Uniform polynomial decay and approximation in control of a weakly coupled thermoelastic wave model

S. NAFIRI

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Team of Modeling and Numerics (MoNum), Ecole Hassania des Travaux Publics (EHTP)

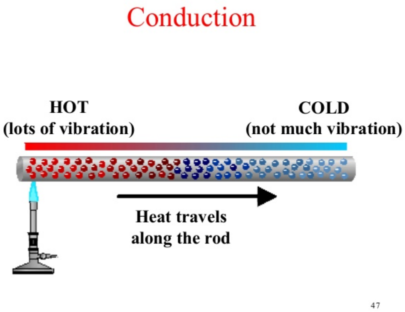
Seminar hosted by the Chair of Computational Mathematics

Outline

1. Introduction and Motivation
2. Asymptotic behaviour by semigroups theory
3. Exponentially and polynomially stable approximations of thermoelastic system
    - Theoretical analysis
    - Numerical experiments
4. Conclusion and open problems

Introduction and motivation

Experiment


Conduction
HOT
(lots of vibration)
COLD
(not much vibration)
Heat travels
along the rod
47

It is natural...

It is natural...

The first paper ever published on thermal stresses and thermoelasticity :

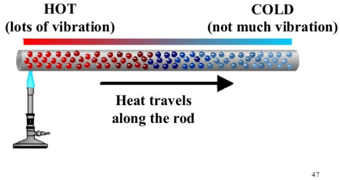
Duhamel, J.-M.-C., Second mémoire sur les phénomènes thermo-mécaniques, J. de l'École Polytechnique, tome 15, cahier 25, 1837, pp. 1–57.

Thermal expansion

Thermal expansion refers to a fractional change in size of a material in response to a change in temperature.

Thermal expansion

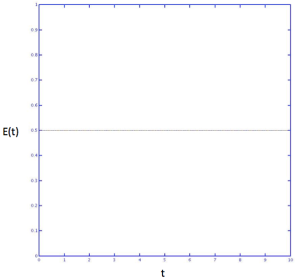
Conduction



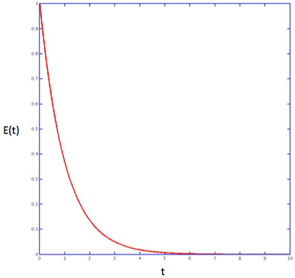
$$u_{tt}(x,t) - \Delta u(x,t) = 0$$

$$\theta_t(x,t) - \Delta\theta(x,t) = 0$$

Decay of energy



$u_{tt}(x,t) - \Delta u(x,t) = 0$



$\theta_t(x,t) - \Delta\theta(x,t) = 0$

Stability of the coupling system

Question : Does the coupling system inherits any kind of stability ?

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$$(S)\left\{\begin{array}{ll} u_{tt}-\Delta u+\gamma B v_1 & =0 \\ \theta_t-\Delta\theta+\gamma C v_2 & =0 \end{array}\right.$$

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- **F. Ammar-Khodja, A. Benabdallah and D. Teniou**, Dynamical stabilizers and coupled systems. ESAIM Proceedings 2, (1997), pp 253-262.
- **F. Ammar-Khodja, A. Bader, A. Benabdallah**, Dynamic stabilization of systems via decoupling techniques. ESAIM Control Optim. Calc. Var. 4, (1999), pp. 577-593.
- **F. Alabau, P. Cannarsa, V. Komornik**, Indirect internal stabilization of weakly coupled evolution equations, J.evol.equ. (2002) 2 : 127.
- **J. Hao and Z. Liu**, Stability of an abstract system of coupled hyperbolic and parabolic equations. Zeitschrift für angewandte Mathematik und Physik, 64, (2013), pp. 1145-1159.

Why studying thermoelastic systems ?

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Hubble space Telescope launched into Earth orbit in 1990.

Why studying thermoelastic systems ?

"The transfer of energy between its mechanical form and heat generally has been ignored as a source of both structural damping and excitation in the vast literature on control of flexible structures. Only a few recent papers have considered control of thermoelastic structures. However, the thermally induced vibrations that hampered the recently launched Hubble space telescope have highlighted the coupling between mechanical vibration and heat transfer and the need to model and control thermoelastic phenomena in flexible structures."

J. S. GIBSON, I. G. ROSEN, AND G. TAO, Approximation in control of thermoelastic systems, SIAM J. Control. Optim., 30 (1992), pp. 1163-1189.

