Large-time dynamics of the particle and kinetic Kuramoto models with frustration

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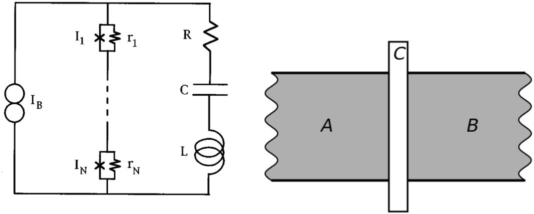
Array of Josephson junctions



Josephson junctions shunted in parallel by an inductor-capacitor-resistor load

Equation of currents

$$\frac{\hbar}{2er_j}\dot{\phi}_j + I_j \sin\phi_j + \dot{Q} = I_B, \quad j = 1, \cdots, N$$

$$L\ddot{Q} + R\dot{Q} + C^{-1}Q = \frac{\hbar}{2e}\sum_{k=1}^{N}\dot{\phi}_k.$$

Kuramoto model with frustration

Averaging on $I_j = \bar{I}(1+\varepsilon\eta_j), r_j = \bar{r}(1+\varepsilon\mu_j)$ leads to the following equations up to first order of ε.

$$\dot{\theta}_j = \Omega_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j + \alpha), \quad t > 0,$$

where

$$K = \frac{N\bar{r}\bar{\omega}(\frac{2e}{\hbar}\bar{r}I_B - \bar{\omega})}{[(L\bar{\omega}^2 - C^{-1})^2 + \bar{\omega}^2(R + N\bar{r})^2]^{1/2}},$$

$$\cos\alpha = \frac{L\bar{\omega}^2 - C^{-1}}{[(L\bar{\omega}^2 - C^{-1})^2 + \bar{\omega}^2(R + N\bar{r})^2]^{1/2}}.$$

This equation is known as the Kuramoto model [Wiesenfeld, Colet, Strogatz (1998)].

Goal of this talk

Today, I would like to address emergent dynamics of the Kuramoto model with frustrations via analysis and numerics.

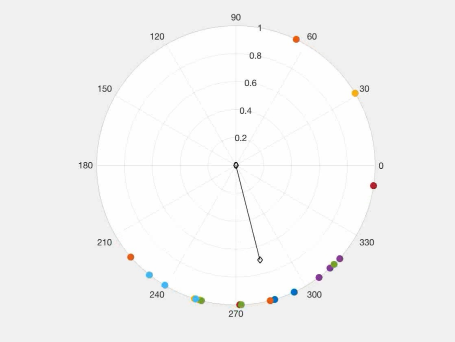
1. Synchronization on small frustration ($|\alpha| \ll 1$)
2. Simulations for subcritical frustrations ($|\alpha| < \pi/2$)
3. Solitary waves under critical frustration ($\alpha = \pi/2$)

Kuramoto model with frustration

Let $\theta_j(t) \in \mathbb{T}$ be the phase of the j-th Kuramoto oscillator, then

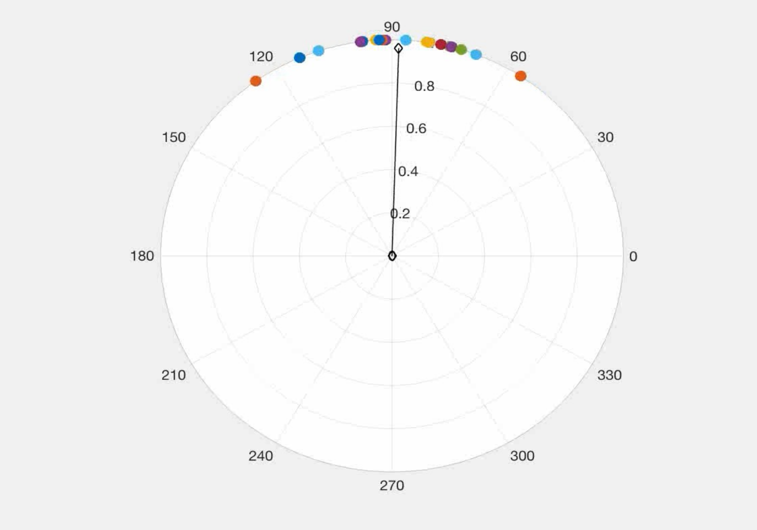
$$\dot{\theta}_j = \Omega_j + \frac{K}{N}\sum_{k=1}^{N} \sin(\theta_k - \theta_j + \alpha), \quad t > 0.$$

