Partitioned multirate domain decomposition waveform relaxation methods for the heat equation

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Outline

1. Motivation

2. A multirate approach

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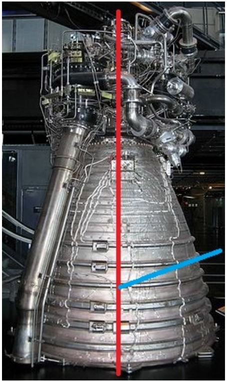
Applications of thermal FSI



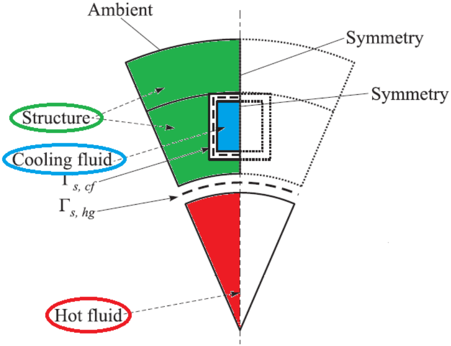
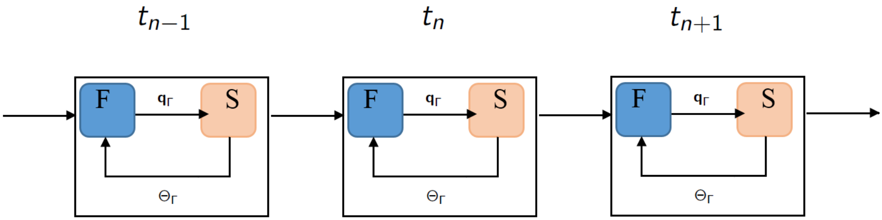


Figure: Left: Vulcain engine for Ariane 5. Right: Sketch of the cooling system; Pline, Wikimedia Commons

Thermal interaction between fluid and structure needs to be modelled

Dirichlet–Neumann coupling

- *Partitioned approach* for the solution of the coupled problem.
- *Fluid Model*: Compressible Navier–Stokes - FVM.
- *Structure Model*: Nonlinear heat equation - FEM.



Limitations

- The subsolvers are sequential.
- Same time integration for both fields.

Limitations of the Dirichlet–Neumann method

List of wishes

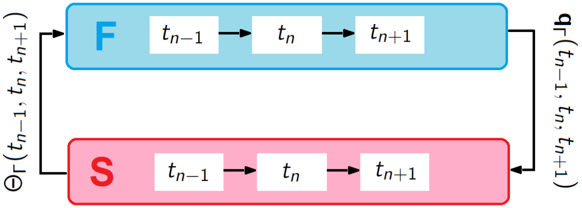
- Two independent time integration schemes.
- High order resolution (at least 2nd order).
- To be able to insert time adaptivity in the framework.

Option 1

Exchange fixed point iteration with time recursion

Option 2

Use a different domain decomposition method



Limitations of the Dirichlet–Neumann method

Option 1

Dirichlet-Neumann Waveform Relaxation (DNWR) algorithm

+ Computationally cheap
– Sequential method

Option 2

Neumann-Neumann Waveform Relaxation (NNWR) algorithm

+ Parallel method
– Computationally expensive

- *DNWR and NNWR introduced by Gander and Kwok 2016: Constant coefficients and one single time integration scheme.*

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Model Problem: Coupled heat equations

$$\alpha_m \frac{\partial u_m(\mathbf{x},t)}{\partial t} - \nabla \cdot (\lambda_m \nabla u_m(\mathbf{x},t)) = 0,$$

$$t \in [t_0, t_f], \quad \mathbf{x} \in \Omega_m \subset \mathbb{R}^d, \; m = 1,2$$

$$u_m(\mathbf{x},t) = 0, \quad \mathbf{x} \in \partial\Omega_m \backslash \Gamma$$

$$u_1(\mathbf{x},t) = u_2(\mathbf{x},t), \quad \mathbf{x} \in \Gamma$$

$$\lambda_2 \frac{\partial u_2(\mathbf{x},t)}{\partial \mathbf{n}_2} = -\lambda_1 \frac{\partial u_1(\mathbf{x},t)}{\partial \mathbf{n}_1}, \quad \mathbf{x} \in \Gamma$$

$$u_m(\mathbf{x},0) = g_m(\mathbf{x}) \quad \mathbf{x} \in \Omega_m$$

where $\alpha_m = \lambda_m / D_m$.

