Synchronization of the Kuramoto model under noise

Dongnam Ko (高東男)

Joint work with Barbara Gentz, Seung-Yeal Ha, Christian Weisel

Seoul National University

January 9, 2017

Overview

1. Emergent behavior of oscillators
   - Dynamics of Oscillators
   - The Kuramoto model
   - Known Results on the Kuramoto model
2. Synchronization in the Stochastic Kuramoto model
   - Synchronization transition time
   - Emergence of synchronization
3. Summary

The Kuramoto model

Kuramoto model (1975) is one of the synchronization model that describes individuals as phase. (Metronomes, Electric signal, Biological cells, $\cdots$)

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i), \quad 1 \le i \le N,\ t > 0,$$

where $\theta_i(t) \in \mathbb{R}$ or S^1 and constants K, Ω_i such that

1. The coupling strength, $K \geq 0$,
2. A natural frequency, Ω_i satisfy $\sum \Omega_i = 0$.

The Kuramoto model

Kuramoto model (1975) is one of the synchronization model that describes individuals as phase. (Metronomes, Electric signal, Biological cells, $\cdots$)

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i), \quad 1 \le i \le N, \ t > 0,$$

where $\theta_i(t) \in \mathbb{R}$ or S^1 and constants K, Ω_i such that

1. The coupling strength, $K \ge 0$,
2. A natural frequency, Ω_i satisfy $\sum \Omega_i = 0$.

Our main concern ; Behavior of stochastic model.

$$d\theta_i = \Big[\Omega_i + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i)\Big]dt + \sigma dB_i(t), \quad 1 \le i \le N, \ t > 0,$$

where σ is a noise strength (infinitesimal standard deviation).

Properties of (deterministic) Kuramoto model

1. Order parameter

$$re^{i\phi} := \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

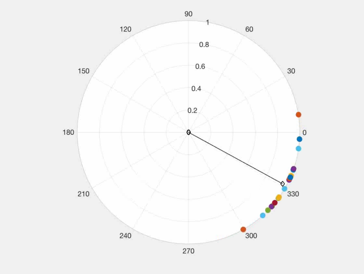
$r = 1$; complete phase synchronization,
$r \simeq 0$; incoherent states.

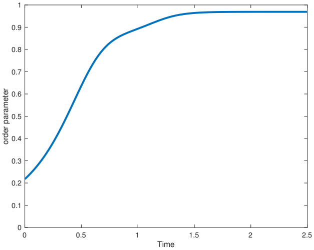
2. Potential flow (Gradient system)

$$\dot{\Theta} = -\nabla V[\Theta], \quad V[\Theta] := -\sum_{k=1}^{N} \Omega_k \theta_k + \frac{K}{2N} \sum_{k,l=1}^{N} \big(1 - \cos(\theta_k - \theta_l)\big).$$

Potential flow stabilizes to equilibrium states or goes to infinity.

Order parameter





Emergence of synchronizaion comes with increasing r.

The Kuramoto model can be represented in terms of order parameter,

$$\frac{d\theta_i}{dt} = \Omega_i - Kr\sin(\theta_i - \phi).$$

Definition of synchronization

Let $\Theta = (\theta_1, \cdots, \theta_N)$ is a system of phase oscillators in Kuramoto model.

Definition

1. Θ tends to a phase-locked state if it goes to an equilibritum solution $(\theta_1^\infty, \cdots, \theta_N^\infty)$:

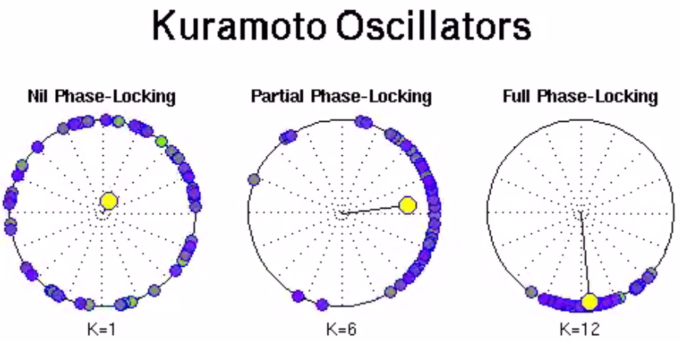
$$\Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j^\infty - \theta_i^\infty) = 0, \quad i = 1, \cdots, N.$$

2. Θ tends to a (frequency) synchronization if it satisfies

$$\lim_{t\to\infty} |\dot\theta_i(t) - \dot\theta_j(t)| = 0, \quad 1 \le i, j \le N.$$

Synchronization $\Leftrightarrow$ Θ converges to a stable phase-locked state,
and not follows an unbounded curve.