Asymptotic behaviour by semigroups theory

Exponential case

Exponential case

Exponential decay

Theorem : Gearhart 1978 and Prüss 1984

Let H be a Hilbert space and A the generator of a C_0-semigroup $\{T(t)\}_{t\geqslant 0}$. $\{T(t)\}_{t\geqslant 0}$ is exponentially stable

$$\Updownarrow$$

$$\sup\{Re\lambda, \lambda \in \sigma(A)\} < 0 := s(A) < 0 + \sup_{Re\lambda\geqslant 0}\{\|(\lambda I - A)^{-1}\|\} < \infty$$

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Theorem : Huang Falun 1985

In the previous result if $\{T(t)\}_{t\geqslant 0}$ is a contraction semigroup, then $T(\cdot)$ is exponentially stable $\Leftrightarrow$ $i\mathbb{R} \subset \rho(A)$ and $\sup_{\beta\in\mathbb{R}}\{\|(i\beta I - A)^{-1}\|\} < \infty$

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Remark : There is also Lyapunov theory.

Asymptotic behaviour

Polynomial case

Asymptotic behaviour

Polynomial case

Theorem. (Borichev and Tomilov, 2010)

Let $T(t)$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that $i\mathbb{R} \subset \rho(A)$. Then for a fixed $\alpha > 0$ the following conditions are equivalent :

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\begin{array}{lll}
(i) & \|(isI - A)^{-1}\| = O(|s|^{\alpha}), & s \to \infty. \\
(ii) & \|T(t)(-A)^{-\alpha}\| = O(t^{-1}), & t \to \infty. \\
(iii) & \|T(t)(-A)^{-1}\| = O(t^{\frac{-1}{\alpha}}), & t \to \infty.
\end{array}
$$

Z. Liu and B. Rao, Characterization of polynomial decay rate for the solution of linear evolution equation, Zeitschrift für angewandte Mathematik und Physik ZAMP, 56, (2005), pp. 630-644.

Bátkai, A., Engel, K.-J., Prüss, J., Schnaubelt, R., Polynomial stability of operator semigroups. Math.Nachr. 279, 1425-1440 (2006).

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- **C. J. K. Batty and T. Duyckaerts**, Non-uniform stability for bounded semigroups on Banach spaces, J. Evol. Equ., 8(4), pp.765-780, 2008.
- **Borichev Alex, Tomilov Yu**, Optimal polynomial decay of functions and operator semigroups (2010).

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$$(S)\left\{\begin{array}{lll} u_{tt}-\Delta u+\gamma(-\Delta)^{\frac{1}{2}}\theta & =0, & \Omega\times(0,+\infty), \\ \theta_t-\Delta\theta+\gamma(-\Delta)^{\frac{1}{2}}u_t & =0, & \Omega\times(0,+\infty), \\ u=0=\theta & =0, & \partial\Omega \\ +I.C & & \end{array}\right.$$

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i.e., $\exists ? C, \alpha > 0$ s.t :

$$E(u,t) \leqslant Ce^{-\alpha t}E(u,0)?$$

$$\Updownarrow$$

i.e., $\exists ? C, \alpha > 0$ s.t :

$$\|T(t)y\| \leqslant Ce^{-\alpha t}\|y\|?$$

Stability of the coupling system

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u_{tt} - \Delta u + \gamma(-\Delta)^{\frac{1}{2}}\theta & = 0, \quad \Omega \times (0, +\infty), \\
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$$
E(t) \underset{t\to\infty}{\longrightarrow} 0 \text{ (Dafermos 1968)}
$$

C. M. Dafermos, On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity. Arch. Rat. Mech. Anal., 29, 1968, pp. 241-271.

Remark : No decay rate was given.

$$
(S)\begin{cases}
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\end{cases}
$$

$\exists C, \alpha > 0$ s.t :

$$
\|T(t)y\|_{D(A)} \leqslant Ce^{-\alpha t}\|y\|_{D(A)} \quad \text{(Slemrod 1981)}
$$

- **Slemrod, M**, Global existence, uniqueness, and asymptotic stability of classical smooth solutionsin one-dimensional non-linear thermoelasticity,Arch. Rational Mech. Anal., 76(1981), 97-133.
- **J.E.M. Rivera**, Energy decay rate in linear thermoelasticity, Funkcial Ekvac., Vol. 35 (1992), pp. 19-30.