Dirichlet-Neumann waveform relaxation (DNWR)

$$(D)\begin{cases}\alpha_1 \frac{\partial u_1^{k+1}(\mathbf{x},t)}{\partial t} - \nabla\cdot(\lambda_1 \nabla u_1^{k+1}(\mathbf{x},t)) = 0, & \mathbf{x}\in\Omega_1,\\ u_1^{k+1}(\mathbf{x},t) = 0, & \mathbf{x}\in\partial\Omega_1\backslash\Gamma,\\ u_1^{k+1}(\mathbf{x},t) = g^k(\mathbf{x},t), & \mathbf{x}\in\Gamma,\\ u_1^{k+1}(\mathbf{x},0) = u_1^0(\mathbf{x}), & \mathbf{x}\in\Omega_1.\end{cases}$$

$$(N)\begin{cases}\alpha_2 \frac{\partial u_2^{k+1}(\mathbf{x},t)}{\partial t} - \nabla\cdot(\lambda_2 \nabla u_2^{k+1}(\mathbf{x},t)) = 0, & \mathbf{x}\in\Omega_2,\\ u_2^{k+1}(\mathbf{x},t) = 0, & \mathbf{x}\in\partial\Omega_2\backslash\Gamma,\\ \lambda_2 \frac{\partial u_2^{k+1}(\mathbf{x},t)}{\partial \mathbf{n}_2} = -\lambda_1 \frac{\partial u_1^{k+1}(\mathbf{x},t)}{\partial \mathbf{n}_1}, & \mathbf{x}\in\Gamma,\\ u_1^{k+1}(\mathbf{x},0) = u_1^0(\mathbf{x}), & \mathbf{x}\in\Omega_1.\end{cases}$$

$$(U)\quad g^{k+1}(\mathbf{x},t) = \Theta u_2^{k+1}(\mathbf{x},t) + (1-\Theta) g^k(\mathbf{x},t), \quad \mathbf{x}\in\Gamma.$$

How to choose the relaxation parameter Θ properly?

Neumann-Neumann waveform relaxation (NNWR)

$$(D_m), m = 1, 2 \begin{cases} \alpha_m \frac{\partial u_m^{k+1}(\mathbf{x},t)}{\partial t} - \nabla \cdot (\lambda_m \nabla u_m^{k+1}(\mathbf{x},t)) = 0, & \mathbf{x} \in \Omega_m, \\ u_m^{k+1}(\mathbf{x},t) = 0, & \mathbf{x} \in \partial\Omega_m \setminus \Gamma, \\ u_m^{k+1}(\mathbf{x},t) = g^k(\mathbf{x},t), & \mathbf{x} \in \Gamma, \\ u_m^{k+1}(\mathbf{x},0) = u_1^0(\mathbf{x}), & \mathbf{x} \in \Omega_m. \end{cases}$$