Synchronization of the Kuramoto model



Nil, partial and full phase-locking in an all-to-all network of Kuramoto oscillators. Phase-locking is governed by the coupling strength K and the distribution of intrinisic frequencies ω. Here, the intrinsic frequencies were drawn from a normal distribution (M=0.5Hz, SD=0.5Hz). The yellow disk marks the phase centroid. Its radius is a measure of coherence.

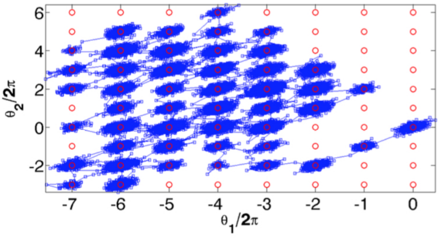
(from wikipedia)

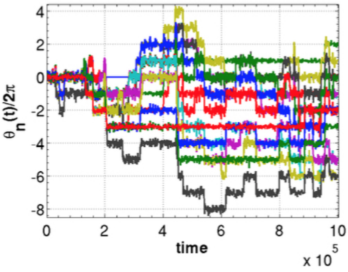
Known Results on the Kuramoto model $(N < \infty)$

- Basic Potential analysis for large K [Aeyels, 2004]

$$\left(\frac{\partial^2 V}{\partial \theta_i \partial \theta_j}\right)_{ij} = \frac{K}{N}(-D + vv^T + ww^T).$$

- Chaotic phenomena when K is less than critical K [Maistrenko, 2005]
- Sync-transition for identical stochastic osc. [Lee Deville, 2011]





Known Results on the Kuramoto model $(N < \infty)$

- Lyapunov analysis starting from half circle, large K [Choi, Ha, Jung, Kim, 2012]

  $\max|\theta_i - \theta_j|$ decreases exponentially, and
  $|\theta_i - \theta_j|$ decreases exponentially when $\max|\theta_i - \theta_j| < \pi/2$.

- Potential analysis especially for identical oscillators [Dong, Xue, 2013]

$$V[\Theta] := -\sum_{k=1}^{N} \Omega_k \theta_k + \frac{K}{2N} \sum_{k,l=1}^{N} \big(1 - \cos(\theta_k - \theta_l)\big).$$

- Synchronization on general initial conditions, large K [Ha, Kim, Ryoo, 2016]

  r has a hierarchy of positively invariant sets.

Stochastic Kuramoto model

What is the difference on Synchronization under noise?

$$d\theta_i = \left[\Omega_i + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i)\right]dt + \sigma dB_i(t), \quad 1 \le i \le N,\ t > 0,$$

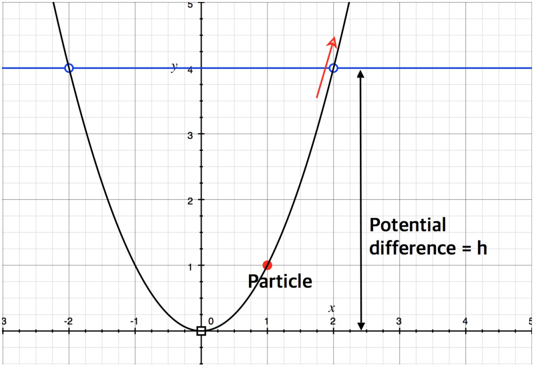
We assume $\sigma^2 \ll K$ since we want to see the small noise effect.
(Large noise will make incoherent state)

Two significant difference from small additive noise.

1. A phase-locked state is no more stable in global time.
   $\rightarrow$ We estimate the transition time, from one to other states.

2. A particle-path can be described only under some probability.
   $\rightarrow$ We want to show asymptotic behavior to a phase-locked state.

