Existence and classification of solutions to the Cahn-Hilliard equation

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The Cahn-Hilliard equation

$$-\varepsilon\Delta u = \varepsilon^{-1}(u - u^3) - \ell_\varepsilon \qquad \text{in } \Omega \tag{1}$$

is the Euler equation of the Ginzburg-Landau energy

$$E_\varepsilon(u,\Omega) := \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{(1-u^2)^2}{4\varepsilon}\right)dx$$

under the constraint

$$\frac{1}{|\Omega|}\int_\Omega u dx = m, \qquad m \in (-1,1) \tag{2}$$

The mean curvature

Given an N-dimensional manifold M and an $(N-1)$-dimensional submanifold $\Sigma \subset M$, the principal curvature $k_j(p)$ of Σ at $p \in \Sigma$ are the eigenvalues of the second fundamental form A_Σ. The mean curvature is by definition

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- We say that Σ is a *minimal submanifold* if $H_\Sigma = 0$.
- We say that Σ is a *constant mean curvature submanifold* if H_Σ is constant.

Variational characterisation

Let $X : D \to \Sigma$ be a parametrisation. Let $N : D \to \mathbf{R}$ be the external unit normal and let $h : D \to \mathbf{R}$ be a smooth function. For $t \in (-\delta, \delta)$, we set

$$X_t(u,v) := X(u,v) + th(u,v)N(u,v).$$

Then

$$A(t,h) = \int_D \sqrt{E_t G_t - F_t^2}\, dudv$$

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Since

$$A'[h] = \frac{d}{dt} A(t, h)|_{t=0} = -2 \int_D hH_\Sigma \sqrt{E_t G_t - F_t^2}\, dudv \qquad \forall h \in C_c^\infty(D),$$

we can see that Σ is a *minimal surface* if and only if $A'[h] = 0$ for any $h \in C_c^\infty(D)$.

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Theorem 1 (Modica).

Let

- $\Omega \subset \mathbf{R}^N$ *be a bounded domain.*
- $\varepsilon_k > 0$ *be such that* $\varepsilon_k \to 0$.
- u_k *be minimisers of the Ginzburg-Landau energy* E_{ε_k} *under the constraint (2) such that* $u_k \to u_0$ *in* $L^1(\Omega)$.

Then $u_0(x) \in \{\pm 1\}$ *for almost every* $x \in \Omega$*, and the boundary in* Ω *of the set* $E := \{x \in \Omega : \; u_0(x) = 1\}$ *has minimal perimeter among all subsets* $F \subset \Omega$ *such that* $|F| = |E|$*, where* $|\cdot|$ *denotes the* N*-dimensional Lebesgue measure.*

Conversely, given a suitable minimal submanifold Σ in a compact Riemaniann manifold M, it is possible to construct a family u_ε of solutions to the Allen-Cahn equation

$$-\varepsilon \Delta u_\varepsilon = \varepsilon^{-1}(u_\varepsilon - u_\varepsilon^3) \qquad \text{in } M \tag{4}$$

such that $u_\varepsilon \to \pm 1$ uniformly on compact subsets of the connected components of $M \backslash \Sigma$. (Pacard and Ritoré)

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It means **non degenerate**, in the sense that the Jacobi operator

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The Jabobi operator arises as the second variation of the area functional, in the sense that

$$A''(\Sigma)[\varphi, \psi] = \int_\Sigma \mathcal{J}_\Sigma \varphi \psi d\sigma.$$

The noncompact case
A graph $u : \Omega \subset \mathbf{R}^N \to \mathbf{R}$ is minimal if and only if

$$div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \qquad \text{in } \Omega. \tag{6}$$

Conjecture 1 (Bernstein).

Let $u : \mathbf{R}^N \to \mathbf{R}$ be a solution to (6) in dimension $N \leq 7$. Then u is affine.

In dimension $N = 8$, there exists a non affine minimal graph (Bombieri-De Giorgi-Giusti).

Conjecture 2 (De Giorgi).

Let u be a solution to the Allen-Cahn equation

$$-\Delta u = u - u^3$$

in $\mathbf{R}^N$ such that $\partial_{x_N} u > 0$. Let $N \leq 8$. Then u just depends on one euclidean variable, that is

$$u(x) = v_\star(x \cdot \nu + a), \qquad \nu \in S^{N-1},\ a \in \mathbf{R},\ v_\star(t) = \tanh(t/\sqrt{2}).$$

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- The conjecture is true in dimension $N = 2, 3$ (Ghoussoub-Gui, Ambrosio-Cabré, Farina-Sciunzi-Valdinoci).
- In dimension $4 \le N \le 8$ (Savin) it is true under the additional assumption that
$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1.$$
- In dimension $N = 9$, Del Pino, Kowalczyk and Wei constructed a monotone solution which is not 1D.

Constant mean curvature surfaces

- The only compact embedded constant mean curvature surface is the round sphere.
- Removing the compactness assumption, the simplest example is the cylinder.
- Delaunay constructed a family of axially symmetric periodic CMC surfaces, depending on a parameter $\tau \in (0, 1]$.

