Controllability of perturbed porous medium flow

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Scope

Let $T > 0$ and $\omega = (a, b) \subsetneq (-1, 1)$ be non-empty.

Null-controllability problem: for "any" y_0, find $u = u(t, x)$ such that the solution y to

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = u\mathbf{1}_\omega + \mathcal{N}(y, \partial_x y) & \text{in } (0, T) \times (-1, 1) \\ (\rho^{\sigma+1}\partial_x y)(t, \pm 1) = 0 & \text{in } (0, T) \\ y(0, \cdot) = y_0(\cdot) & \text{in } (-1, 1) \end{cases}$$

satisfies $y(T, \cdot) = 0$ in $(-1, 1)$. Here

- $\sigma > -1$,
- $\rho(x) = \frac{1}{2}\left(1 - x^2\right)$,
- $\mathcal{N}(y, \partial_x y) = \rho F - \rho^{-\sigma}\partial_x(\rho^{\sigma+1} x F)$, $\qquad F(y, \partial_x y) = \frac{(\partial_x y)^2}{1+y+x\partial_x y}$.

Motivation

For $m > 1$,

$$\partial_t h - \partial_z^2(h^m) = 0,$$

$h \geq 0$ is a gas density or height of thin film.

- *Nonlinear, degenerate* diffusion:

$$h = 0 \quad \Longrightarrow \quad \partial_z(h^{m-1}\partial_z h) = 0.$$

- Finite speed of propagation $\Longrightarrow$ *free boundary* $\partial\{h(t) > 0\}$.

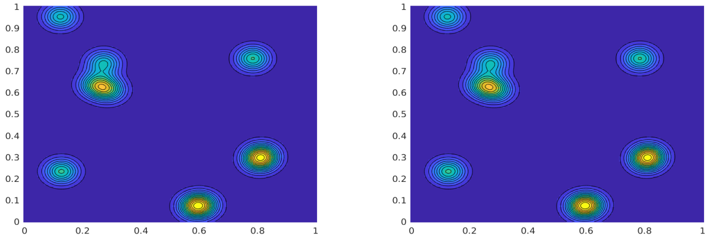

Figure: Linear (fast) versus nonlinear (slow) diffusion.

Motivation

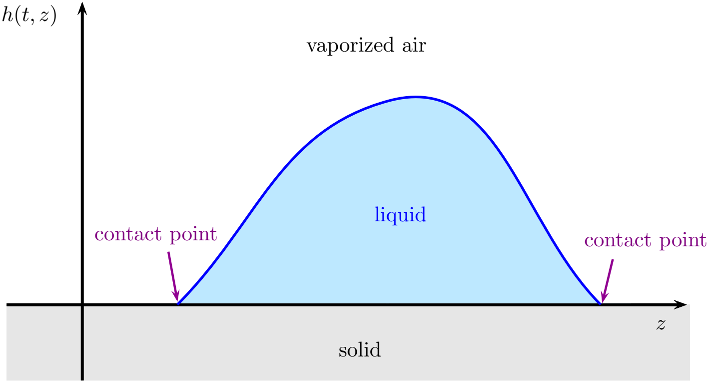

Figure: A droplet spreading along a solid surface.

Motivation

We wish to control the solution and its interface to those of the Barenblatt self-similar solution:

$$h_B(t,z) = (t+1)^{-\frac{1}{m+1}} \left(1 - \frac{m-1}{2m(m+1)} \frac{z^2}{(t+1)^{\frac{2}{m+1}}}\right)^{\frac{1}{m-1}} \quad \text{in } \{h_B > 0\}$$

It is more convenient to consider the problem in self-similar coordinates and pressure variable:

$$\partial_t v - v\partial_z^2 v - (\sigma+1)((\partial_z v)^2 + z\partial_z v) - v = 0 \quad \text{in } \{v > 0\}$$

- Barenblatt is now the parabola $\rho(z) = \frac{1}{2}(1-z^2)$ in $\{\rho > 0\}$.
- Lagrangian-like change of variables (von-Mises transform, Koch '99, Seis '15) to fix the moving domain to $\text{supp}(\rho) = (-1,1)$.
- C^1-diffeomorphism for $C_t^{0,1}C_x^{0,1}$ solutions
- Controllability to Barenblatt in moving domain $\iff$ controllability to zero in fixed.

Setting

For $T > 0$, consider the linear degenerate-parabolic equation

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = f & \text{in } (0,T)\times(-1,1) \\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,\cdot) = y_0(\cdot) & \text{in } (-1,1). \end{cases} \tag{1}$$

- For $k \geq 0$, weighted Sobolev $\mathcal{H}^k$ consists of all $f \in L^1_{\text{loc}}(-1,1)$ s.t.

$$\|f\|^2_{\mathcal{H}^k} := \sum_{j=0}^{k} \int_{-1}^{1} \rho^{\sigma+j}(\partial_x^j f)^2 dx < \infty.$$

- Hence $\mathcal{H}^0 = L^2((-1,1), \rho^\sigma dx)$.
- $C^\infty([-1,1])$ are dense subspaces w.r.t. the above norm.
- Null-controllability works for similar problems considered by Cannarsa, Martinez, Fragnelli, Vancostenoble, . . .