Large-deviation theory on potential flow

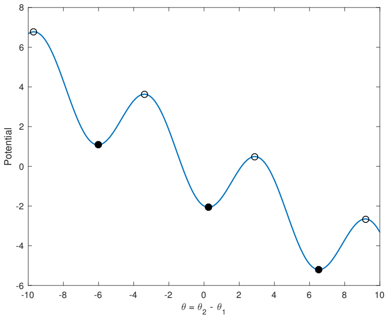


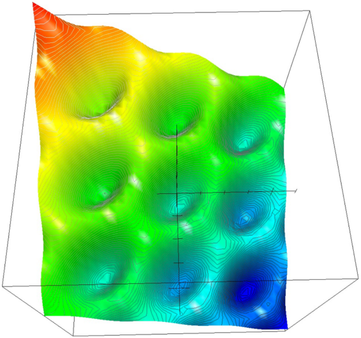
A probability P of escaping the potential wall of h is given by Large-deviation principle:

$$\mathbb{P}\{\text{escape before time } T\} \sim T \exp\{-h/\sigma^2\}, \qquad \text{for } h/\sigma^2 \gg 1.$$

We need to classify eq. states and find h.

Potential in terms of the Order parameter





Left : two oscillators, Right : three oscillators.

Order parameter r plays a key role in the analysis of potential.

$$V[\Theta] = -\sum_{i=1}^{N} \Omega_i \theta_i - \frac{KN}{2} r^2.$$

Example : Two oscillator case

Define $\theta = \theta_2 - \theta_1$, $\Omega = \Omega_2 - \Omega_1$, then

$$\frac{d\theta}{dt} = \Omega - K \sin\theta.$$

This equation has a bifurcation at $K = \Omega$.

If we assume $K > \Omega > 0$, then there exist stable points,

one of the stable point is

$$\theta_{\text{stable}} = \arcsin\left(\frac{\Omega}{K}\right),$$

and its corresponding lowest saddle point exists at

$$\theta_{\text{unstable}} = \pi - \arcsin\left(\frac{\Omega}{K}\right).$$

Example : Two stochastic oscillator case

If we consider the stochastic version,

$$d\theta = (\Omega - K \sin\theta)dt + \sqrt{2}\sigma W_t,$$

where W_t is a Brownian motion.

Since

$$V(\theta) = -\Omega\theta - K\cos(\theta),$$

the potential difference between eq. points is

$$\Delta V = -\Omega\left(\pi - 2\arcsin\left(\frac{\Omega}{K}\right)\right) - K\cos\arcsin\left(\frac{\Omega}{K}\right).$$

Therefore, for large K,

$$\mathbb{P}\{\text{escape before time } T\} \sim T\exp\left(-\frac{\Delta V}{\sigma^2}\right) \sim T\exp(-K/\sigma^2).$$

Potential analysis (N oscillators)

Classification of equilibrium points

Kuramoto model has a fixed point if and only if there exists $r \in [0,1]$ such that

$$\frac{|\Omega_j|}{Kr} \leq 1, \quad r = \frac{1}{N}\sum_{j=1}^{N} s_j \sqrt{1 - \left(\frac{\Omega_j}{Kr}\right)^2}, \tag{1}$$

for some choice of $s = (s_j)_{j=1}^{N} \in \{-1, 1\}^N$.
In this case, the equilibrium point up to rotation is

$$\theta_j = \begin{cases} \arcsin\left(\frac{\Omega_j}{Kr}\right), & s_j = 1, \\ sgn(\Omega_j)\pi - \arcsin\left(\frac{\Omega_j}{Kr}\right), & s_j = -1. \end{cases}$$

Theorem [Aeyels, 2004]

The only possible stable point comes from $s_j = 1$ for all j.

Potential analysis

Theorem 1 [Gentz, Ha, K, Weisel]

There exists a constant $K_0 = K_0(\Omega, N)$ such that for any $K > K_0$, there exists fixed points with $(\tau > 0)$ for each $(s_j)_{j=1}^N$.

