# Turnpike in optimal shape design

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**Abstract:** We investigate the turnpike problem in optimal control, in the context of time-evolving shapes. We focus here on the heat equation model where the shape acts as a source term, and we search the optimal time-varying shape, minimizing a quadratic criterion. We first establish existence of optimal solutions under some appropriate sufficient conditions. We provide necessary conditions for optimality in terms of usual adjoint equations and then, thanks to strict dissipativity properties, we prove that state and adjoint satisfy a measure-turnpike property, meaning that the extremal time-varying solution remains essentially close to an optimal solution of an associated static problem. We illustrate the turnpike phenomenon in shape design with several numerical simulations.

*Keywords:* Optimal shape design, turnpike, strict dissipativity, direct methods, heat equation

## 1. INTRODUCTION

We start with an informal presentation of the turnpike phenomenon for general dynamical optimal shape problems. Let $T > 0$. We consider the problem of determining a time-varying shape $t \mapsto \omega(t)$ (viewed as a control) minimizing the cost functional

$$J_T(\omega) = \frac{1}{T}\int_0^T f^0\big(y(t),\omega(t)\big)\,dt + g\big(y(T),\omega(T)\big) \tag{1}$$

under the constraints

$$\dot y(t) = f\big(y(t),\omega(t)\big), \qquad R\big(y(0),y(T)\big) = 0 \tag{2}$$

where (2) may be a partial differential equation.

We associate to the dynamical problem (1,2) a *static* problem, not depending on time,

$$\min_{\omega} f^0(y,\omega), \qquad f(y,\omega) = 0 \tag{3}$$

According to the well known turnpike phenomenon, one expects that, for $T$ large enough, optimal solutions of (1,2) remain most of the time "close" to an optimal (stationary) solution of the static problem (3).

The turnpike phenomenon was first observed and investigated by economists for discrete-time optimal control problems (see, e.g., Dorfman et al. (1958); McKenzie (1963)). There are several possible notions of turnpike properties, some of them being stronger than the others. For continuous-time problem, exponential turnpike properties have been established in Trélat and Zuazua (2015); Porretta and Zuazua (2013, 2016); Trélat et al. (2018) for the optimal triple resulting of the application of Pontryagin's maximum principle, ensuring that the extremal solution (state, adjoint and control) remains exponentially close to an optimal solution of the corresponding static controlled problem, except at the beginning and at the end of the time interval, as soon as $T$ is large enough. This follows from the hyperbolicity feature of the Hamiltonian flow. For discrete-time problems it has been shown in Damm et al. (2014); Grüne and Müller (2016) that exponential turnpike is closely related to strict dissipativity.

Measure-turnpike is a weaker notion of turnpike, meaning that any optimal solution, along the time frame, remains close to an optimal solution of the associated static optimization problem except along a subset of times that is of

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small Lebesgue measure. It has been proved in Faulwasser et al. (2017); Trélat and Zhang (2018) that measure-turnpike follows from strict dissipativity or from strong duality.

Applications of turnpike in practice are numerous. Indeed, the knowledge of a static optimal solution is a way to reduce significantly the complexity of the dynamical optimal control problem. For instance it has been shown in Trélat and Zuazua (2015) that this is a way to successfully initialize a shooting method, when trying to compute numerically an optimal solution. In shape design and despite of the industrial progress, it is easier to design pieces which do not evolve with time. Turnpike can legitimate such decisions for large-time evolving systems.

## 2. SHAPE TURNPIKE FOR THE HEAT EQUATION

Throughout the paper, we denote by:

- $|Q|$ the Lebesgue measure of a subset $Q \subset \mathcal{R}^N$, $N \geq 1$.
- $\big(p,q\big)$ for $p,q$ in $L^2(\Omega)$ is the scalar product in $L^2(\Omega)$.
- $\|y\|$ for y $\in L^2(\Omega)$ is the $L^2$-norm.
- $\chi_\omega$ is the indicator (or characteristic) function of a subset $\omega \in \mathcal{R}^N$.

