Lack of Null Controllability of Viscoelastic Flows.

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Outline

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Linear Viscoelastic flows

Fluid flow with velocity u , constant density $\rho > 0$, pressure p and stress tensor τ .

Conservation of mass and momentum:

$$\rho \partial_t u + \nabla p = \mathrm{div} \tau, \quad \mathrm{div} u = 0.$$

Constitutive Law: Relates the stress tensor to the motion.

Newtonian Fluid : $\tau_{New} = \eta(\nabla u + (\nabla u)^T), \quad \eta > 0,$

Linear viscoelastic fluid : Stress depends on the current motion as well as the history.
Maxwell model:

$$\tau_t + \lambda \tau = \kappa \Big(\nabla u + (\nabla u)^T \Big),$$

$\lambda > 0,\ \kappa > 0,\ 1/\lambda =$ Stress relaxation time.

Jeffreys and Maxwell Models

Jeffreys model: Stress is a linear combination of Newtonian and Maxwell stress:

$$\text{Total Stress} = \tau_{New} + \tau_{Max}.$$

The control system : $\Omega \subset \mathbb{R}^d$**,** $d = 2, 3$

$$\begin{aligned}
&\rho u_t \; - \; \eta \Delta u + \nabla p = \mathrm{div}\tau + f\chi_{\mathcal{O}} \quad \text{in } \Omega \times (0, T), \\
&\mathrm{div}\; u = 0 \quad \text{in } \Omega \times (0, T), \quad u = 0 \quad \text{in } \partial\Omega \times (0, T), \\
&\tau_t + \lambda \tau = \kappa(\nabla u + (\nabla u)^T) \quad \text{in } \Omega \times (0, T), \\
&u(\cdot, 0) = u_0, \quad \tau(\cdot, 0) = \tau_0 \quad \text{in } \Omega,
\end{aligned}$$

f is a control localized in $\mathcal{O}$, an open subset of Ω.

- For $\eta > 0$, Jeffreys system,
- For $\eta = 0$, Maxwell system.

Relaxation modes

Experimental data for viscoelastic fluids indicate :
several relaxation modes are needed to fit the data.

- Single relaxation mode:

$$\tau_t + \lambda\tau = \kappa\Big(\nabla u + (\nabla u)^T\Big), \quad \lambda > 0, \quad \kappa > 0,$$

- Several relaxation mode: $\tau = \sum_{i=1}^{N} \tau_i,$, for $i = 1, \cdots, N$,

$$(\tau_i)_t + \lambda_i\tau_i = \kappa_i\Big(\nabla u + (\nabla u)^T\Big), \quad \lambda_i > 0, \quad \kappa_i > 0,$$

- Infinite relaxation mode: If $\tau = \sum_{i=1}^{\infty} \tau_i,$, each τ_i satisfying the above ODE.

Question: Are these systems Controllable ?

Outline

Known results

- **Renardy, (Syst. Control Lett. 54, 2005):** 1D models. Exact controllability for single-mode Maxwell and approximate controllability for multimode Maxwell and Jeffreys fluids.
- **Doubova, Fernández-Cara, (Syst. Control Lett. 61, 2012):** Multi-D model - single mode Jeffreys - approximate controllability result for velocity only.
- **Boldrini et. al. , (SICON, 2012):** -single mode Maxwell - exact controllability result is obtained under an underdamped assumption on the system.
- **Chowdhury et. al. , (JMFM, 2017):** Several improvements : Approximate controllability for both variables, (Jeffreys and Maxwell). Exact controllability of Single mode Maxwell without the undamped condition.

Question: Null controllability of Jeffreys and several mode Maxwell fluids.

Outline

Our Result

Theorem (M, Mitra, Renardy, ESAIM COCV, 2018)

$\mathcal{O} \subset \Omega$. *Jeffreys system (single mode or multimode) and multi-mode Maxwell system are not null controllable in any time* $\tau > 0$.