Moreover, the lowest saddle comes from $(s_i = -1)$ when $|\Omega_i| \geq |\Omega_j|$ for all j. Then the potential height from stable point is described by

$$\Delta V = 2K\frac{N-1}{N} - |\Omega_i|\pi + O(K^{-1}).$$

Idea of proof

For the existence of equilibrium points, define

$f(r) = r - \frac{1}{N}\sum_{j=1}^N s_j\sqrt{1-\left(\frac{\Omega_j}{Kr}\right)^2}$, and use the convexity of function.

To see the stability, we need spectral analysis on Jacobian $H = -\frac{\partial^2 V}{\partial x_i \partial x_j}$.

Conclusion of Potential analysis

Corollary : Synchronization transition time

A stable phase-locked state is no longer stable in global time,

From Theorem 1; if we start from a stable point,
The expectation of transition time is the order of $\exp\{K/\sigma^2\}$.

Conclusion of Potential analysis

Corollary : Synchronization transition time

A stable phase-locked state is no longer stable in global time,

From Theorem 1; if we start from a stable point,
The expectation of transition time is the order of $\exp\{K/\sigma^2\}$.

Next question : Emergent behavior

Prove the emergence of synchronization for large K/σ^2,
under the non-escaping probablity.

$$1 - \mathbb{P}\{\text{escape before time } T\} \sim 1 - T\exp\{-K/\sigma^2\},$$

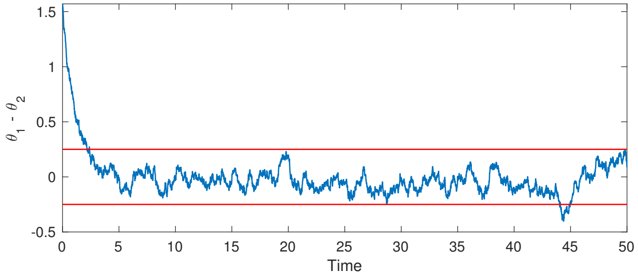
Definition of entrainment

Question : How we can define a phase-locked state under noise.

Achieving entrainment

Two oscillators θ_1, θ_2 achieves entrainment, if there exist small probability p on time $0 < T_1 < T_2$ and a bounded domain $[b_1, b_2]$ such that

$$\mathbb{P}\{ \ \forall \ t \in [T_1, T_2] \ \ : \theta_t \in [b_1, b_2] \ \} \geq 1 - p, \qquad \text{where } \theta_t = \theta_2(t) - \theta_2(t).$$



Definition of entrainment

Question : How we can define a phase-locked state under noise.

Achieving entrainment

Two oscillators θ_1, θ_2 achieves entrainment, if there exist small probability p on time $0 < T_1 < T_2$ and a bounded domain $[b_1, b_2]$ such that

$$\mathbb{P}\{ \ \exists \ t \in [T_1, T_2] \ \ : \theta_t \notin [b_1, b_2] \ \} \leq p, \qquad \text{where } \theta_t = \theta_2(t) - \theta_2(t).$$

Emergence of synchronization

The emergence of synchronization in stochastic model occurs when the all oscillators achieves entrainment with small p,

$$p \sim T_2 \exp(-K/\sigma^2).$$

Two oscillator case

Question : How we can get time-marching behavior of stochastic oscillators?

We start from two oscillators and

Linearize it.

Linearization of oscillators leads to the Ornstein-Uhlenbeck Process.

$$d\theta_t = (\Omega - K\sin\theta_t)dt + \sqrt{2}\sigma dW_t,$$

$$\text{Linearize : } d\theta_t^K = (\Omega - K\theta_t^K)dt + \sqrt{2}\sigma dW_t$$

$$\Rightarrow \theta_t^K = \frac{\Omega}{K} + \left(\theta_0 - \frac{\Omega}{K}\right)e^{-Kt} + \sqrt{2}\sigma Z_t^K,$$

where the Ornstein-Uhlenbeck $Z_t^K := \int_0^t e^{-K(t-s)}dW_s$.