### *2.1 Problem*

Let $\Omega \subset \mathbf{R}^N (N \in \mathbf{N}^*)$ be a bounded domain with a smooth boundary $\partial\Omega$. Let $L \in (0,1)$. We define the set of admissible shapes

$$\mathcal{U}_L = \{\omega \subset \Omega \text{ measurable } \mid |\omega| \leq L|\Omega|\} \tag{4}$$

where the set of subsets $\omega \subset \Omega$ is a measured space, endowed with the Lebesgue measure $|\cdot|$.

*Dynamical optimal shape design problem.* Let $y_0 \in L^2(\Omega)$ be arbitrary. We consider the Dirichlet heat equation controlled by a (measurable) time-varying map $t \mapsto \omega(t)$ of subdomains

$$\frac{\partial y}{\partial t} - \triangle y = \chi_{\omega(\cdot)}, \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0 \tag{5}$$

Given $T > 0$ and $y_d \in L^2(\Omega)$, we consider the dynamical optimal shape design problem $(\mathbf{OSD})_\mathbf{T}$ of determining a measurable map of shapes $t \to \omega(t) \in \mathcal{U}_L$ that minimizes the cost functional

$$J_T(\omega(\cdot)) = \frac{1}{T}\int_0^T \|y(t) - y_d\|^2\, dt \tag{6}$$

where $y$ is the solution of (5) corresponding to $\omega(\cdot)$.

Besides, for the same target function $y_d \in L^2(\Omega)$, we consider an associated static shape design problem $(\mathbf{SSD})$:

*Static problem.*

$$\min_{\omega\in\mathcal{U}_L} \|y - y_d\|^2, \quad \triangle y + \chi_\omega = 0, \quad y_{|\partial\Omega} = 0 \tag{7}$$

We want to compare the solution of $(\mathbf{OSD})_\mathbf{T}$ and $(\mathbf{SSD})$.

### *2.2 Preliminaries*

*Convexification.* Given any measurable subset $\omega \subset \Omega$, we identify $\omega$ with the function $\chi_\omega \in L^\infty(\Omega;\{0,1\})$ and we identify as well $\mathcal{U}_L$ with a subset of $L^\infty(\Omega)$. The convex closure of $\mathcal{U}_L$ in $L^\infty$ star topology is

$$\overline{\mathcal{U}}_L = \Big\{a \in L^\infty\big(\Omega;[0,1]\big) \;\mid\; \int_\Omega a(x)\,dx \leq L|\Omega|\Big\}$$

which is also weak star compact. We define the convexified optimal control problem $(\mathbf{ocp})_\mathbf{T}$ of determining a control $t \to a(t)$ such that $a.e.\, t \in [0,T], a(t) \in \overline{\mathcal{U}}_L$ and minimizing

$$J_T(a) = \frac{1}{T}\int_0^T \|y(t) - y_d\|^2\, dt$$

under the constraints

$$\frac{\partial y}{\partial t} - \triangle y = a, \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0 \tag{8}$$

The corresponding convexified static optimization problem $(\mathbf{sop})$ is

$$\min_{a\in\overline{\mathcal{U}}_L} \|y - y_d\|^2, \qquad \triangle y + a = 0, \qquad y_{|\partial\Omega} = 0 \tag{9}$$

We recall some useful inequalities to study existence and turnpike. First the energy inequality. There exists $C > 0$ such that for any solution $y$ of (8), for $a.e.\, t \in [0,T]$,

$$\frac{1}{2}\|y(t)\|^2 + \int_0^t \|\nabla y(s)\|^2\, ds \leq \frac{1}{2}\|y_0\|^2 + C\int_0^t \|a(s)\|^2\, ds \tag{10}$$

We can improve this inequality by using the Poincaré inequality and the Gronwall lemma to get for $a.e.\, t \in [0,T]$,

$$\|y(t)\|^2 \leq\; \|y_0\|^2 e^{-\frac{t}{C}} + C\int_0^t e^{-\frac{t-s}{C}}\|a(s)\|^2\, ds \tag{11}$$

The constant $C > 0$ depends only on the domain $\Omega$ and comes from the Poincaré inequality.

Taking a minimizing sequence and by classical arguments of functional analysis (see, e.g., Lions (1968)), it is not difficult to prove existence of solutions $a_T$ and $\bar{a}$ respectively of $(\mathbf{ocp})_\mathbf{T}$ and $(\mathbf{sop})$.

