would yield a controllability of the pressure equation to the parabolc shape ($v(T, \cdot) = \rho(\cdot)$).

Functional setting

For $T > 0$, consider the linear degenerate-parabolic equation

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = f & \text{in } (0,T)\times(-1,1) \\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,\cdot) = y_0 & \text{in } (-1,1). \end{cases} \tag{2}$$

- For $k \geq 0$, the weighted Sobolev space $\mathcal{H}^k_\sigma$ consists of all f satisfying

$$\|f\|^2_{\mathcal{H}^k_\sigma} = \sum_{j=0}^{k} \int_{-1}^{1} \rho^{\sigma+j}(\partial_x^j f)^2 dx < \infty,$$

  and denote $\mathcal{H}^0_\sigma = L^2_\sigma = L^2((-1,1), \rho^\sigma dx)$.
- Spaces can be realized as closure of $C^\infty([-1,1])$ w.r.t. the above norm.

The linear differential operator

Well-posedness of the linearized problem will follow from semigroup theory after analysis of the operator $\mathcal{A} = -\rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x)$.

Lemma

Let $k \geq 1$, $\ell \geq 0$ and $\alpha \geq \frac{\sigma+1+\ell-k}{2}$ with $\alpha > 0$. There exists $C(k,\alpha) > 0$ such that

$$\|(1-x^2)^\alpha \partial_x^\ell f\|_{C^0([-1,1])} \leq C\|f\|_{\mathcal{H}_\sigma^{k+\ell}} \qquad \text{for all } f \in C^\infty([-1,1]).$$

True for $\alpha = \sigma + 1$, $\ell = 1$ and $k = 1$ in particular, whence any $f \in \mathcal{H}_\sigma^2$ satisfies $(\rho^{\sigma+1}\partial_x y)(\pm 1) = 0$.

Lemma

The operator $\mathcal{A} : \mathcal{H}_\sigma^2 \to L_\sigma^2$ is self-adjoint, nonnegative, and has compact resolvents.

Well-posedness

In view of what precedes, $\mathcal{A}:\mathcal{H}^2_\sigma\to L^2_\sigma$ generates an analytic semigroup on L^2_σ, and thus

Corollary

For every $y_0\in L^2_\sigma$ and $f\in L^2(0,T;L^2_\sigma)$, there exists a unique weak solution

$$y\in C^0([0,T];L^2_\sigma)\cap L^2(0,T;\mathcal{H}^1_\sigma)$$

to Problem (2)*. If moreover $y_0\in\mathcal{H}^1_\sigma$, the unique solution y is a strong solution and enjoys maximal regularity*

$$y\in L^2(0,T;\mathcal{H}^2_\sigma)\cap H^1(0,T;L^2_\sigma)\cap C^0([0,T];\mathcal{H}^1_\sigma).$$

Controllability of linear equations

Let X, U be two Hilbert spaces, $A : \mathcal{D}(A) \to X$ generates a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ on X and $B \in \mathcal{L}(U, X)$. Consider

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t) & \text{in } (0,T) \\ y(0) = y_0 \in X. \end{cases}$$

Definition (Null-controllability)

We say that (A, B) is null-controllable at time $T > 0$ if for every $y_0 \in X$, there exists $u \in L^2(0,T;U)$ such that the solution $y \in C^0([0,T];X)$ satisfies

$$y(0) = y_0 \quad \text{and} \quad y(T) = 0.$$

For null-controllable (A, B), we call *control cost* the quantity

$$\kappa(T) = \sup_{\|y_0\|_X = 1} \inf_u \|u\|_{L^2(0,T;U)}.$$

Controllability of linear equations

Lemma

Assume A non-positive operator, with an ONB of eigenfunctions $\{\varphi_k\}_{k=0}^{\infty}$ and decreasing sequence of eigenvalues $\{-\lambda_k\}_{k=0}^{\infty}$ satisfying

$$\inf_{k\geq 0}(\lambda_{k+1}-\lambda_k)>0$$

$$\lambda_k = rk^2 + O(k)$$

for some $r>0$ as $k\to\infty$. Assume U separable Hilbert space and there exists $m>0$ such that

$$\|B^*\varphi_k\|_U \geq m$$

for all $k\in\mathbb{N}$. Then (A,B) is null-controllable in any time $T>0$.