Methods :

- Several Mode Maxwell : Construct initial data so that the solution is not C^∞ at time T.
- Jeffreys Fluid : $\mathrm{Ran}\mathbb{T}_T \not\subset \mathrm{Ran}\Phi_T$.

Relation with memory type problem

The stress tensor is

$$\tau(x,t) = \tau_0 + \int_0^t e^{-\lambda(t-s)}\Big(\nabla u + (\nabla u)^T\Big)ds.$$

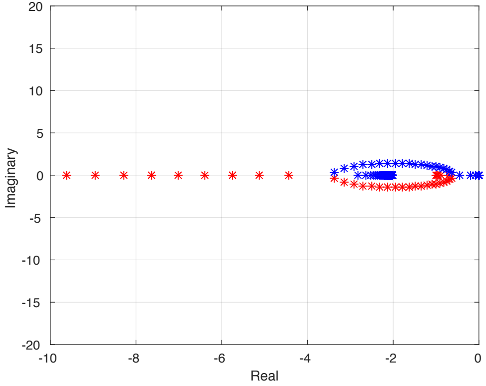
If $\tau_0 = 0$, then Jeffreys system can be seen as Stokes equation with memory :

$$\partial_t u - \eta\Delta u + \nabla p - \kappa \int_0^t e^{-\lambda(t-s)}\Delta u(s) = \chi_{\mathcal{O}} f, \quad \mathrm{div}\ u = 0.$$

- Lack of null controllability is expected.
- Accumultion point in the spectrum.

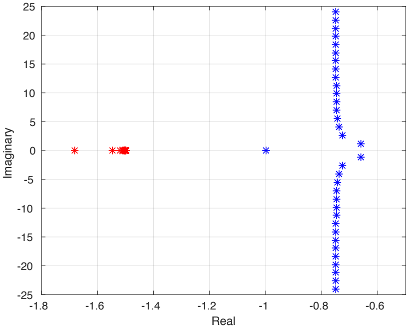
Spectrum of single mode Jeffreys system

- $\eta > 0$.



Spectrum of multimode Maxwell system

- $N = 2$, $\eta = 0$.



• In general $(N-1)$ convergent real eigenvalue and one pair of complex eigenvalue (like wave). • $N = 1 \sim$ Wave. (Exact Controllability)

Suitable space for null controllability

The stress tensor is

$$\tau(x,t) = \tau_0 + \int_0^t e^{-\lambda(t-s)}\Big(\nabla u + (\nabla u)^T\Big)\,ds.$$

- Necessarily, $\tau_0 \in R_0 := \left\{\nabla f + \nabla f^\top | f \in H_0^1, \mathrm{div}\ f = 0\right\}.$
- Hence we seek $\tau = (\nabla v + (\nabla v)^T) = 2Dv$, for divergence free v vanishing on the boundary.

Thus the new system for divergence free (u, v) vanishing on boundary,

$$\partial_t u \ = \ \eta \Delta u + \sum_{i=1}^N \Delta v_i - \nabla p + f\chi_{\mathcal{O}} \quad \text{in } \Omega \times (0,T), \quad i = 1, \cdots, N,$$

$$\partial_t v_i + \lambda_i v_i = \kappa_i u \quad \text{in } \Omega \times (0,T), \quad i = 1, \cdots, N,$$

$$\mathrm{div}\ u = \mathrm{div}\ v = 0.$$

Sketch of the proof : Double mode Maxwell ($\eta = 0$)

- The solution can not be smooth at any time T. (Propagation of singularities, uniqueness theorem ...)
- Set $a = (\operatorname{curl} u)_1$, and $b_i = (\operatorname{curl} v_i)_1$.
- (a, b_i) solves

$$\begin{cases} \partial_t a \ = \ \displaystyle\sum_{i=1}^{N} \Delta b_i + (\operatorname{curl} f)_1 & x \in \Omega \\ \partial_t b_i + \lambda_i b_i = \kappa_i a & x \in \Omega \\ a(\cdot, 0) = a_0 = \operatorname{curl} u_0, \quad b_i(\cdot, 0) = b_{i,0} = \operatorname{curl} v_{i,0}. \end{cases}$$

- a can not be smooth in any time T.