*Necessary optimality conditions for* $(\mathbf{ocp})_\mathbf{T}$. Applying the Pontryagin maximum principle in (Lions, 1968, Chapitre 3, Théorème 2.1), for any optimal solution $(y_T, a_T)$ of $(\mathbf{ocp})_\mathbf{T}$ there exists an adjoint state $p_T \in L^2(0,T;\Omega)$ such that

$$\begin{aligned} \frac{\partial y_T}{\partial t} - \triangle y_T = a_T, \quad & y_{T|\partial\Omega} = 0, \quad y_T(0) = y_0 \\ \frac{\partial p_T}{\partial t} + \triangle p_T = 2(y_T - y_d), \quad & p_{T|\partial\Omega} = 0, \quad p_T(T) = 0 \end{aligned} \tag{12}$$

$$\forall a \in \overline{\mathcal{U}}_L, \text{for a.e. } t \in [0,T], \quad \big(p_T(t), a_T(t) - a\big) \geq 0 \tag{13}$$

*Necessary optimality conditions for* $(\mathbf{sop})$. Similarly, applying (Lions, 1968, Chapitre 2, Théorème 1.4), for any optimal solution $(\bar{y}, \bar{a})$ of $(\mathbf{sop})$ there exists an adjoint state $\bar{p} \in L^2(\Omega)$ such that

$$\triangle \bar{y} + \bar{a} = 0, \qquad \bar{y}_{|\partial\Omega} = 0$$
$$\triangle \bar{p} = 2(\bar{y} - y_d), \qquad \bar{p}_{|\partial\Omega} = 0 \tag{14}$$
$$\forall a \in \overline{\mathcal{U}}_L, \qquad (\bar{p}, \bar{a} - a) \geq 0 \tag{15}$$

Using the bathtub principle (see, e.g., (Lieb and Loss, 2001, Theorem 1.14)), (13) and (15) give

$$a_T(\cdot) = \chi_{\{p_T(\cdot) > s_T(\cdot)\}} + c_T(\cdot)\chi_{\{p_T(\cdot) = s_T(\cdot)\}} \tag{16}$$
$$\bar{a} = \chi_{\{\bar{p} > \bar{s}\}} + \bar{c}\chi_{\{\bar{p} = \bar{s}\}} \tag{17}$$

with

$a.e.\, t \in [0,T]$, $c_T(t) \in L^\infty(\Omega;[0,1])$ and $\bar{c} \in L^\infty(\Omega;[0,1])$

$$s_T(\cdot) = \inf\left\{\sigma \in \mathbf{R} \ \mid \ |\{p_T(\cdot) > \sigma\}| \leq L|\Omega|\right\}$$
$$\bar{s} \ = \inf\left\{\sigma \in \mathbf{R} \ \mid \ |\{\bar{p} > \sigma\}| \leq L|\Omega|\right\}$$

It is important to note that, if $|\{\bar{p} = \bar{s}\}| = 0$, then it follows from (17) that the static optimal control $\bar{a}$ is actually the characteristic function of a shape $\bar{\omega} \in \mathcal{U}_L$.

### 2.3 Main results

*Existence of solutions.* Proving existence of solutions for $(\mathbf{OSD})_\mathbf{T}$ and $(\mathbf{SSD})$ is not an easy task. We can find cases where there is no existence for $(\mathbf{SSD})$ in (Henrot and Pierre, 2005, Section 4.2, Example 2): this is the relaxation phenomenon. This is why some assumptions are required on the target function $y_d$.

First, using maximum principle for elliptic (see Evans (1998) sec. 6.4) and parabolic equations (see Evans (1998) sec. 7.1.4) we introduce :

- $y^{T,0}$ and $y^{T,1}$ are solutions of (8) with respectively $a(\cdot) = 0$ and $a(\cdot) = 1$
- $y^{s,0}$ and $y^{s,1}$ solutions of (9) with respectively $a = 0$ and $a = 1$
- $y^0 = \min\left(y^{s,0}, \min\limits_{t\in(0,T)} y^{T,0}(t)\right)$
- $y^1 = \max\left(y^{s,1}, \max\limits_{t\in(0,T)} y^{T,1}\right)$

*Theorem 1.* If either $y_d$ verifies $y_d < y^0$ or $y_d > y^1$ or $y_d$ convex then we have existence and uniqueness of optimal solutions for both $(\mathbf{SSD})$ and $(\mathbf{OSD})_\mathbf{T}$.