Delaunay surfaces in $\mathbf{R}^3$

We rotate the graph of the periodic function $\rho(t)$ around a fixed axes.

$$X(\vartheta, t) = (\rho(t)\cos\vartheta, \rho(t)\sin\vartheta, t), \qquad (\vartheta, t) \in [0, 2\pi) \times \mathbf{R}.$$

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We determine ρ in such a way that the mean curvature is constant, that is

$$\partial_t^2 \rho - \frac{1}{\rho}(1 + \partial_t \rho^2) + (1 + \partial_t \rho^2)^{3/2} = 0.$$

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Proposition 1.

For any $\tau \in (0, 1]$, there exists a periodic solution ρ_τ of period T_τ such that

- $\rho_\tau(0) = 1 - \sqrt{1 - \tau^2}$,
- $\partial_t \rho_\tau(0) = 0$,
- $1 - \sqrt{1 - \tau^2} \le \rho_\tau(t) \le 1 + \sqrt{1 - \tau^2}$, *for any $t \in [0, T_\tau]$.*

We are interested in the set $\mathcal{M}_{k,g}$ of *complete Alexandrov embedded constant mean curvature surfaces of genus g with k ends*, that is

$$\Sigma \cap (\mathbf{R}^3 \backslash B_R) = \cup_{j=1}^{k} E_j,$$

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for some $R > 0$.

- $\mathcal{M}_{0,g}$ consists of the round sphere.
- $\mathcal{M}_{1,g}$ is empty.
- $\mathcal{M}_{2,g}$ consists of the cylinder and Delaunay surfaces.

Each of the ends is asymptotic to a translated and rotated copy to a Delaunay surface D_{τ_j}, of axes $\mathbf{c}_j \in S^2$.

The radial case

Let $f(u) := u - u^3$. If u is a solution to (1) in $\mathbf{R}^N$, then $v(x) := u(\varepsilon x)$ solves

$$-\Delta v = f(v) - \delta, \qquad \delta = \varepsilon \ell_\varepsilon \qquad \text{in } \mathbf{R}^N. \tag{7}$$

If $\delta > 0$ is small enough, then there exist

$$z_-(\delta) < -1 < 0 < z_0(\delta) < z_+(\delta) < 1$$

such that $f(z_i(\delta)) = \delta$, $i = 1, 2, 3$. It is known (Dancer, Peletier-Serrin) that, if $\delta > 0$ is small enough, then there exists a unique solution v_δ to(7) in $\mathbf{R}^N$ such that

- $v_\delta < z_+(\delta)$ in $\mathbf{R}^N$,
- $v_\delta(x) \to z_+(\delta)$ as $|x| \to \infty$.

Moreover, this solution is radially symmetric.

Theorem 2 (R.).

Let $\delta \in (-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}})$ *and let* u *be a solution to the Cahn-Hilliard equation*

$$-\Delta u = f(u) - \delta$$

in $\mathbf{R}^N$. *Then* $z_-(\delta) \leq f(u_\delta) \leq z_+(\delta)$. *If, in addition,* $u > z_0(\delta)$ *outside a ball, then*

- *for* $\delta \in (-\frac{2}{3\sqrt{3}}, 0]$, *we have* $u \equiv z_+(\delta)$.
- *for* $\delta \in (0, \frac{2}{3\sqrt{3}})$, *we have either* $u = v_\delta$ *or* $u \equiv z_+(\delta)$. *In particular,* u *is radially symmetric.*

The periodic case

Let $\tau \in (0,1)$ and let D_τ be the corresponding Delaunay surface in $\mathbf{R}^3$. We denote the exterior and the interior of D^τ by $D_\tau^{\pm}$ respectively.

Theorem 3 (Hernández, Kowalczyk).

For $\tau \in (0,1)$ and $\varepsilon > 0$ small enough, there exists a solution u_ε to (1) in $\mathbf{R}^3$ such that

- *u_ε is periodic in x_3, of period T_τ.*
- *u_ε is radially symmetric in $x' = (x_1, x_2)$.*
- *$u_\varepsilon(x', x_3) \to z_+(\varepsilon \ell_\varepsilon)$ as $|x'| \to \infty$, uniformly in x_3 and ε.*
- *$u_\varepsilon(x) \to \pm 1$ as $\varepsilon \to 0$ uniformly on compact sets of $D_\tau^{\pm}$.*

The Jacobi operator

$$J_{D_\tau} = \Delta_{D_\tau} + |A_{D_\tau}|^2$$

of D_τ has 6 linearly independent Jacobi fields.

- the ones related to translations, denoted by Φ_τ^{T,e_j}, $1 \leq j \leq 3$;
- the ones related to rotations about the x_j axes, $j = 1, 2$, denoted by Φ_τ^{R,e_j}, $1 \leq j \leq 2$;
- the one related to the Delaunay parameter, denoted by Φ_τ^{D}.