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$\exists C,\alpha>0$ s.t :

$$\|T(t)y\|_H\leqslant Ce^{-\alpha t}\|y\|_H \quad \text{(Hansen 1992)}$$

- **S. W. Hansen**, Exponential energy decay in a linear thermoelastic rod. J. Math. Anal. Appli.,167, 1992, pp. 429-442.
- **Z. Liu and S.M. Zheng**, Exponential stability of the semigroup associated with a thermoelastic system, Quart. Appl. Math. 51 (1993), pp. 535-545.
- **Z.Y. Liu and S. Zheng**, Semigroups Associated with Dissipative Systems. Chapman & Hall/CRC Research Notes in Mathematics Series (1999).

Question :

If a system of PDE decays polynomially (res. exponentially) to zero, what are conditions for which its numerical approximation still decreases polynomially (resp. exp) to zero uniformly with respect to the step size ?

Exponentially and polynomially stable approximations of thermoelastic system

Question :

$\exists ? C, \alpha > 0$ independent of n s.t :

$$\|T_n(t)x\| \leqslant Ce^{-\alpha t}\|x\|$$

Uniform exponential case

- **H. T. Banks, K. Ito and C.Wang**. Exponentially stable approximations of weakly damped wave equations. Internat. Ser. Numer. Math. 100, Birkhäuser, (1991), pp. 1-33.
- **J. A. Infante and E. Zuazua**. Boundary observability for the space semidiscretizations of the $1-d$ wave equation. ESAIM : Mathematical Modelling and Numerical Analysis, 33, (1999), pp. 407-438.
- **L. I. Ignat and E. Zuazua**, A two-grid approximation scheme for nonlinear Schrödinger equations : dispersive properties and convergence, C. R. Math. Acad. Sci. Paris, 341, (2005), pp. 381-386.
- **K. Ramdani, T. Takahashi and M. Tucsnak**, Uniformly exponentially stable approximations for a class of second order evolution equations application to LQR problems. ESAIM Control. Optim. Calc. Var., 13, (2007), pp. 503-527.
- **S. Ervedoza and E. Zuazua**, Uniform exponential decay for viscous damped systems. Progr. Nonlinear Differential Equations Appl. 78, (2009), pp. 95-112.

(WE) with internal damping

$$(S)\begin{cases} u_{tt}-\Delta u+\gamma u_t & =0,\Omega \\ u & =0,\partial\Omega \\ +I.C & \end{cases}$$

(WE) with boundary damping

$$(S)\begin{cases} u_{tt}-\Delta u+u_t & =0,\Omega \\ u & =0,\partial\Omega_1 \\ \frac{\partial u}{\partial \nu}+\gamma u_t & =0,\partial\Omega_2 \\ +I.C & \end{cases}$$

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Remark

For the wave equation with internal or boundary friction damping, the dissipation is relatively strong so that the energy method can be applied to obtain the exponential stability as well as the uniformly exponential stability for the approximation. However, the dissipation in the thermoelastic system, due to heat conduction, is much weaker.

Approximation of thermoelastic system by SM, FEM, MFE

$$
(S)\begin{cases}
u_{tt} - \Delta u + \gamma(-\Delta)^{\frac{1}{2}}\theta & = 0, \quad \Omega \times (0, +\infty),\\
\theta_t - \Delta\theta + \gamma(-\Delta)^{\frac{1}{2}}u_t & = 0, \quad \Omega \times (0, +\infty),\\
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Z. Y. Liu and S. Zheng, Uniform exponential stability and approximation in control of a thermoelastic system. SIAM J. Control Optim. 32, (1994), pp. 1226-1246.

Theorem : Z. Y. Liu and S. Zheng
Let $T_n(t)$ $(n = 1, \ldots)$ be a sequence of semigroups of contraction on the Hilbert spaces H_n and A_n be the corresponding infinitesimal generators. Then $T_n(t)$ $(n = 1, \ldots)$ are uniformly exponentially stable if and only if

$$\sup_{n \in \mathbb{N}} \{Re\lambda;\ \lambda \in \sigma(A_n)\} = \sigma_0 < 0$$

and

$$\sup_{Re\lambda \geqslant 0, n \in \mathbb{N}} \{\|(\lambda I - A_n)^{-1}\|\} < \infty$$

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hold.