Thanks to Theorem 1, hereafter we denote by

- $(y_T, p_T, \omega_T)$ an optimal triple of $(\mathbf{OSD})_\mathbf{T}$.
- $(\bar{y}, \bar{p}, \bar{\omega})$ an optimal triple of $(\mathbf{SSD})$.
- $J_T = \dfrac{1}{T}\displaystyle\int_0^T \|y_T(t) - y_d\|^2$ and $\bar{J} = \|\bar{y} - y_d\|^2$.

*Measure-turnpike.*

*Definition 1.* We say that $(y_T, p_T)$ satisfies the **state-adjoint measure-turnpike property** if for every $\epsilon > 0$ there exists $\Lambda(\epsilon) > 0$, independent of $T$, such that

$$|P_{\epsilon,T}| < \Lambda(\epsilon), \qquad \forall T > 0$$

where

$$P_{\epsilon,T} = \left\{t \in [0,T] \ \mid \ \|y_T(t) - \bar{y}\| + \|p_T(t) - \bar{p}\| > \epsilon\right\}$$

We refer to Carlson et al. (1991); Faulwasser et al. (2017); Trélat and Zhang (2018) (and references therein) for similar definitions. Here $P_{\epsilon,T}$ is the set of times at which the optimal couple state-adjoint solution $\big(y_T(\cdot), p_T(\cdot)\big)$ stays outside an $\epsilon$-neighborhood of $(\bar{y}, \bar{p})$ in $L^2$ topology.

We next recall the notion of dissipativity (see Willems (1972)).

*Definition 2.* We say that $(\mathbf{OSD})_\mathbf{T}$ is **strictly dissipative** at an optimal stationary point $(\bar{y}, \bar{\omega})$ of (7) with respect to the **supply rate function**

$$w(y, \omega) = \|y - y_d\|^2 - \|\bar{y} - y_d\|^2$$

if there exists a **storage function** $S : E \to \mathbf{R}$ locally bounded and bounded from below and a $\mathcal{K}$**-class function** $\alpha(\cdot)$ such that, for any $T > 0$ and any $0 < \tau < T$, the strict dissipation inequality

$$S(y(\tau)) + \int_0^\tau \alpha(\|y(t) - \bar{y}\|)dt < S(y(0)) + \int_0^\tau w\big(y(t), \omega(t)\big)dt \tag{18}$$

is satisfied for any couple $\big(y(\cdot), \omega(\cdot)\big)$ solution of (5).

*Theorem 2.*
- (i) $(\mathbf{OSD})_\mathbf{T}$ is strictly dissipative in the sense of Definition 2.
- (ii) If $y_d$ is convex then the unique optimal solution of $(\mathbf{OSD})_\mathbf{T}$ satisfies the measure-turnpike property.

The measure-turnpike property is here a nice-to-have. We nonetheless get the stronger internal turnpike property which implies the previous one.

*Integral turnpike.*

*Theorem 3.* If $y_d$ is convex then there exists $M > 0$ such that

$$\forall T > 0, \qquad \int_0^T \big(\|y_T(t) - \bar{y}\|^2 + \|p_T(t) - \bar{p}\|^2\big)dt < M$$

*Exponential turnpike.* The exponential turnpike property is a stronger property and can be either on the state, the adjoint or the control or even the three together. Based on the numerical simulations presented in Section 3 we conjecture:

*Conjecture 4.* If $y_d$ is convex then there exist $C_1 > 0$ and $C_2 > 0$ independent of $T$ such that, for a.e. $t \in [0,T]$,

$$\|y_T(t) - \bar{y}\| \leq C_1\Big(e^{-C_2 t} + e^{-C_2(T-t)}\Big)$$
$$\|p_T(t) - \bar{p}\| \leq C_1\Big(e^{-C_2 t} + e^{-C_2(T-t)}\Big)$$
$$\|\chi_{\omega_T(t)} - \chi_{\bar{\omega}}\| \leq C_1\Big(e^{-C_2 t} + e^{-C_2(T-t)}\Big)$$

### 2.4 Sketch of proof

*Sketch of proof of Theorem 1.* We give the idea for the static problem $(\mathbf{SSD})$.