The linear differential operator

Well-posedness of the linearized problem will follow from semigroup theory after analysis of the operator $\mathcal{A} = -\rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x)$.

Lemma

Let $k \geq 1$, $\ell \geq 0$ and $\alpha \geq \frac{\sigma+1+\ell-k}{2}$ with $\alpha > 0$. Then

$$\|(1-x^2)^\alpha \partial_x^\ell f\|_{C^0([-1,1])} \lesssim_{k,\alpha} \|f\|_{\mathcal{H}^{k+\ell}} \qquad \textit{for all } f \in C^\infty([-1,1]).$$

True for $\alpha = \sigma + 1$, $\ell = 1$ and $k = 1$ in particular, whence any $f \in \mathcal{H}^2$ satisfies $(\rho^{\sigma+1}\partial_x y)(\pm 1) = 0$.

Proposition

The operator $\mathcal{A} : \mathcal{H}^2 \to \mathcal{H}^0$ is self-adjoint, nonnegative, and has compact resolvents.

Well-posedness

In view of what precedes, $\mathcal{A} : \mathcal{H}^2 \to \mathcal{H}^0$ generates an analytic semigroup on $\mathcal{H}^0$, and thus

Corollary

For every $y_0 \in \mathcal{H}^0$ and $f \in L^2(0,T;\mathcal{H}^0)$, there exists a unique weak solution

$$y \in L^2(0,T;\mathcal{H}^1) \cap C^0([0,T];\mathcal{H}^0)$$

to Problem (1). *If moreover $y_0 \in \mathcal{H}^1$, the unique solution y is a strong solution and*

$$y \in L^2(0,T;\mathcal{H}^2) \cap C^0([0,T];\mathcal{H}^1).$$

Controllability of linear equations

Let X, U be two Hilbert spaces, $A : \mathcal{D}(A) \to X$ generates a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ on X and $B \in \mathcal{L}(U, X)$. Consider

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t) & \text{in } (0, T) \\ y(0) = y_0 \in X. \end{cases}$$

Definition

For null-controllable (A, B), we call *control cost* the quantity

$$\kappa(T) = \sup_{\|y_0\|_X = 1} \inf_u \|u\|_{L^2(0,T;U)}.$$

Controllability of linear equations

Lemma (Fattorini-Russell)

Assume A self-adjoint, non-negative operator, with an ONB of eigenfunctions $\{\varphi_k\}_{k=0}^{\infty}$ and decreasing sequence of eigenvalues $\{-\lambda_k\}_{k=0}^{\infty}$ satisfying

$$\inf_{k\geq 0}(\lambda_{k+1}-\lambda_k)>0$$

$$\lambda_k = rk^2 + O(k)$$

for some $r>0$ as $k\to\infty$. Assume U separable Hilbert space and there exists $\mu>0$ such that

$$\|B^*\varphi_k\|_U \geq \mu$$

for all $k\geq 0$. Then (A,B) is null-controllable in any time $T>0$.

Recall that we are interested in proving the null-controllability of the linearized problem

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,\cdot) = y_0(\cdot) & \text{in } (-1,1). \end{cases} \tag{2}$$

Theorem

For any $y_0 \in \mathcal{H}^0$, Problem (2) is null-controllable. That is to say, there exists $u \in L^2((0,T)\times\omega)$ such that $y \in C^0([0,T];\mathcal{H}^0)$ satisfies

$$y(T,\cdot) = 0 \quad \textit{in } (-1,1).$$

Theorem (Angenent '90, Seis '14)

The spectrum of $\mathcal{A}$ *consists of simple nonnegative eigenvalues* $\{\lambda_k\}_{k=0}^{\infty}$*, given by*

$$\lambda_k = \frac{k^2}{2} + \frac{k}{2}(1+2\sigma)$$

for $k \geq 0$*. The corresponding eigenfunctions* $\{\varphi_k\}_{k=0}^{\infty}$ *are of the form*

$$\varphi_k(x) = {}_2F_1\Big(-\frac{k}{2}, \sigma + \frac{k}{2} + \frac{1}{2}, \frac{1}{2}, x^2\Big) \quad \text{if } k \text{ is even}$$

and

$$\varphi_k(x) = {}_2F_1\Big(-\frac{k-1}{2}, \sigma + \frac{k}{2} + 1, \frac{3}{2}, x^2\Big)x \quad \text{if } k \text{ is odd}$$

for $x \in (-1,1)$*.*
In particular, $\lambda_0 = 0$ *with associated eigenfunction* $\varphi_1(x) = 1$ *since constants are in the domain of* $\mathcal{A}$*.*

Controllability in spite of the source term

Now consider

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = u\mathbf{1}_\omega + f & \text{in } (0,T)\times(-1,1) \\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,\cdot) = y_0(\cdot) & \text{in } (-1,1) \end{cases} \tag{3}$$

for non-zero source terms f.