Remark : This theorem generalize the already stated result by Gearhart, Prüss and Huang for a single semigroup $T(\cdot)$.

Numerical simulation for exp.case

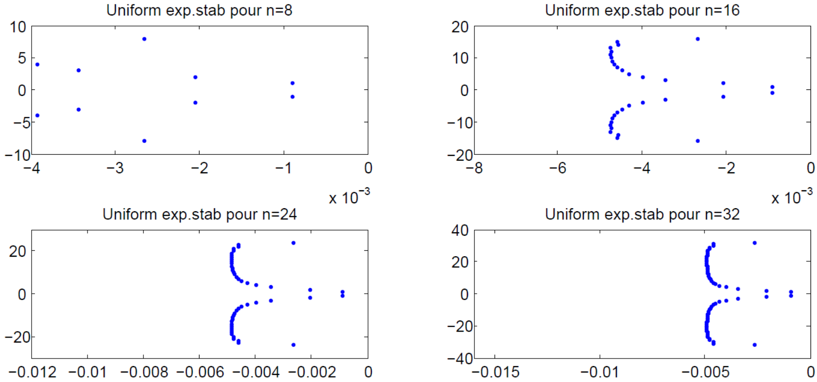

Table – Distance between $\sigma(A_n)$ and the imaginary axis for the spectral method in the case of Dirichlet-Dirichlet boundary conditions.

n	min{-Reλ, $\lambda \in \sigma(A_n)$}
8	8.9227×10^{-4}
16	8.9383×10^{-4}
24	8.9402×10^{-4}
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Theorem : Hansen 1992

If $\gamma < 1/2$. Eigenvalues of the generators A_n $(z_n' = A_n z_n, z_n(0) = z_{n0})$ satisfy

$$\sup_{\lambda \in \sigma(A_n) - \{0\}} Re\lambda \leqslant -\frac{\gamma^2}{2}.$$

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Farid Ammar Khodja, Assia Benabdallah, and Djamel Teniou, Stability of coupled systems, Abstr. Appl. Anal. Volume 1, Number 3 (1996), 327-340.

Remark : There are situations in practice where a semigroup can decays polynomially to 0, but not exponentially, in the following sense

$$\|tT(t)(-A)^{-\alpha}\| \leq C, \qquad \forall t > 0.$$

Uniform polynomial case case

$\exists ? C, \alpha > 0$ independent of n s.t :

$$\|tT_n(t)A_n^{-\alpha}x\| \leq C\|x\|$$

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- **Louis Tebou**, Stabilization of some coupled hyperbolic/parabolic equations. Discrete & Continuous Dynamical Systems B, 2010, 14 (4) : 1601-1620.
- **J. Hao and Z. Liu**, Stability of an abstract system of coupled hyperbolic and parabolic equations. Zeitschrift für angewandte Mathematik und Physik,64, (2013), pp. 1145-1159.

Polynomial Stability of of a single semigroup $T(\cdot)$

Theorem. (Borichev and Tomilov, 2009)

Let $T(t)$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that $i\mathbb{R} \subset \rho(A)$. Then for a fixed $\alpha > 0$ the following conditions are equivalent :

$$
\begin{array}{lrl}
(i) & \|R(is, A)\| = O(|s|^{\alpha}), & s \to \infty. \\
(ii) & \|T(t)(-A)^{-\alpha}\| = O(t^{-1}), & t \to \infty. \\
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- **Borichev Alex, Tomilov Yu**, Optimal polynomial decay of functions and operator semigroups (2009).

Uniform Polynomial Stability of $T_n(\cdot)$

Theorem : S. N and L. Maniar 2016

Let $T_n(t)$ $(n = 1, \ldots)$ be a uniformly bounded sequence of C_0-semigroups on the Hilbert spaces H_n and let A_n be the corresponding infinitesimal generators, such that $i\mathbb{R} \subset \rho(A_n)$ and $\sup\limits_{n\in\mathbb{N}} ||A_n^{-1}|| < \infty$. Then for a fixed $\alpha > 0$ the following conditions are equivalent :

1. $$\sup_{s,\ n\in\mathbb{N}} |s|^{-\alpha}\|R(is, A_n)\| < \infty.$$
2. $$\sup_{t\geqslant 0,\ n\in\mathbb{N}} \|tT_n(t)A_n^{-\alpha}\| < \infty.$$
3. $$\sup_{t\geqslant 0,\ n\in\mathbb{N}} \|t^{\frac{1}{\alpha}}T_n(t)A_n^{-1}\| < \infty.$$

L. Maniar and S. Nafiri, Approximation and uniform polynomial stability of C_0-semigroups, ESAIM : COCV 22 (2016), pp. 208–235.