We suppose $y_d > y^1$ (we proceed similarly for $y_d < y^0$). Having in mind (14) and (17) we get $-\triangle\bar{y} = \bar{c}$ on $\{\bar{p} = \bar{s}\}$. By contradiction, if $\bar{c} \leq 1$ on $\{\bar{p} = \bar{s}\}$, let us consider the solution $y^*$ of (9) with the same $\bar{a}$ verifying (17) except that $\bar{c} = 1$ on $\{\bar{p} = \bar{s}\}$. Then, by application of maximum principle (see Evans (1998) sec. 6.4), we get $y_d \geq y^* \geq \bar{y}$ and so $\|y^* - y_d\| \leq \|\bar{y} - y_d\|$. That means $\bar{a}$ verifying

(17) with $\bar{c} = 1$ is an optimal control. We conclude after with the uniqueness. We use similar reasoning for $(\mathbf{OSD})_\mathbf{T}$ solution's existence.

Now if $y_d$ is convex, we have $\triangle y_d \geq 0$ on $\Omega$. Having in mind (14) and (17), we assume by contradiction that $|\{\bar{p} = \bar{s}\}| > 0$. By (Le Dret, 2013, Theorem 3.2), we have $\triangle \bar{p} = 0$ on $\{\bar{p} = \bar{s}\}$. We infer that $\triangle y_d = -\bar{a}$ on $\{\bar{p} = \bar{s}\}$, which contradicts $\triangle y_d \geq 0$. Hence $|\{\bar{p} = \bar{s}\}| = 0$ and thus $\bar{a} = \chi_{\bar{\omega}}$ for some $\bar{\omega} \in \mathcal{U}_L$. Existence of solution for (**SSD**) is proved.

Uniqueness of $\bar{a} = \bar{\omega}$ comes from the fact that the problem (**sop**) is strictly convex. Uniqueness of $\bar{y}$ and $\bar{p}$ follows by application of (11).

*Remark 1.* Proving existence for $(\mathbf{OSD})_\mathbf{T}$ is more difficult. Anyway, if one replaces the Lagrange cost functional (6) with the Mayer cost functional

$$J_T(\omega) = \|y(T) - y_d\|^2$$

then the optimality system becomes

$$\begin{aligned} &\frac{\partial y_T}{\partial t} - \triangle y_T = a_T,\ y_{T|\partial\Omega} = 0,\ y_T(0) = y_0 \\ &\frac{\partial p_T}{\partial t} + \triangle p_T = 0,\ p_{T|\partial\Omega} = 0,\ p_T(T) = 2\big(y_d - y_T(T)\big) \end{aligned} \tag{19}$$

with (13) unchanged. It follows that $p_T$ is analytic on $(0, T) \times \Omega$ and that all level sets $\{p_T(t) = \alpha\}$ have zero Lebesgue measure. We conclude that the optimal control $a_T$ satisfying (13,19) is such that $a_T(t) = \chi_{\omega_T(t)}$ with $\omega_T(t) = \{p_T(t) > s_T(t)\}$ for a.e. $t \in (0, T)$. Hence, for a Mayer problem, existence of an optimal time-shape is proved.

*Proof of Theorem 2.* We follow Trélat and Zhang (2018) and the idea that *strict dissipativity* implies *measure-turnpike*.

(i) Strict dissipativity is established thanks to the storage function $S(y) = \big(y, \bar{p}\big)$ where $\bar{p}$ is the static optimal adjoint. Indeed, we consider an admissible pair $(y(\cdot), \chi_\omega(\cdot))$ satisfying (5), we multiply it by $\bar{p}$ and we integrate over $\Omega$. Then we integrate in time on $(0, T)$, we use the static optimality conditions (14) and we get a strict dissipation inequality (18) with $\alpha : s \to s^2$.