Two oscillator case

To track the two oscillator phase difference θ, we consider the estimation,

$$-K\theta < -K\sin\theta < -K\frac{\sin D_\infty}{D_\infty}\theta \ =: -KR_\infty\theta, \quad \text{for } 0 < \theta < D_\infty < \pi.$$

Define the stopping time

$$\tau_b^{\text{hit}} = \inf\{\ t > 0 \mid \theta_t = b\ \},$$

then we can conclude from comparison principle,

$$\theta_t^K \le \theta_t \le \theta_t^{KR_\infty}, \quad \text{for } t \le \tau_0^{\text{hit}} \wedge \tau_{D_\infty}^{\text{hit}}.$$

Therefore, the problem is on estimation of θ_t^ν.

$$\theta_t^\nu = \frac{\Omega}{\nu} + \left(\theta_0 - \frac{\Omega}{\nu}\right)e^{-\nu t} + \sqrt{2}\sigma Z_t^\nu.$$

Estimation of θ_t^ν

The estimation on θ_t^ν starts from the Ornstein-Uhlenbeck process.

Lemma : Estimation on Ornstein-Uhlenbeck process

For any constants $\varepsilon, \sigma > 0$ and time T, we have

$$\mathbb{P}\{\sup_{0\le t\le T} \sqrt{2}\sigma|Z_t^\nu| \ge \varepsilon\} \le 2[\nu T + 1] \exp\left(-\frac{\nu\varepsilon^2}{2\sigma^2 e^2}\right)$$

Estimation of θ_t^ν

The estimation on θ_t^ν starts from the Ornstein-Uhlenbeck process.

Lemma : Estimation on Ornstein-Uhlenbeck process

For any constants $\varepsilon, \sigma > 0$ and time T, we have

$$\mathbb{P}\{\sup_{0\le t\le T} \sqrt{2}\sigma|Z_t^\nu| \ge \varepsilon\} \le 2[\nu T + 1] \exp\left(-\frac{\nu\varepsilon^2}{2\sigma^2 e^2}\right)$$

Lemma : Estimation on θ_t^ν

For any constants $\varepsilon, \sigma > 0$ and time T, we have

$$\mathbb{P}\{\sup_{0\le t\le T} |\theta_t^\nu - [\theta_t^\nu]_{\text{det}}| \le \varepsilon\} \ge 1 - 2[\nu T + 1] \exp\left(-\frac{\nu\varepsilon^2}{2\sigma^2 e^2}\right),$$

where $[\theta_t^\nu]_{\text{det}} = \frac{\Omega}{\nu} + \left(\theta_0 - \frac{\Omega}{\nu}\right) e^{-\nu t}$.

Two particle result

Theorem 2 [Gentz, Ha, K, Weisel]

Assume $\sigma^2 \ll K$, $K > \max\left\{\frac{4\sigma^2}{R_\infty \varepsilon^2}, \frac{4|\Omega|}{R_\infty \varepsilon}\right\}$, $0 \le \theta_0 < D_\infty < \pi$, and $\varepsilon < 8\min\{D_\infty/9, D_\infty - \theta_0, D_\infty - \frac{\Omega}{KR_\infty}\}$, where $R_\infty = \sin D_\infty / D_\infty$. Then there exist constants $C, c > 0$ and $T_* = \frac{1}{K}\log\frac{16\pi}{\varepsilon}$ s.t.

$$\mathbb{P}\{\,\exists s \in [T_*, t] :\ \theta_s \notin [-\varepsilon, \varepsilon]\} \le C(KR_\infty t + |\log\varepsilon| + 1)e^{-c\frac{\varepsilon^2 KR_\infty}{\sigma^2}}.$$

Therefore, the two-oscillator system has emergence of synchronization.

Two particle result

Theorem 2 [Gentz, Ha, K, Weisel]

Assume $\sigma^2 \ll K$, $K > \max\left\{\frac{4\sigma^2}{R_\infty \varepsilon^2}, \frac{4|\Omega|}{R_\infty \varepsilon}\right\}$, $0 \le \theta_0 < D_\infty < \pi$, and $\varepsilon < 8\min\{D_\infty/9, D_\infty - \theta_0, D_\infty - \frac{\Omega}{KR_\infty}\}$, where $R_\infty = \sin D_\infty / D_\infty$. Then there exist constants $C, c > 0$ and $T_* = \frac{1}{K}\log\frac{16\pi}{\varepsilon}$ s.t.

$$\mathbb{P}\{\ \exists s \in [T_*, t]:\ \theta_s \notin [-\varepsilon, \varepsilon]\} \le C(KR_\infty t + |\log\varepsilon| + 1)e^{-c\frac{\varepsilon^2 KR_\infty}{\sigma^2}}.$$

Therefore, the two-oscillator system has emergence of synchronization.