We may define the real order parameter R to see the synchronization.



$$R = \frac{1}{N}\left|\sum_{j=1}^{N} e^{\theta_j\sqrt{-1}}\right|$$

Kuramoto model with frustration ($|\alpha| < \pi/2$)



Synchronization result ($\alpha = 0$)

Synchronization Theorem [Ha, Kim, Ryoo (2016)]

Suppose that natural frequencies are distributed and initial configuration satisfy

$$R^0 = \frac{1}{N}\Big|\sum_k e^{i\theta_k^0}\Big| > 0, \quad \theta_j^0 \neq \theta_k^0, \quad 1 \le j \neq k \le N.$$

Then there exists a phase-locked state Θ^∞ and a constant $K_\infty > 0$ such that if $K \ge K_\infty$, then the solution goes to Θ^∞,

$$\lim_{t\to\infty} ||\Theta(t) - \Theta^\infty||_{\ell^\infty} = 0.$$

Main idea : The model is a gradient flow on $\mathbb{R}^N$.

1) Monotonicity of R,

2) Existence of positively invariant domain,

3) Linearization near equilibrium.

Synchronization on small frustration ($\alpha \ll 1$)

Synchronization Theorem [Ha, K., Zhang (2018) SIAM J Appl Dyn Syst.]

Let Θ be a solution with initial data Θ^0 satisfying conditions:

$$|\alpha| < \arctan\left(\frac{1}{2\sqrt{N}}\right), \quad R^0 > \frac{1}{N} + \frac{\cos\alpha + \sin|\alpha|}{1+\cos\alpha}, \quad \theta_j^0 \neq \theta_k^0 \text{ for all } j \neq k.$$

Then there exists a phase-locked state Θ^∞ such that the solution goes to Θ^∞,

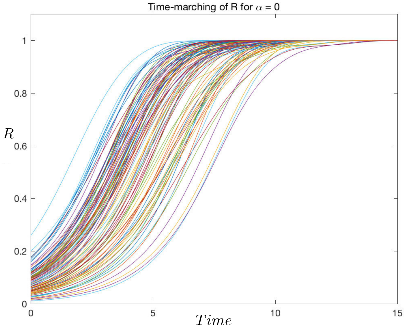
$$\lim_{t\to\infty} ||\Theta(t) - \Theta^\infty||_{\ell^\infty} = 0.$$

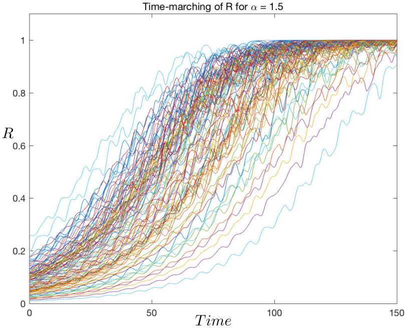
Main idea : This model is no longer a gradient flow.

1) Existence of lower bounds of R,
2) Existence of positively invariant domain for large R,
3) Linearization near equilibrium.

Simulations along frustration

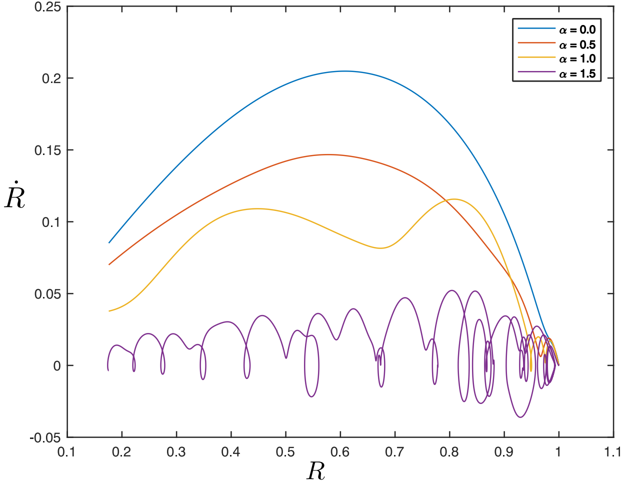
Dynamics of R for $\alpha = 0, 1.5$





Monotonicity breaks for $\alpha \neq 0$.