(ii) Following the argument of Trélat and Zhang (2018), we prove that strict dissipativity implies measure-turnpike. Applying (18) to the optimal solution $(y_T, \omega_T)$ we get

$$\frac{1}{T}\int_0^T \|y_T(t) - \bar{y}\|^2\, dt \leq J_T - \bar{J} + \frac{(y(0) - y(T), \bar{p})}{T} \tag{20}$$

Considering then the solution $y_s$ of (5) with $\omega(\cdot) = \bar{\omega}$ and $J_s = \frac{1}{T}\int_0^T \|y_s(t) - y_d\|^2$, we have $J_T - J_s < 0$ and we show that $J_s - \bar{J} \leq \frac{1 - e^{-CT}}{CT}$, then

$$\frac{1}{T}\int_0^T \|y_T(t) - \bar{y}\|^2\, dt \leq \frac{M}{T} \tag{21}$$

To add the adjoint dependence, we apply (10) to the quantity $\psi(\cdot) = p_T(T - \cdot) - \bar{p}$ combined with the optimality conditions (12,14) and get

$$\frac{1}{2C}\int_0^T \|p_T(t) - \bar{p}\|^2\, dt \leq C\int_0^T \|y_T(t) - \bar{y}\|^2\, dt + \frac{\|p_T(0) - \bar{p}\|^2 - \|p_T(T) - \bar{p}\|^2}{2}$$

Using again the strict dissipativity equation (18) we get $\frac{\epsilon^2 |P_{\epsilon,T}|}{T} \leq \frac{M}{T}$. Hence we can find a constant $M > 0$ which does not depend on $T$ such that $|P_{\epsilon,T}| \leq \frac{M}{\epsilon^2}$.

*Proof of Theorem 3.* We consider the triples $(y_T, p_T, \chi_{\omega_T})$ and $(\bar{y}, \bar{p}, \chi_{\bar{\omega}})$ satisfying the optimality conditions (12) and (14). Since $\chi_{\omega_T}$ is bounded at each time $t \in [0, T]$ and by application of (11) to $y_T$ and $p_T$ we can find a constant C depending on $y_0, y_d, \Omega, L$ such that

$$\forall T > 0, \quad \|y_T(T)\|^2 \leq C \quad \text{and} \quad \|p_T(0)\|^2 \leq C \tag{22}$$

We set $\tilde{y} = y_T - \bar{y},\ \tilde{p} = p_T - \bar{p},\ \tilde{a} = \chi_{\omega_T} - \chi_{\bar{\omega}}$ which verify

$$\frac{\partial \tilde{y}}{\partial t} - \triangle \tilde{y} = \tilde{a}, \quad \tilde{y}_{|\partial\Omega} = 0, \quad \tilde{y}(0) = y_0 - \bar{y} \tag{23}$$

$$\frac{\partial \tilde{p}}{\partial t} - \triangle \tilde{p} = 2\tilde{y}, \quad \tilde{p}_{|\partial\Omega} = 0, \quad \tilde{p}(T) = -\bar{p} \tag{24}$$

First, using (12) and (14) one can show that

$$a.e.t \in [0, T], \quad \big(\tilde{p}(t), \tilde{a}(t)\big) \geq 0 \tag{25}$$

Multiplying then (23) by $\tilde{p}$, (24) by $\tilde{y}$ and then adding them we get

$$\big(\bar{y} - y_0, \tilde{p}(0)\big) - \big(\tilde{y}(T), \bar{p}\big) = \int_0^T \big(\tilde{p}(t), \tilde{a}(t)\big)\, dt + \int_0^T \|\tilde{y}(t)\|^2\, dt$$

We apply then Cauchy-Schwarz inequality and (22) to find $C > 0$ such that

$$\frac{1}{T}\int_0^T \|\tilde{y}(t)\|^2\, dt + \frac{1}{T}\int_0^T \big(\tilde{p}(t), \tilde{a}(t)\big)\, dt \leq \frac{C}{T} \tag{26}$$

The two terms at the left-hand side are positive and using the inequality (10) with $\tilde{p}(T - t)$ we finally get

$$\frac{1}{T}\int_0^T \big(\|y_T(t) - \bar{y}\|^2 + \|p_T(t) - \bar{p}\|^2\big)\, dt \leq \frac{M}{T} \tag{27}$$

Note again that the integral turnpike property is stronger than the measure-turnpike property.