Proof)

We have $\theta_t^{KR_\infty} \le \theta_t \le \theta_t^K$ for $0 \le \theta_t \le D_\infty$, and estimate θ_t^K and $\theta_t^{KR_\infty}$.
The only problems are on the stopping times $\tau_0^{\text{hit}}, \tau_{D_\infty}^{\text{hit}}$.

Sketch of proof

Lemma 1 (From θ_0 to $\varepsilon/2$)

In the same condition on Theorem 2, there exist constants $C, c > 0$ such that

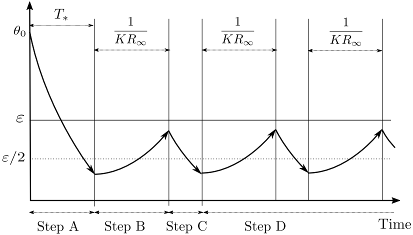
$$\mathbb{P}\left\{\theta_s \geq \frac{\varepsilon}{2} : \ \forall s < \frac{1}{K}\log\frac{16\pi}{\varepsilon}\right\} \leq 2C(|\log\varepsilon| + 1)e^{-c\frac{\varepsilon^2 K}{\sigma^2}}.$$

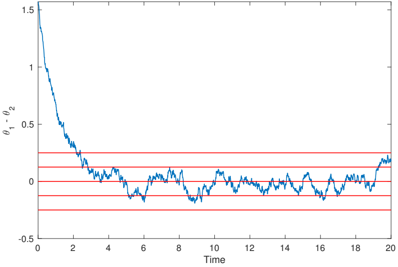
Lemma 2 (From $\varepsilon/2$ to ε)

Assume $|\theta_0| \leq \frac{\varepsilon}{2}$ and the same condition of Theorem 2. Then there exists constant $c > 0$ such that

$$\mathbb{P}\left\{|\theta_s| \geq \varepsilon : \ \exists s < \frac{1}{KR_\infty}\right\} \leq e^{-c\frac{\varepsilon^2 KR_\infty}{\sigma^2}}.$$

Sketch of proof





Step A : Using Lemma 1, we can confirm that θ is in $[-\varepsilon/2, \varepsilon/2]$ within time $t = T_* := \frac{1}{K}\log\frac{16\pi}{\varepsilon}$, in a probability of order $\exp\{-\frac{\varepsilon^2 K}{\sigma^2}\}$

Step B : From Lemma 2, θ does not escape $[-\varepsilon, \varepsilon]$ until time $t = T_*$ in a same order of probability.

Step C, D : Iterate Step A, B.

Proof of lemma 1

Lemma 1 (From θ_0 to $\varepsilon/2$)

In the same condition on Theorem 2, there exist constants $C, c > 0$ s.t.

$$\mathbb{P}\left\{\theta_s \geq \frac{\varepsilon}{2} : \ \forall s < \frac{1}{K}\log\frac{16\pi}{\varepsilon}\right\} \leq 2C(|\log\varepsilon| + 1)e^{-c\frac{\varepsilon^2 K}{\sigma^2}}.$$

Proof) Consider the probability,

$$\mathbb{P}\left\{\theta_s \geq \frac{\varepsilon}{2} : \ \forall s < t \wedge \tau^{\text{hit}}_{D_\infty}\right\}.$$

From comparison principle, $\theta_t^{KR_\infty} \leq \theta_t \leq \theta_t^K$ for $0 \leq \theta_t \leq D_\infty$,

$$\mathbb{P}\left\{\theta_s \geq \frac{\varepsilon}{2} : \ \forall s < t \wedge \tau^{\text{hit}}_{D_\infty}\right\} \leq \mathbb{P}\left\{\theta_s^{KR_\infty} \geq \frac{\varepsilon}{2} : \ \forall s < t \wedge \tau^{\text{hit}}_{D_\infty}\right\}.$$

We can use comparison only before time $\tau^{\text{hit}}_{D_\infty}$, but we want to get rid of $\tau^{\text{hit}}_{D_\infty}$ on the estimation of θ_t^{ν}.