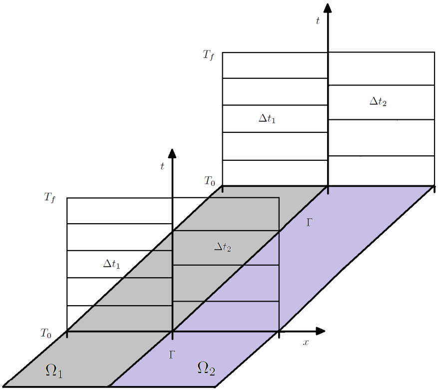
$$(N_m), m = 1, 2 \begin{cases} \alpha_m \frac{\partial \psi_m^{k+1}(\mathbf{x},t)}{\partial t} - \nabla \cdot (\lambda_m \nabla \psi_m^{k+1}(\mathbf{x},t)) = 0, & \mathbf{x} \in \Omega_m, \\ \psi_m^{k+1}(\mathbf{x},t) = 0, & \mathbf{x} \in \partial\Omega_m \setminus \Gamma, \\ \lambda_m \frac{\partial \psi_m^{k+1}(\mathbf{x},t)}{\partial \mathbf{n}_1} = \lambda_1 \frac{\partial u_1^{k+1}(\mathbf{x},t)}{\partial \mathbf{n}_1} + \lambda_2 \frac{\partial u_2^{k+1}(\mathbf{x},t)}{\partial \mathbf{n}_2}, & \mathbf{x} \in \Gamma, \\ \psi_m^{k+1}(\mathbf{x},0) = 0, & \mathbf{x} \in \Omega_m. \end{cases}$$

$$(U) \quad g^{k+1}(\mathbf{x},t) = g^k(\mathbf{x},t) - \Theta(\psi_1^{k+1}(\mathbf{x},t) + \psi_2^{k+1}(\mathbf{x},t)), \quad \mathbf{x} \in \Gamma.$$

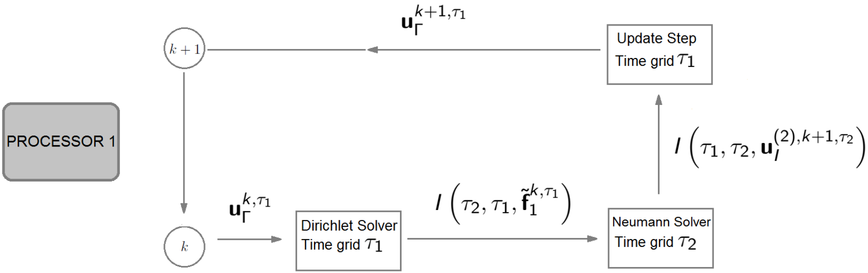
How to choose the relaxation parameter Θ properly?

Choices

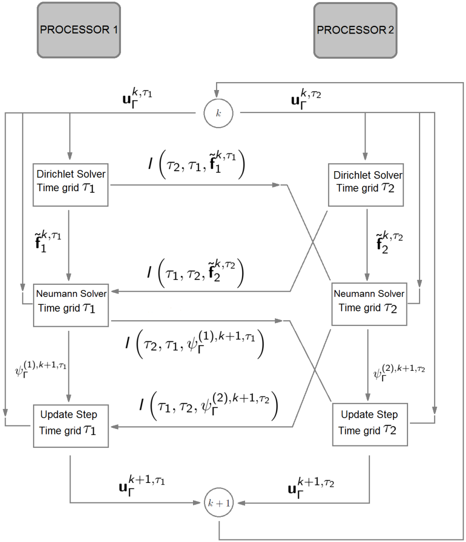
- *Space discretization*: 1D and 2D finite elements
- *Time discretization*: Implicit Euler and SDIRK2
- *Matching space grid* at the interface, unknowns on interface
- *Nonmatching time grids* at the interface, linear interpolation



Multirate DNWR Algorithm



Multirate NNWR Algorithm



Relaxation parameter

- *Space discretization*: 1D equidistant FE/FE.
- *Time integration*: nonmultirate Implicit Euler.
- Use eigendecomposition of the tridiagonal Toeplitz matrices $\mathbf{M}/\Delta t + \mathbf{A}$ to compute the spectral radius of the iteration matrix w.r.t. $\mathbf{u}_\Gamma$.