Application 1 : General thermoelastic model

$0 \leq \tau < \frac{1}{2}$:

$$\left\{ \begin{array}{l} \ddot{u}_n + \rho B_n u_n - \mu B_n^\tau \theta_n = 0, \\ \dot{\theta}_n + \kappa B_n \theta_n + \sigma B_n^\tau \dot{u}_n = 0, \\ u_n(0) = u_{0n},\ \dot{u}_n(0) = u_{1n},\ \theta_n(0) = \theta_{0n}, \end{array} \right.$$

$$\Updownarrow$$

$$\left\{ \begin{array}{l} x_n' = \mathcal{A}_{\tau,n} x_n, \\ x_n(0) = x_{n0}, \end{array} \right.$$

Hypothesis

- H_n family of Hilbert spaces.
- $B_n : D(B_n) \subset H_n \to H_n$, selfadjoint, positive definite, B_n^{-s} compact for positive s, $0 \in \rho(B_n)$ and $\sup\limits_{n \in \mathbb{N}} || B_n^{\frac{-1}{2}} || < \infty$.
- $\mathcal{H}_n = D(B_n^{\frac{1}{2}}) \times H_n \times H_n$.

Consequences

- $\mathcal{A}_{\tau,n}$ generates a family of C_0-semigroupes of contraction $S_{\tau,n}(t)$.
- $i\mathbb{R} \subset \rho(\mathcal{A}_{\tau,n}), \quad n \in \mathbb{N}$.
- $\sup\limits_{n\in\mathbb{N}} ||\mathcal{A}_{\tau,n}^{-1}|| < \infty$.

Theorem

Assume $0 \leq \tau < \frac{1}{2}$. Then, the semigroup generated by $\mathcal{A}_{\tau,n}$ is uniform. poly. stable with order $\alpha = 2(1-2\tau)$.

Since $\mathcal{A}_{\tau,n}$ verifies the hypothesis of the main theorem, then

$$\sup_{|\beta|\geq 1,\ n\in\mathbb{N}} \frac{1}{|\beta|^{2(1-2\tau)}} \|(i\beta I_{3n} - \mathcal{A}_{\tau,n})^{-1}\| < \infty.$$

$$\Updownarrow$$

$$\exists C > 0, \alpha > 0 : \|tT_n(t)(-A_n)^{-\alpha}\| \leq C, \qquad \forall t > 0, \forall n \in \mathbb{N}.$$

Application 2

$$(S)\begin{cases} u_{tt}(x,t)-u_{xx}(x,t)+\gamma\theta(x,t)=0 \quad in\ (0,\pi)\times(0,\infty),\\ \theta_t(x,t)-k\theta_{xx}(x,t)-\gamma u_t(x,t)=0 \quad in\ (0,\pi)\times(0,\infty),\\ u(x,t)\mid_{x=0,\pi}=0=\theta(x,t)\mid_{x=0,\pi} \qquad on\ (0,\infty),\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ \theta(x,0)=\theta_0(x) \quad on\ (0,\pi), \end{cases}$$

- F. A. Khodja, A. Benabdallah and D. Teniou, Dynamical stabilizers and coupled systems, ESAIM Proceedings, Vol. 2 (1997), 253-262.
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By introducing new variable (velocity)

$$v = u_t, \tag{1}$$

system (S) can be reduced to the following abstract first order evolution equation :

$$(S)\left\{\begin{array}{l} \frac{dz}{dt} = \mathcal{A}z \\ z(0) = z_0 \end{array}\right.$$

with

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ D^2 & 0 & -\gamma \\ 0 & \gamma & kD^2 \end{pmatrix}$$

$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ the state space equipped with the norm

$$\|z\|_{\mathcal{H}} = \left(\|Dz_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|z_3\|_{L^2}^2\right)^{\frac{1}{2}},$$

Here we have used the notation $D = \partial/\partial x,\ D^2 = \partial^2/\partial x^2$.