Proof of lemma 1

Hence we need the estimation of stopping time $\tau_{D_\infty}^{\text{hit}}$.

$$\begin{aligned}
&\mathbb{P}\left\{\theta_s \geq \frac{\varepsilon}{2} :\ \forall s < t \wedge \tau_{D_\infty}^{\text{hit}}\right\} \\
\leq &\mathbb{P}\left\{\tau_{D_\infty}^{\text{hit}} \leq t \wedge \tau_{\varepsilon/2}^{\text{hit}}\right\} \\
&\quad + \mathbb{P}\left\{\theta_s^{KR_\infty} \geq \frac{\varepsilon}{2} :\ \forall s < t \wedge \tau_{D_\infty}^{\text{hit}} \text{ and } \tau_{D_\infty}^{\text{hit}} > t \wedge \tau_{\varepsilon/2}^{\text{hit}}\right\}.
\end{aligned}$$

The first term can be obtained from the estimation on $\theta_t^{KR_\infty}$.

$$\begin{aligned}
\mathbb{P}\left\{\tau_{D_\infty}^{\text{hit}} \leq t \wedge \tau_{\varepsilon/2}^{\text{hit}}\right\} &\leq \mathbb{P}\left\{\theta_s \geq D_\infty :\ \exists\ s < t \wedge \tau_{\varepsilon/2}^{\text{hit}}\right\} \\
&\leq \mathbb{P}\left\{\theta_s^{KR_\infty} \geq D_\infty :\ \exists\ s < t\right\}.
\end{aligned}$$

Proof of lemma 1

Since we know

$$\mathbb{P}\left\{\theta_s^{KR_\infty} - [\theta_s^{KR_\infty}]_{\mathsf{det}} \geq \varepsilon : \ \exists\, s < t\right\} \leq 2(KR_\infty t + 1)\exp(-\frac{\varepsilon^2 KR_\infty}{2\sigma^2 e^2}),$$

$\mathbb{P}\left\{\tau_{D_\infty}^{\mathsf{hit}} \leq t \wedge \tau_{\varepsilon/2}^{\mathsf{hit}}\right\}$ is less then this probability when

$$[\theta_s^{KR_\infty}]_{\mathsf{det}} + \varepsilon < D_\infty.$$

Therefore, we need the assumption

$$\max\{\theta_0, \frac{\Omega}{KR_\infty}\} + \varepsilon < D_\infty.$$

Proof of lemma 1

The second term can be calculated in the same way.

$$\mathbb{P}\left\{\theta_s^{KR_\infty} \geq \frac{\varepsilon}{2} :\ \forall s < t \wedge \tau_{D_\infty}^{\text{hit}} \text{ and } \tau_{D_\infty}^{\text{hit}} > t \wedge \tau_{\varepsilon/2}^{\text{hit}}\right\}$$
$$= \mathbb{P}\left\{\theta_s^{KR_\infty} \geq \frac{\varepsilon}{2} :\ \forall s < t\right\}.$$

Here also we use

$$\mathbb{P}\left\{\theta_s^{KR_\infty} - [\theta_s^{KR_\infty}]_{\text{det}} \geq \frac{\varepsilon}{8} :\ \exists\ s < t\right\} \leq 2(KR_\infty t + 1)\exp(-\frac{\varepsilon^2 KR_\infty}{32\sigma^2 e^2}),$$

In order to use the above inequality, we need

$$[\theta_s^{KR_\infty}]_{\text{det}} + \frac{\varepsilon}{8} = \frac{\Omega}{KR_\infty} + \left(\theta_0 - \frac{\Omega}{KR_\infty}\right) e^{-KR_\infty s} < \frac{\varepsilon}{2} \quad \text{for some } s \leq t.$$

Set $s = t = T_* = \frac{1}{K}\log\frac{16\pi}{\varepsilon}$, then the assumption $\frac{|\Omega|}{KR_\infty} < \frac{\varepsilon}{4}$ leads to the conclusion.