## 3. NUMERICAL SIMULATIONS: OPTIMAL SHAPE DESIGN FOR THE 2D HEAT EQUATION

We set $\Omega = [-1, 1]^2$, $L = \frac{1}{8}$, $T = 5$, $y_d = \text{Cst} = 0.1$ and $y_0 = 0$. We consider the minimization problem

$$\min_{\omega(\cdot)} \int_0^5 \int_{[-1,1]^2} |y(t, x) - 0.1|^2\, dx\, dt \tag{28}$$

under the constraints

$$\frac{\partial y}{\partial t} - \triangle y = \chi_\omega, \qquad y(0, \cdot) = 0, \qquad y_{|\partial\Omega} = 0 \tag{29}$$

We compute numerically a solution by solving the equivalent convexified problem $(\mathbf{ocp})_\mathbf{T}$ thanks to a direct method in optimal control (see Trélat (2005)). We discretize here with an implicit Euler method in time and with a decomposition on a finite element meshing of $\Omega$ using `FREEFEM++` (see Hecht (2012)). We express the problem as a quadratic programming problem in finite dimension. We use then the routine `IpOpt` (see Wächter and Biegler (2006)) on a standard desktop machine.

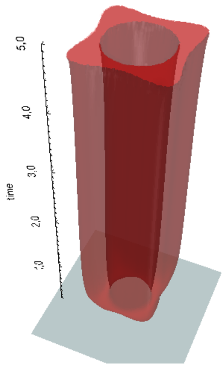


Fig. 1. Time optimal shape's evolution cylinder

We plot in Fig. 1 the evolution in time of the shape $t \to \omega(t)$ which appears like a cylinder whose section at time $t$ represents the shape $\omega(t)$. At the beginning $(t = 0)$ we notice that the shape concentrate at the middle of $\Omega$ in order to warm as soon as possible near to $y_d$. Once it is acceptable the shape stabilizes during a long time. Finally close to the final time the shape moves to the boundary of $\Omega$ in order to flatten the state $y_T$ because $y_d$ is here taken as a constant.

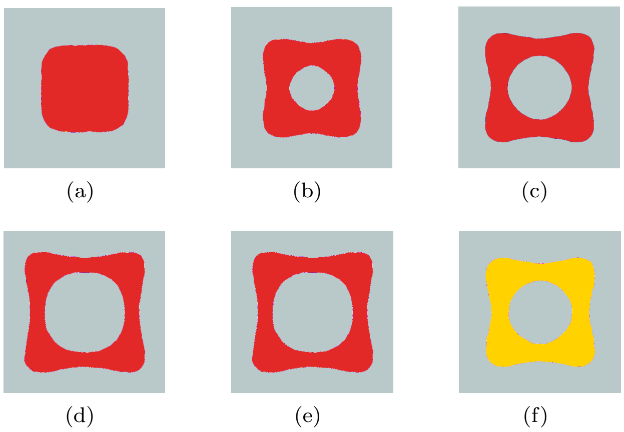


Fig. 2. Time optimal shape - Static shape: (a) $t = 0$; (b) $t = 0.5$; (c) $t \in [1, 4]$; (d) $t = 4.5$; (e) $t = T$; (f) static shape

We plot in Fig. 2 the comparison between the optimal shape at several times (in red) and the optimal static shape (in yellow). We see the same behavior when $t = \frac{T}{2}$.

Now in order to mirror the turnpike phenomenon we plot the evolution in time of the distance between the optimal dynamic triple and the optimal static one $t \to \|y_T(t) - \bar{y}\| + \|p_T(t) - \bar{p}\| + \|\chi_{\omega_T(t)} - \chi_{\bar{\omega}}\|$.

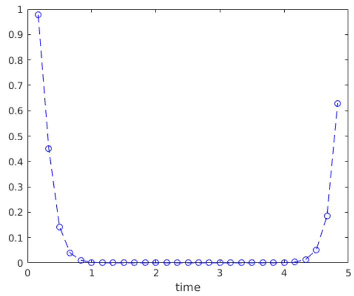


Fig. 3. Error between time optimal triple and static one

In Fig. 3 we observe that the function is exponentially close to 0. This behavior lets us think that the exponential turnpike property should be verified in our case.