Simulations along frustration



Phase diagram $R - \dot{R}$ graph for each $\alpha = 0, 0.5, 1.0, 1.5$

Sketch of proof ($\alpha \ll 1$)

Existence of lower bound of R

For any positive odd $k < N$, we have

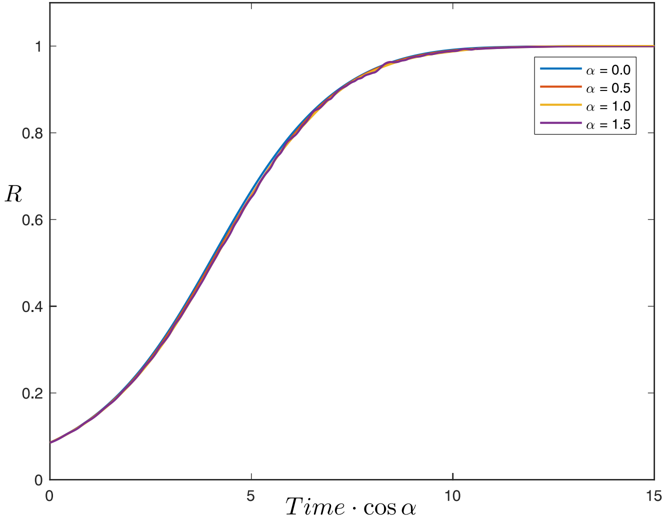
$$\dot{R}\Big|_{R=1-k/N} > 0.$$

Sketch of proof

Equation can be described as

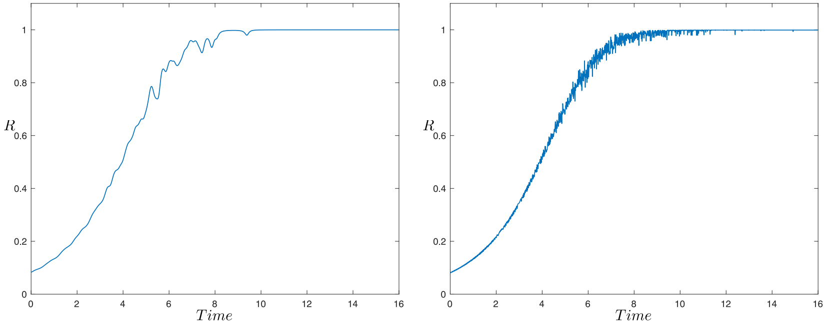
$$\dot{\theta}_j = K\cos\alpha \underbrace{\frac{1}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j)}_{\text{Synchronization term}} + K\sin\alpha \underbrace{\frac{1}{N}\sum_{k=1}^{N}\cos(\theta_k - \theta_j)}_{\text{Wave-generating term}}.$$

Simulations along frustration



Averages of order parameter R on $\alpha = 0.0, 0.5, 1.0, 1.5$.

Simulations near critical points



Order parameter R for $\alpha = \pi/2 - 1,\ \pi/2 - 0.001$

Summary for subcritical frustration ($|\alpha| < \pi/2$)

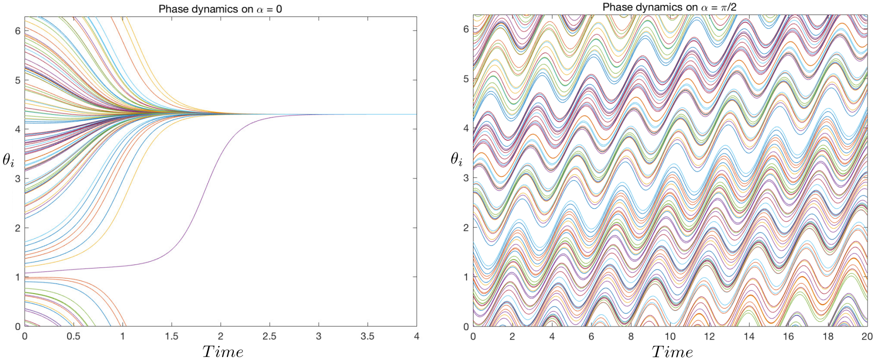
Remarks

- Synchronization can be proved similarly on small frustration.
- Synchronization is observed for generic initial data.