Sketch of the proof

Step 1. Propagation of Singularities in $\mathbb{R}^d$.

$$\begin{cases} \rho \widetilde{a}_t = \sum_{i=1}^{N} \triangle \widetilde{b}_i & \text{in } (0,\tau)\times\mathbb{R}^d, \\ (\widetilde{b}_i)_t + \lambda_i \widetilde{b}_i = \kappa_i \widetilde{a} & \text{in } (0,\tau)\times\mathbb{R}^d, \\ \widetilde{a}(0) = \widetilde{a}_0, \quad \widetilde{b}_i(\cdot,0) = \widetilde{b}_{i,0}. \end{cases}$$

- $x_0 \in \Omega \setminus \mathcal{O}$. Construct $(\widetilde{a}_0, \widetilde{b}_{i,0})$ such that $\widetilde{a}(t,\cdot)$ has singularity at x_0 for any $t > 0$. In particular, we show that $(x_0, t, \xi, 0) \in WF(\tilde{a})$.
- Choice of $\widetilde{a}_0$ and $\widetilde{b}_{i,0}$:

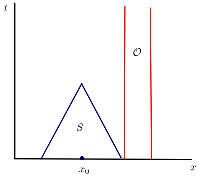
$$\mathcal{F}(\widetilde{a}_0) = i\xi_2 \frac{e^{-ix_0\cdot\xi}}{(1+|\xi|^2)^k}, \quad \mathcal{F}(\widetilde{b}_{i,0}) = \mathcal{F}(\widetilde{a}_0) e_j(\xi),$$

where $e_j(\xi)$ being the eigenfunction corresponding to the accumulating eigenvalue.

Sketch of the proof

Step 2. Singularity of a from singularity of $\widetilde{a}$. Holmgreen's uniqueness theorem.

If a_0 and $\widetilde{a}_0$ agrees in a small neighbourhood around x_0, then there exists a region S such that $a = \widetilde{a}$ in S.



- $(x_0, t, \xi, 0) \in WF(a)$ for $t \in (0, T_0)$.

Sketch of the proof

Step 3. Singularity of a propagates for all the time. Due to Hörmander.

Theorem (Hormander)

Let $P(D)$ be of real principal type and principal symbol p_m. If $u \in \mathcal{D}'(\Omega \times (0, T))$, $P(D)u = f$ and $((x_0, t_0), (\xi, \tau)) \in WF(u) \backslash WF(f)$, then $p_m(\xi, \tau) = 0$ and $I \times \{(\xi, \tau)\} \subset WF(u)$, provided $I \subset \Omega \times (0, T)$ is a line segment containing (x_0, t_0) with direction $\nabla p_m(\xi, \tau)$ such that $I \times \{(\xi, \tau)\}$ does not meet $WF(f)$.

- $\nabla p_m(\xi, \tau) \sim (0, 1)$. The line segment (x_0, t) does not meet the observation region. In particular $(x_0, t, \xi, 0) \in WF(a)$ for any $t > 0$.

Step 4. Construct $u_0, v_{i,0}$ such that $a_0 = \operatorname{curl} u_0, ...$

Sketch of the proof : Jeffreys model ($\eta > 0$)

Recall:

$$\partial_t u \;=\; \eta\Delta u + \Delta v - \nabla p + f\chi_{\mathcal{O}} \quad \text{in } \Omega \times (0, T),$$
$$\partial_t v + \lambda v = \kappa u \quad \text{in } \Omega \times (0, T),$$
$$\text{div } u = \text{div } v = 0.$$

- $\mathrm{Ran}\Phi_\tau$ contains smooth functions outside $\mathcal{O}$.
- Consider the free dynamics
- We can find initial data such that

$$u(t,x) = c_0(t)u_0(x) + \text{ regular terms }, \qquad c_0 \in C^\infty[0,\tau].$$

- Possible because of accumulating eigenvalue.
- Similar result holds for for several mode Jeffreys system.

Thank You