To complete this work, we need to clarify the existence of optimal shapes for $(\mathbf{OSD})_\mathbf{T}$ when $y_d$ is convex. We see numerically in fig. 2 the time optimal shape's existence for $y_d$ convex on $\Omega$. Otherwise we can sometimes observe a relaxation phenomenon due to the presence of $\bar{c}$ and $c_T(\cdot)$ in the optimality conditions (12,14). We consider the same problem $(\mathbf{ocp})_\mathbf{T}$ in 2D with $\Omega = [-1, 1]^2$, $L = \frac{1}{8}, T = 5$ and the static one associated $(\mathbf{sop})$. We take a target function $y_d(x, y) = -\frac{1}{20}(x^2 + y^2 - 2)$.

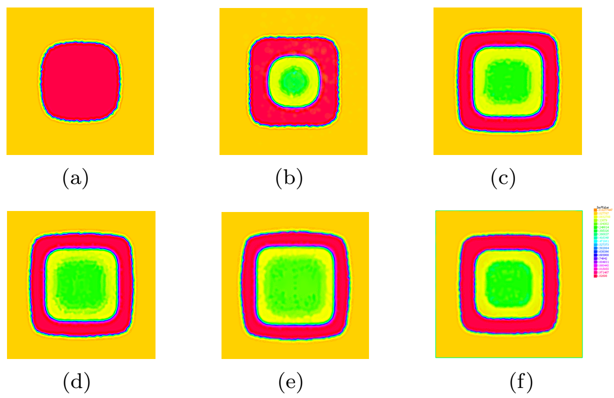


Fig. 4. Relaxation phenomenon : (a) $t = 0$; (b) $t = 0.5$; (c) $t \in [1, 4]$; (d) $t = 4.5$; (e) $t = T$; (f) static shape

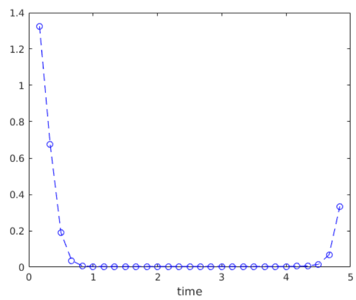


Fig. 5. Error between time optimal triple and static one (Relaxation case)

In Fig. 4 we see that optimal control $(a_T, \bar{a})$ of $(\mathbf{ocp})_\mathbf{T}$ and $(\mathbf{sop})$ are in $(0, 1)$ in the middle of $\Omega$. This illustrates that relaxation occurs for some $y_d$. It was choosen to verify $-\triangle y_d \in (0, 1)$. Here we calibrate the previous parameter $L$ to observe this phenomenon, but for same $y_d$ and smaller $L$, optimal solutions are both shapes. Despite the relaxation we see Fig. 5 that turnpike still occurs.

## 4. COMMENTS AND FURTHER WORKS

Numerical simulations when $y_d$ is convex motivates us to conjecture the existence of optimal shape for $(\mathbf{OSD})_\mathbf{T}$, because we have never observed relaxation in that case.

Moreover our simulations and particularly Fig. 3 indicate the occurence of the exponential turnpike property.

The work that we presented here is focused on the heat equation. It seems reasonable to extend our results to general parabolic operators, because we did not use any of the specific properties of the Laplacian operator. We consider here a linear partial differential equation which gives us the uniqueness of the solution thanks to the strict convexity of the criterion. As in Trélat and Zhang (2018), the notion of measure-turnpike seems to be a good and soft way to obtain turnpike results.

To go further with the numerical simulations, our objective will be to find optimal shapes evolving in time, solving dynamical shape design problems for more difficult real-life partial differential equations which play a role in fluid mechanics for example. We can find in the recent literature articles on the optimization of a wavemaker (e.g., Dalphin and Barros (2017); Nersisyan et al. (2015)). It is natural to ask for what happens if we consider a wavemaker whose shape can evolve in time.

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