Let

$$E_j = \begin{pmatrix} \phi_j \\ 0 \\ 0 \end{pmatrix}, \quad E_{n+j} = \begin{pmatrix} 0 \\ \psi_j \\ 0 \end{pmatrix}, \quad E_{2n+j} = \begin{pmatrix} 0 \\ 0 \\ \xi_j \end{pmatrix}, \qquad j = 1, ..., n$$

be a basis for the finite dimensional space $\mathcal{H}_n = H_1^n(\Omega) \times H_2^n(\Omega) \times H_3^n(\Omega) \subset H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \subset \mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. The inner product on $\mathcal{H}_n$ is the one induced by the $\mathcal{H}$ product. We consider the approximation to system (S) of the form

$$z_n = \sum_{j=1}^{3n} \tilde{z}_j(t) E_j(x),$$

which is required to satisfy the following variational system :

$$(\dot{z}_n, E_i)_{\mathcal{H}} = (\mathcal{A} z_n, E_i)_{\mathcal{H}}, \qquad i = 1, ..., 3n.$$

$$M_n \dot{\tilde{z}}_n = \begin{bmatrix} M_n^{(1)} & & \\ & M_n^{(2)} & \\ & & M_n^{(3)} \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}}_n^{(1)} \\ \dot{\tilde{z}}_n^{(2)} \\ \dot{\tilde{z}}_n^{(3)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \tilde{D}_n^T & 0 \\ -\tilde{D}_n & 0 & -\gamma \tilde{F}_n \\ 0 & \gamma \tilde{F}_n^T & -G_n \end{bmatrix} \begin{bmatrix} \tilde{z}_n^{(1)} \\ \tilde{z}_n^{(2)} \\ \tilde{z}_n^{(3)} \end{bmatrix} = \tilde{A}_n \tilde{z}_n$$

with

$$(M_n^{(1)})_{ij} = (D\phi_i, D\phi_j)_{L^2}, \quad (M_n^{(2)})_{ij} = (\psi_i, \psi_j)_{L^2}, \quad (M_n^{(3)})_{ij} = (\xi_i, \xi_j)_{L^2},$$

$$(\tilde{D}_n)_{ij} = (D\phi_i, D\psi_j)_{L^2}, \quad (\tilde{F}_n)_{ij} = (\xi_i, \psi_j)_{L^2}, \quad (G_n)_{ij} = (D\xi_i, D\xi_j)_{L^2}$$

and

$$\tilde{z}_n^{(i)} = (\tilde{z}_{(i-1)n+1}, ..., \tilde{z}_{in})^T, \qquad i = 1, 2, 3.$$

By construction, the matrix $M_n^{(i)}$ is symmetric and positive definite. Therefore, there exists a lower triangle matrix $L_n^{(i)}$ such that

$$M_n^{(i)} = (L_n^{(i)})^T (L_n^{(i)}),$$

and denote $L_n \tilde{z}_n$ by $\bar{z}_n$,
then

$$\dot{\bar{z}}_n = A_n \bar{z}_n$$

with

$$A_n = \begin{bmatrix} 0_{\mathbb{C}^n} & (L_1^T)^{-1} \tilde{D}_n^T L_2^{-1} & 0_{\mathbb{C}^n} \\ -(L_2^T)^{-1} \tilde{D}_n L_1^{-1} & 0_{\mathbb{C}^n} & -\gamma (L_2^T)^{-1} \tilde{F}_n L_3^{-1} \\ 0_{\mathbb{C}^n} & \gamma (L_3^T)^{-1} \tilde{F}_n^T L_2^{-1} & -(L_3^T)^{-1} G_n L_3^{-1} \end{bmatrix}.$$

It is easy to see that

$$(A_n \bar{z}_n, \bar{z}_n)_{\mathbb{C}^{3n}} = -(G_n L_3^{-1} \bar{z}_n^{(3)}, L_3^{-1} \bar{z}_n^{(3)})_{\mathbb{C}^n} \leq 0$$

provided that G_n is semipositive definite.
$\Longrightarrow A_n$ generates a $C0$-semigroup $T_n(t)$ of contraction on $\mathcal{H}_n$