None of these Jacobi fields is in $L^2(D_\tau)$.

Theorem 4 (R.).
Let $\delta \in (0, 2/3\sqrt{3})$ and let u_δ be a non constant solution to

$$-\Delta u_\delta = f(u_\delta) - \delta$$

in $\mathbf{R}^N$ such that $u_\delta > z_0(\delta)$ outside a cylinder $C_{R(\delta)}$, for some $R(\delta) > 0$. If u_δ is periodic in x_N, then

- *u_δ is radially symmetric in x', that is, up to a translation, $u_\delta(x) = w_\delta(|x'|, x_N)$.*
- *u_δ is radially increasing, in the sense that $(\nabla u_\delta(x), (x', 0)) > 0$, for any $x = (x', x_N) \in \mathbf{R}^N \setminus \{0\}$.*

The k-ended case
We consider surfaces $\Sigma \in \mathcal{M}_{k,g}$, with $k \geq 3$, $g \geq 0$.

- We say that such a surface is non degenerate if the Jacobi operator

$$J_\Sigma = \Delta_\Sigma + |A_\Sigma|^2$$

  has no kernel in $L^2(\Sigma)$.
- Not two of the ends are parallel, in the sense that, if $\mathbf{c}_i = \lambda \mathbf{c}_j$, $1 \leq i \neq j \leq k$, then $\lambda = -1$.

Theorem 5 (Kowalczyk, R.).

Let $g \geq 0$, $k \geq 3$. Let $\Sigma \in \mathcal{M}_{k,g}$ be such that

- *Σ is non degenerate,*
- *Not two of the ends are parallel.*

Let $\Sigma^{\pm}$ be the exterior and the interior of Σ respectively. Then, for $\varepsilon > 0$ small enough, there exists a solution u_ε to (7) such that $u_\varepsilon \to \pm 1$ as $\varepsilon \to 0$ on compact subsets of $\Sigma^{\pm}$ respectively.

The proof of Theorem 5 We look for a solution of the form $u = v + w$, where v is an approximate solution and w is a correction.

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We start from the unique solution to the ODE

$$\begin{cases} -v_\star'' = v_\star - v_\star^3 \\ v_\star(0) = 0, \\ \lim_{t\to\pm\infty} v_\star = \pm 1, \end{cases} \tag{8}$$

that is $v_\star(t) = \tanh(t/\sqrt{2})$.

We define the Fermi coordinates $(y, z) \in \Sigma \times \mathbf{R}$ by the relation

$$x = y + z\nu_\Sigma(y),$$

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A first, rough idea could be to put $v(\phi)(x) \simeq v_\star(z)$ near Σ.

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Then we set $z = t + \phi(\varepsilon y)$, where ε is small and $\phi : \Sigma \to \mathbf{R}$ is a small shift function.
The approximate solution is $v(\phi)(x) \simeq v_\star(t)$. The correction w and the shift function ϕ are the unknowns of the problem, and they are determined by a Lyapunov-Schmidt reduction.

The Lyapunov-Schmidt reduction

We have to solve a nonlinear equation $N(u) = 0$. We look for a solution $u = v(\phi) + w$.

We Taylor-expand

$$0 = N(v(\phi) + w) = N(v(\phi)) + N'(v(\phi))[w] + Q_{v(\phi)}(w).$$

We project along $X := ker(N'(v(\phi)))^{\perp}$ and $X^{\perp}$

$$N'(v(\phi))[w] = -\Pi_X\big(N(v(\phi)) + Q_{v(\phi)}(w)\big) \tag{9}$$

$$\Pi_{X^{\perp}}\big(N(v(\phi)) + Q_{v(\phi)}(w)\big) = 0. \tag{10}$$

The correction w is determined solving (9) for any fixed ϕ, exploiting the coercivity of the quadratic form

$$\int_{\Sigma\times\mathbf{R}} |\nabla_y w|^2 + (\partial_t w)^2 + (3v_\star^2 - 1)w^2 \, dydt$$

on the space of functions satisfying the orthogonality condition

$$\int_{\mathbf{R}} w(y,t)v_\star'(t)dt = 0, \qquad \forall y \in \Sigma.$$

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ϕ is determined solving (10), which is equivalent to a non linear equation of the form

$$J_\Sigma \phi = \mathcal{F}_\varepsilon(y,\phi),$$

which is solvable thanks to non degeneracy.

Problem

The error obtained by the standard ansatz about the approximate solution is not decaying along the surface.

Solution:

- along each of the ends we need to add a correction, also determined by a Lyapunov-Schmidt reduction.
- the decay is guaranteed by the fact that not two of the ends are parallel.

Kowalczyk, M., Rizzi, M. Multiple Delaunay ends solutions of the Cahn-Hilliard equation, accepted by *Communications in Partial differential equations.*
Rizzi, M. Radial and cylindrical symmetry of solutions to the Cahn-Hilliard equation, submitted to *Calculus of variations and PDEs.*