Difficulties

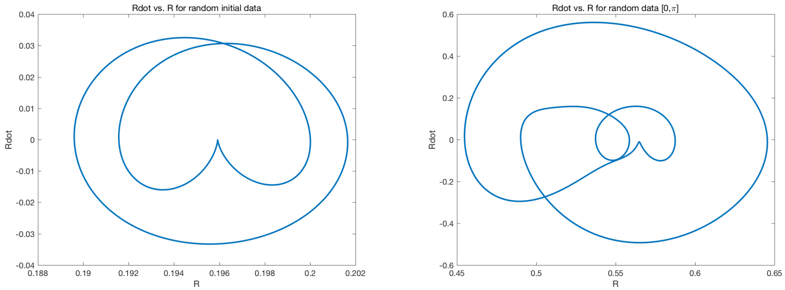
- Simulation near $\alpha = \pi/2$ needs smaller time steps and large final time.
- Computational costs depend on N^2 from nonlocal interactions.

Kuramoto model with frustration ($|\alpha| = \pi/2$)



Phases along time, left : $\alpha = 0$, right : $\alpha = \pi/2$.

Simulations along frustration



Phase diagram $R - \dot{R}$ graph for $\alpha = \pi/2$

Questions

- Are there nontrivial periodic solutions?
- Is every nontrivial solution periodic? Numerics says "Yes", but...

Integrability of the system for identical oscillators

At $\alpha = \pi/2$, the model reduces to,

$$\dot{\theta}_j = \frac{1}{N}\sum_{k=1}^{N} \cos(\theta_k - \theta_j).$$

Integrability at ($\alpha = \pi/2$) [Watanabe, Strogatz (1993)]

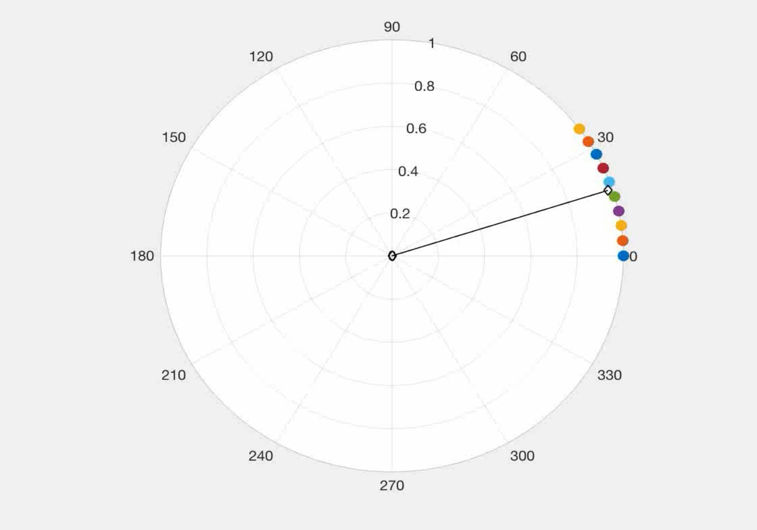
Let $S_{i,j} := \sin[(\theta_i - \theta_j)/2]$ and

$$J = S_{1,2}S_{2,3}\cdots S_{N-1,N}S_{N,1} \text{ is a constant of motion.}$$

Precisely, $N-2$ constant of motions are functionally independent.

2 degrees of freedom: every solution is periodic, up to rotational symmetries of S^1. (quasiperiodic)

Particle simulation



Kinetic Kuramoto model

When we apply mean-field limit ($N \to \infty$) on the Kuramoto particle model, we have the equation of a distribution function $f = f(\theta, \Omega, t)$;

$$
\begin{aligned}
&\partial_t f + \partial_\theta \cdot (\mathcal{V}(f) f) = 0, \\
&\mathcal{V}(f)(\theta, \Omega, t) = \Omega - K \int_{\mathbb{R}} \int_0^{2\pi} \sin(\psi - \theta + \alpha) f(\psi, \Omega_*, t) d\psi d\Omega_*, \\
&f(\theta, \Omega, 0) = f_0(\theta, \Omega), \quad \int_0^{2\pi} f(\theta, \Omega, 0) d\theta = g(\Omega), \\
&\rho(\theta, t) := \int_{\mathbb{R}} f(\theta, \Omega, t) d\Omega.
\end{aligned}
$$

This limit process let us know the motion of density function $\rho(\theta, t)$, which doesn't distinguish individuals.