DNWR Algorithm

$$\Theta_{opt} = \left(1 + \frac{\mathbf{S}^{(1)}}{\mathbf{S}^{(2)}}\right)^{-1},$$

NNWR Algorithm

$$\Theta_{opt} = \left(2 + \frac{\mathbf{S}^{(1)}}{\mathbf{S}^{(2)}} + \frac{\mathbf{S}^{(2)}}{\mathbf{S}^{(1)}}\right)^{-1},$$

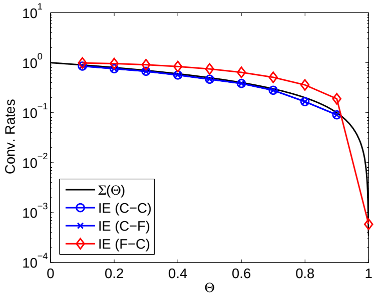
with

$$\mathbf{S}^{(m)} = (6\Delta x(\alpha_m \Delta x^2 + 3\lambda_m \Delta t) - (\alpha_m \Delta x^2 - 6\lambda_m \Delta t)^2 s_m).$$

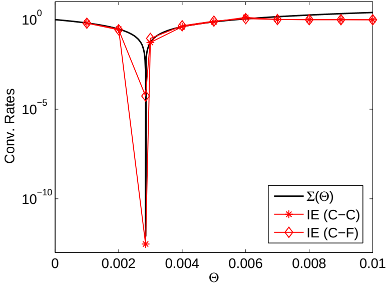
Θ_{opt} gives the optimal parameter for any coupled materials!

Convergence Rates: Air-Water coupling

DNWR (1D case)



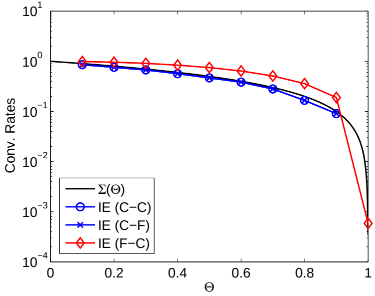
NNWR (1D case)



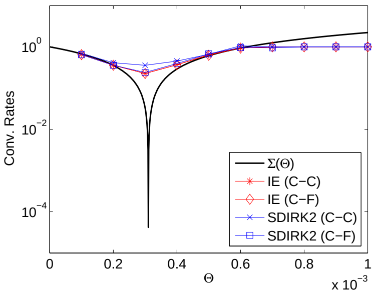
Θ_{opt} hits the optimal convergent performance.

Convergence Rates: Air-Water coupling

DNWR (2D case)

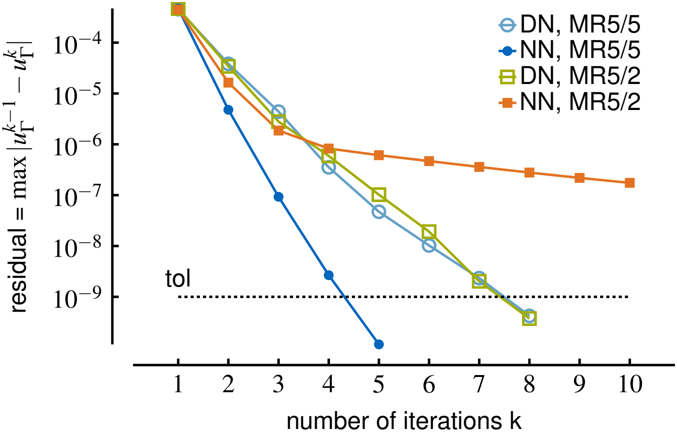


NNWR (2D case)



The NNWR method is more sensitive than the DNWR method.

Comparison DNWR and NNWR



NNWR uses half the iterations than DNWR but is less robust!

Conclusions

- Multirate parallel method for coupled parabolic problems (NNWR).
- Multirate sequential method for coupled parabolic problems (DNWR).
- We have performed a 1D analysis to find Θ_{opt}.
- Θ_{opt} is dependent on Δt, Δx, λ_m and α_m, $m = 1, 2$.
- Θ_{opt} works for 2D, multirate and **time adaptivity**.
- Θ_{opt} is more sensitive for NNWR than for DNWR.

More at:

1. A. Monge, P. Birken, SISC, accepted (arXiv:1805.04336)
2. A. Monge, P. Birken, Proceedings of 25th Domain Decomposition Conference, accepted.
3. A. Monge, P. Birken, PAMM 19, submitted.

Thank you!