Let
$\phi_j = \sqrt{\frac{2}{\pi}}\frac{1}{j}\sin jx, \quad \psi_j = \sqrt{\frac{2}{\pi}}\sin jx, \quad \xi_j = \sqrt{\frac{2}{\pi}}\sin jx, \qquad j = 1, ..., n.$
the eigenvalues of (S), then

$$A_n = \begin{bmatrix} 0 & D_n & 0 \\ -D_n & 0 & -\gamma \\ 0 & \gamma & -D_n^2 \end{bmatrix}$$

with

$$D_n = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & n \end{bmatrix}.$$

Uniform Polynomial Stability with spectral element method

Theorem

The semigroups generated by A_n are uniformly polynomially stable. Moreover, we have :

$$\sup_{|\beta|\geq 1,\ n\in\mathbb{N}} \frac{1}{\beta^2}\|(i\beta I - A_n)^{-1}\| < \infty,$$

$$\Updownarrow$$

$$\sup_{t\geqslant 0,\ n\in\mathbb{N}} \|t^{\frac{1}{2}} T_n(t) A_n^{-1}\| < \infty.$$

Finite difference semi-discretization

$$x_0 = 0 < x_1 = \Delta < \cdots < x_{n-1} = (n-1)\Delta < x_n = \pi$$

$$\begin{cases} \ddot{u}_j(t) + \frac{1}{\Delta^2}[u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)] + \gamma\theta_j(t) = 0, t > 0, j = 1, \ldots, n-1 \\ \dot{\theta}_j(t) + \frac{1}{\Delta^2}[\theta_{j+1}(t) - 2\theta_j(t) + \theta_{j-1}(t)] - \gamma\dot{u}_j(t) = 0, t > 0, j = 1, \ldots, n-1 \\ u_0(t) = u_n(t) = \theta_0(t) = \theta_n(t) = 0,\ t > 0 \\ u_j(0) = u_{0j}, \quad \dot{u}_j(0) = u_{1j}, \quad \theta_j(0) = \theta_{0j},\ j = 0, \ldots, n \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \ddot{\mathbf{u}}_n + B_n\mathbf{u}_n + \gamma\boldsymbol{\theta}_n = 0, \\ \dot{\boldsymbol{\theta}}_n + B_n\boldsymbol{\theta}_n - \gamma\dot{\mathbf{u}}_n = 0, \\ \mathbf{u}_n(0) = \mathbf{u}_{0n}, \quad \dot{\mathbf{u}}_n(0) = \mathbf{u}_{1n}, \quad \boldsymbol{\theta}_n(0) = \boldsymbol{\theta}_{0n}, \end{cases}$$

$$\frac{dU_n}{dt} = \mathcal{A}_n U_n, \quad U_n(0) = U_{0n},$$

$$\mathcal{A}_n = \begin{bmatrix} 0 & I_n & 0 \\ -B_n & 0 & -\gamma I_n \\ 0 & \gamma I_n & -B_n \end{bmatrix}, B_n = \frac{1}{\Delta^2} \begin{bmatrix} 2 & -1 & & & \mathbf{0} \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ \mathbf{0} & & & -1 & 2 \end{bmatrix}$$

Finite element semi-discretization

$$\begin{cases} \ddot{\mathbf{u}}_n + B_n \mathbf{u}_n + \gamma \boldsymbol{\theta}_n = 0, \\ \dot{\boldsymbol{\theta}}_n + B_n \boldsymbol{\theta}_n - \gamma \dot{\mathbf{u}}_n = 0, \\ \mathbf{u}_n(0) = \mathbf{u}_{0n}, \quad \dot{\mathbf{u}}_n(0) = \mathbf{u}_{1n}, \quad \boldsymbol{\theta}_n(0) = \boldsymbol{\theta}_{0n}, \end{cases}$$

where $B_n = (M_n^{(2)})^{-1} M_n^{(1)}$.

$$M_n^{(1)} = \frac{1}{\Delta} \begin{bmatrix} 2 & -1 & & & \mathbf{0} \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ \mathbf{0} & & & -1 & 2 \end{bmatrix}, M_n^{(2)} = \Delta \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & & & \mathbf{0} \\ \frac{1}{6} & \frac{2}{3} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \frac{2}{3} & \frac{1}{6} \\ \mathbf{0} & & & \frac{1}{6} & \frac{2}{3} \end{bmatrix}$$

Uniform Polynomial Stability with finite difference et finite element method

Theorem

The semigroups generated by A_n are uniformly polynomially stable. Moreover, we have :