Kinetic Kuramoto model

The existence and uniqueness of measure-valued (or classical) solutions can be treated by Neunzert's theory since velocity field $\mathcal{V}(f)$ is bounded and Lipshitz on θ, Ω, f. [Lancelloti 2005]

$$\partial_t f + \partial_\theta \cdot (\mathcal{V}(f) f) = 0,$$

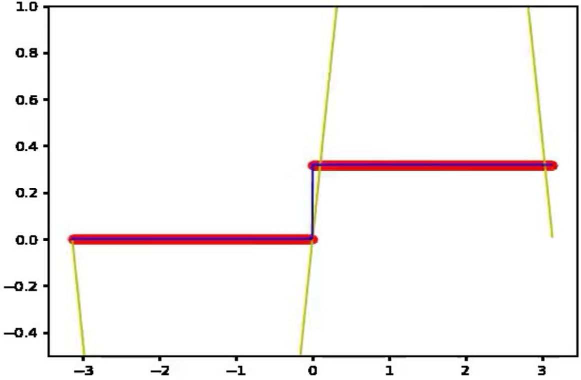
$$\mathcal{V}(f)(\theta, \Omega, t) = \Omega - K \int_{\mathbb{R}} \int_0^{2\pi} \sin(\psi - \theta + \alpha) f(\psi, \Omega_*, t) d\psi d\Omega_*.$$

If it has a smoothing heat kernal, incoherent solutions are stable for large conductivity. [Qinghua, Ha 2016] for $\alpha = 0$, [Ha, Kim, Lee, Zhang 2018] for $\alpha \neq 0$.

$$\partial_t f + \partial_\theta \cdot (\mathcal{V}(f) f) = \sigma \partial_\theta^2 f,$$

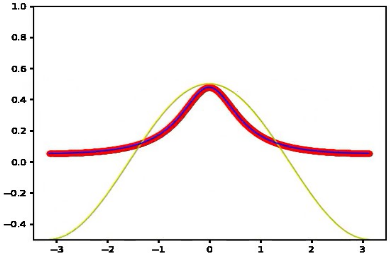
$$\mathcal{V}(f)(\theta, \Omega, t) = \Omega - K \int_{\mathbb{R}} \int_0^{2\pi} \sin(\psi - \theta + \alpha) f(\psi, \Omega_*, t) d\psi d\Omega_*.$$

Kinetic simulations



Spacial discretize : ENO 3rd upwind scheme.
Temporal derivative : RK 3rd,
Positivity guaranteed for cfl < 1. 1000 mesh points.

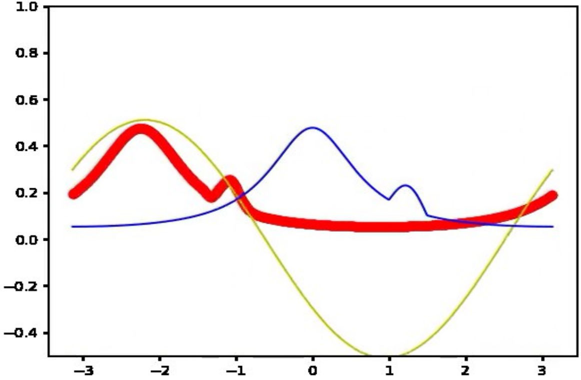
Kinetic simulations



$$\rho(\theta, t) = \frac{1}{2\pi} \frac{V}{U - A\cos(\theta - Ut)}, \quad \mathcal{V}(\rho) = A\cos(\theta - Ut),$$

are known periodic solutions for $U = \frac{1+A^2}{2}$, $V = \frac{1-A^2}{2}$, and $A \in [0, 1]$.

Kinetic simulations



Every solution seems to be quasi-periodic.

Summary for the critical frustration

Observations on critical frustration

1. Nonlinear waves are generated.
2. There exists 1-parameter family of solitary waves.
3. No proper schemes for nonlocal kinetic equations in literature.

Remarks on the critical frustration

1. Numerically $R^0 \neq 0$ and $\alpha \neq \pi/2$ lead to the synchronization.
2. Periodicity (Quasi-periodicity) in kinetic model is still open.
3. Main difficulty is that it is not a gradient flow.

Thank you