Key idea. Null-controllability $\iff$ observability inequality:

$$\|e^{tA^*}y_0\|_X^2 \leq \mathcal{C}(T)^2\int_0^T \|B^*e^{tA^*}y_0\|_U^2 dt.$$

Recall that we are interested in proving the null-controllability of the problem

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = u\mathbb{1}_\omega + f & \text{in } (0,T)\times(-1,1) \\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,\cdot) = y_0(\cdot) & \text{in } (-1,1). \end{cases} \quad (3)$$

Let us first assume $f \equiv 0$.

Theorem

For any $y_0 \in L^2_\sigma$, Problem (3) with $f \equiv 0$ is null-controllable. That is to say, there exists $u \in L^2((0,T)\times\omega)$ such that $y \in C^0([0,T];L^2_\sigma)$ satisfies

$$y(0,\cdot) = y_0 \quad \textit{and} \quad y(T,\cdot) = 0.$$

Sketch of proof. Seis ('15, preprint) computes the spectrum of $\mathcal{A}$:

$$\lambda_k = \begin{cases} 2k^2 + k(1+2\sigma) & \text{if } k \text{ is even} \\ 2k^2 + k(1+2\sigma) + (\sigma + 2k + 1) & \text{if } k \text{ is odd} \end{cases}$$

and

$$\varphi_k(x) = c_k \begin{cases} P_k^{(-\frac{1}{2},\sigma)}(1-2x^2) & \text{if } k \text{ is even} \\ xP_k^{(\frac{1}{2},\sigma)}(1-2x^2) & \text{if } k \text{ is odd} \end{cases}$$

for $k \geq 0$, where

$$c_k^2 = \begin{cases} \dfrac{2^\sigma k!(2k+\sigma+\frac{1}{2})\Gamma(k+\sigma+\frac{1}{2})}{\Gamma(k+\frac{1}{2})\Gamma(k+\sigma+1)} & \text{if } k \text{ is even} \\ \dfrac{2^\sigma k!(2k+\sigma+\frac{3}{2})\Gamma(k+\sigma+\frac{3}{2})}{\Gamma(k+\frac{3}{2})\Gamma(k+\sigma+1)} & \text{if } k \text{ is odd.} \end{cases}$$

$P_k^{(\pm\frac{1}{2},\sigma)}$ are *Jacobi polynomials*, in this instance polynomials of order $2k$ and $2k+1$ respectively.

The eigenvalues satisfy both conditions from the previous Lemma.

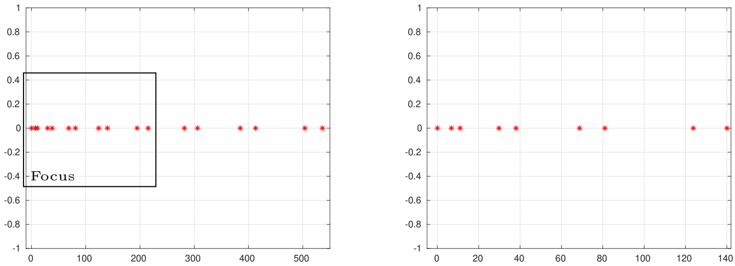

Figure: The gap between consecutive eigenvalues is $\geq 2 + \sigma$ and increases as k grows.

Here $Bu = u\mathbb{1}_\omega$, hence $B^*\varphi = \varphi|_\omega$, where $\omega = (a,b) \subset (-1,1)$. It remains to be seen whether

$$\int_a^b \varphi_k^2 dx \geq m$$

for some $m = m(a,b,\sigma) > 0$ independent of k.