- To keep the controllability result from the homogeneous problem, we will need f with decay quick enough near the final time compared to the control cost in small time.
- Let $\theta_{\mathcal{F}}, \theta_0 : [0,T] \to [0,\infty)$ be two continuous, non-increasing functions s.t. $\theta_{\mathcal{F}}(T) = \theta_0(T) = 0$, constructed from the control cost.
- Consider

$$\mathcal{F} = \left\{ f \in L^2(\mathcal{H}^0) \colon \frac{f}{\theta_{\mathcal{F}}} \in L^2(\mathcal{H}^0) \right\}$$

$$\mathcal{U} = \left\{ u \in L^2(L^2(\omega)) \colon \frac{u}{\theta_0} \in L^2(L^2(\omega)) \right\}.$$

The source-term method

Theorem (Liu, Takahashi, Tucsnak (COCV '13))

There exists $C_T > 0$ and a continuous linear map $\mathfrak{L} : \mathcal{H}^1 \times \mathcal{F} \to \mathcal{U}$ s.t. for any $y_0 \in \mathcal{H}^1$ and $f \in \mathcal{F}$, the solution y of (3) *with control $u = \mathfrak{L}(y_0, f)$ satisfies*

$$\left\| \frac{y}{\theta_0} \right\|_{C^0([0,T];\mathcal{H}^1)} + \left\| \frac{y}{\theta_0} \right\|_{L^2(0,T;\mathcal{H}^2)} + \|u\|_{\mathcal{U}} \leq C_T \big(\|f\|_{\mathcal{F}} + \|y_0\|_{\mathcal{H}^1} \big).$$

Since θ_0 is continuous and $\theta_0(T) = 0$, this yields $y(T, \cdot) = 0$.

Has since been adapted by Le Balch '18, Beauchard - Marbach '18 . . .

The nonlinear problem

With only an $L^2(L^2(\omega))$-regular control, we cannot ensure that $y \in C^{0,1}([0,T] \times [-1,1])$ so to control the denominator in

$$\mathcal{N}(y, \partial_x y) = \rho F - \rho^{-\sigma} \partial_x (\rho^{\sigma+1} x F), \qquad F(y, \partial_x y) = \frac{(\partial_x y)^2}{1 + y + x \partial_x y}.$$

What can be done?

- Let $\chi : [0, \infty) \to [0,1]$ be a smooth cut-off function, supported on $[0,2)$ with $\chi(x) \equiv 1$ on $[0,1]$.
- Fix $\varepsilon, \delta > 0$ with $2(\varepsilon + \delta) < 1$ and for $p, q \in \mathbb{R}$ set

$$F_{\varepsilon,\delta}(p,q) = \chi\left(\frac{p^2}{\delta^2}\right) \chi\left(\frac{q^2}{\varepsilon^2}\right) F(p,q),$$

The cut-off is inactive whenever y is small enough in $C^{0,1}([0,T] \times [-1,1])$.

The nonlinear problem

We consider:

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = \rho F_{\varepsilon,\delta}(y,\partial_x y) + u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,x) = y_0(x) & \text{in } (-1,1). \end{cases} \tag{4}$$

Theorem

Let $\sigma \in (-1,0)$. There exists $r > 0$ such that for every $y_0 \in \mathcal{H}^1$ satisfying $\|y_0\|_{\mathcal{H}^1} \le r$, there exists a control $u \in L^2(0,T;L^2(\omega))$ for which the unique solution $y \in L^2(0,T;\mathcal{H}^2) \cap C^0([0,T];\mathcal{H}^1)$ of (4) satisfies

$$y(T,\cdot) = 0.$$

Key ingredients in proof:

- $\frac{\theta_0^2}{\theta_{\mathcal{F}}}$ is continuous on $[0,T]$
- $\|\sqrt{\rho}\,\partial_x y\|_{C^0[-1,1]} \lesssim_\sigma \|y\|_{\mathcal{H}^2}$ for $\sigma \in (-1,0)$.

Perspectives

Recap: This is the most we can do with L^2-regular controls.

- Existence of a regular control (L^∞ at least) in order to ensure the required regularity (if we have maximal $L^p(L^q)$ regularity) of the state to remove the cut-off and control the full nonlinear problem;
- The Lipschitz regularity is also sufficient to invert the transformation and deduce a controllability result for the free boundary problem;
- Higher dimensional problem will likely require a Carleman estimate.

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Thank you for your attention.