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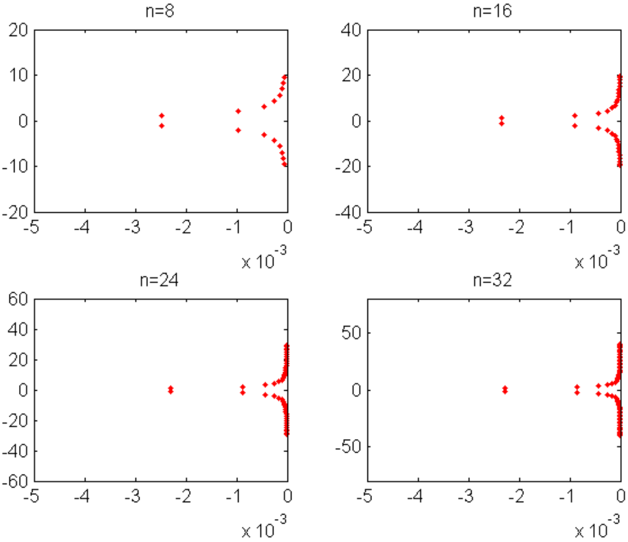
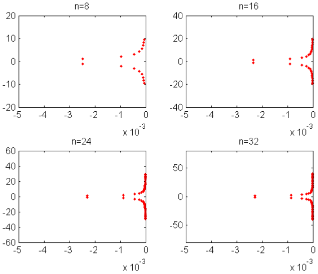
$$\Updownarrow$$

$$\sup_{t\geqslant 0,\ n\in\mathbb{N}} \|t^{\frac{1}{2}} T_n(t) A_n^{-1}\| < \infty.$$

Numerical experiments





Theorem : Batkai and al 2006, Borichev and Tomilov 2010
If A is the generator of a contraction polynomially stable (of order $\alpha > 0$) C_0-semigroup on a Hilbert space X. Fix $\delta > 0$ s.t $[0, \delta] \subset \rho(A)$. Then we have for some constant C

$$|Im\lambda| \geqslant C(Re\lambda)^{-\frac{1}{\alpha}} \text{ for all } \lambda \in \sigma(A) \text{ with } Re\lambda \leqslant \delta.$$

Influence of regularity on the decay of energy !

For $T = 100$ and $\Delta t = 10^{-2}$, we consider the following initila value :

$$u(x,0) = 0, \qquad \theta(x,0) = 0, \qquad u_t(x,0) = \sqrt{\frac{2}{\pi}} sin(jx),\ j = 1,2,3.$$

Influence of regularity on the decay of energy !

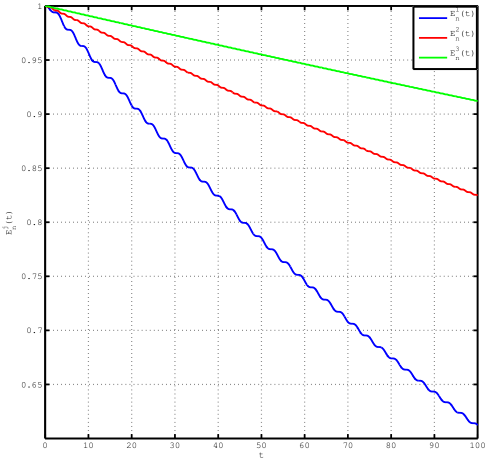


Figure – Effect of smoothness of the initial data on the rate of decay of energy.

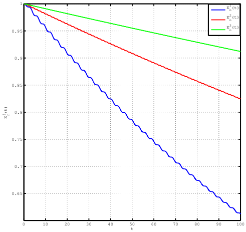


Figure – Effect of smoothness of the initial data on the rate of decay of energy.

Theorem : Batkai and al 2006

If A is the generator of a contraction C_0-semigroup on a Banach space X, with $0 \in \rho(A)$. Then we have the equivalence with $s > 0$

(a) $\|T(t)A^{-s}\| = O(t^{-r})$, $t \to +\infty$

(b) $\|T(t)A^{-s\xi}\| = O(t^{-r\xi})$, $t \to +\infty,\ \xi > 0.$

Conclusion and open problems

- Motivation behind the study of thermoelastic systems.
- Asymptotic behaviour of the solutions of thermoelastic models.
- Numerical simulations for uniform exponential and uniform polynomial decay.
- If we consider other **B.C**.
- Numerical study when $d = 2, 3$.
- Consider non autonomous case $A(t),\ t > 0$
- Characterize polynomial decay in Banach spaces.
- etc...

Thank you for your attention
Gracias