1. As φ_k are polynomials, the lower bound holds for some $m = m(a,b,\sigma,k) > 0$.
2. We study the behavior of this quantity as $k \to \infty$. Using the asymptotic formula

$$P_k^{(\pm\frac{1}{2},\sigma)}(\cos\theta) = \frac{1}{\sqrt{k}}\Big[\frac{1}{\sqrt{\pi}}\sin^{\mp\frac{1}{2}-\frac{1}{2}}(\frac{\theta}{2})\cos^{-\sigma-\frac{1}{2}}(\frac{\theta}{2})\cos(\theta\gamma(k)) + O(k^{-\frac{3}{2}}),$$

with $\gamma(k) = k + \frac{1}{2}(\sigma + 1 \pm \frac{1}{2})$, properties of $\Gamma(z)$ and weak-* L^∞ convergence of $\{\cos(n\cdot)\}_{n\in\mathbb{N}}$ to 0, we are able to conclude.

Controllability in spite of the source term

Now consider the linearized problem

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = u\mathbb{1}_\omega + f & \text{in } (0,T)\times(-1,1) \\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,\cdot) = y_0(\cdot) & \text{in } (-1,1) \end{cases} \tag{4}$$

for non-zero source terms f.

- To keep the controllability result from the homogeneous problem, we will need f to decay sufficiently fast w.r.t. the control cost κ near the final time T.
- Let $\rho_{\mathcal{F}}, \rho_0 : [0,T] \to [0,\infty)$ be two continuous, non-increasing functions satisfying $\rho_{\mathcal{F}}(T) = \rho_0(T) = 0$, constructed from the control cost $\kappa(t)$.
- Consider

$$\mathcal{F} = \left\{ f \in L^2(0,T;(\mathcal{H}^1_\sigma)^*) \colon \frac{f}{\rho_{\mathcal{F}}} \in L^2(0,T;(\mathcal{H}^1_\sigma)^*) \right\}$$

$$\mathcal{U} = \left\{ u \in L^2((0,T)\times\omega) \colon \frac{u}{\rho_0} \in L^2((0,T)\times\omega) \right\}.$$

Adapted from Liu et al. (ESIAM COCV '13):

Theorem

There exists $C = C(T) > 0$ *and a continuous linear map* $\mathfrak{L} : L^2_\sigma \times \mathcal{F} \to \mathcal{U}$ *such that for any* $y_0 \in L^2_\sigma$ *and* $f \in \mathcal{F}$, *the solution* y *of* (4) *with control* $u = \mathfrak{L}(y_0, f)$ *satisfies*

$$\left\| \frac{y}{\rho_0} \right\|_{C^0([0,T];L^2_\sigma)} + \|u\|_{\mathcal{U}} \leq C(\|f\|_{\mathcal{F}} + \|y_0\|_{L^2_\sigma}).$$

In particular, since ρ_0 *is continuous and* $\rho_0(T) = 0$, *the above yields* $y(T, \cdot) = 0$.

Idea of proof.

- Time splitting: $T_k = T - \frac{T}{q^k}$ for fixed $q > 1$.
- Consider equation on (T_k, T_{k+1}) and split $y = y_1 + y_2$ where
  - y_1 solves uncontrolled problem with zero initial data and the source f
  - y_2 solves the controlled problem without f (use previous Theorem).
- The control $u = \sum_{k=0}^{\infty} u_k \mathbb{1}_{[T_k, T_{k+1})}$ is shown to suffice.

The nonlinear problem

Recall that the nonlinear term is of the form

$$f[y] = \rho F[y] - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}xF[y])$$

where $F[y] = \frac{(\partial_x y)^2}{1+y+x\partial_x y}$.

- Let $\eta : [0,\infty) \to [0,1]$ be a smooth cut-off function, supported on $[0,2)$ with $\eta(x) \equiv 1$ on $[0,1]$.
- Fix $0 < \varepsilon, \delta < 1$ and for $p, q \in \mathbb{R}$ set

$$\eta_{\varepsilon,\delta}(p,q) = \eta\Big(\frac{p^2}{\delta^2}\Big)\eta\Big(\frac{p^2}{\varepsilon^2}\Big),$$

  and plug $F_{\varepsilon,\delta} = \eta_{\varepsilon,\delta}F$ in the nonlinear term. The cut-off is inactive whenever y is small enough in $C^{0,1}([0,T]\times[-1,1])$, and allows for contraction estimates.
- Expected result: local null-controllability (small initial data) of the truncated nonlinear problem.

Thank you for your invitation and attention.