Proof of Lemma 2

Lemma 2 (From $\varepsilon/2$ to ε)

Assume $|\theta_0| \leq \frac{\varepsilon}{2}$ and the same condition of Theorem 2. Then there exists constant $c > 0$ such that

$$\mathbb{P}\left\{|\theta_s| \geq \varepsilon : \ \exists s < \frac{1}{KR_\infty}\right\} \leq e^{-c\frac{\varepsilon^2 KR_\infty}{\sigma^2}}.$$

proof) The linearization comparison does not work near $\theta_t = 0$.
In this part, we use $(\theta_t)^2$. For $t \leq \tau_\varepsilon := \inf\{t \geq 0 : |\theta_t| > \varepsilon\}$,

$$\begin{aligned} d\theta_t^2 &= 2\theta_t d\theta_t + d\theta_t \cdot d\theta_t \\ &= 2\theta_t\Big((-K\sin\theta_t + \Omega)dt + \sqrt{2}\sigma dB_t\Big) + 2\sigma^2 dt \\ &\leq \Big[-2KR_\varepsilon\theta_t^2 + 2\Omega\theta_t + 2\sigma^2\Big]dt + 2\sqrt{2}\sigma\theta_t dB_t. \end{aligned}$$

Proof of Lemma 2

Our object is to estimate $\mathbb{P}\{\tau_\varepsilon < T\}$ for some proper time T.

Using Itô formula,

$$\theta_t^2 e^{2KR_\varepsilon t} \leq \; 2\sqrt{2}\sigma \int_0^t \theta_s e^{2KR_\varepsilon s} dB_s + \left(\frac{\sigma^2}{KR_\varepsilon} + \frac{|\Omega|\varepsilon}{KR_\varepsilon} \right) (e^{2KR_\varepsilon t} - 1) + \theta_0^2.$$

The first term can be estimated by Martingale property of B_s.
In conclusion, using assumptions $K > \max\left\{\frac{4\sigma^2}{\varepsilon^2 R_\varepsilon}, \frac{4|\Omega|}{\varepsilon R_\varepsilon}\right\}$,

$$\mathbb{P}\{\tau_\varepsilon < T\} \leq \exp\left[-\frac{\varepsilon^2 KR_\varepsilon}{64\sigma^2 (e^{4KR_\varepsilon T} - 1)} \right].$$

We want the order of $\exp(-K/\sigma^2)$. Therefore, we choose $T = \frac{1}{KR_\infty}$ and conclude Lemma.

N particle result when initially on the half circle

Theorem 3 [Gentz, Ha, K, Weisel]

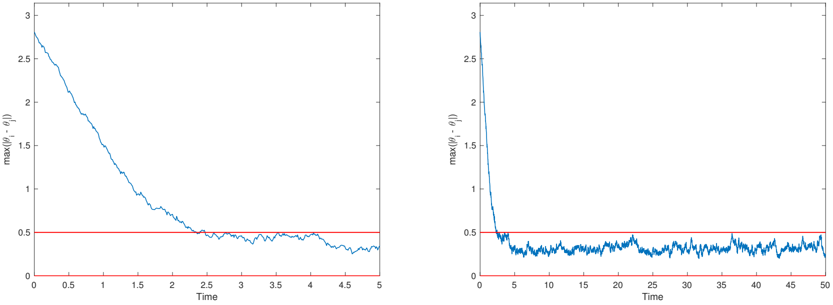
Define $D(\theta) := \max_j \theta_j - \min_j \theta_j$ in the case of $D(\theta)(0) < D_\infty < \pi$.

Assume $\sigma^2 \ll K$, $K > D(\Omega)\varepsilon^{-1}R_\infty^{-1}$, and
$\varepsilon < (D_\infty - \max\{D(\theta)(0), D(\Omega)/KR_\infty\})$.
Then there exists some constants $C, c > 0$ only depend on D_∞ such that

$$\mathbb{P}\left\{\exists\ s \leq t : D(\theta)(s) > D(\theta)(0)e^{-KR_\infty s} + \frac{D(\Omega)}{KR_\infty}\left(1 - e^{-KR_\infty s}\right) + \varepsilon\right\}$$
$$\leq CN\left[Kt + 1\right]e^{-c\frac{\varepsilon^2 KR_\infty}{\sigma^2}}.$$

Therefore, the emergence of synchronization occurs for $K/\sigma^2 \gg 1$, until the time $\ll \exp(K/\sigma^2)$.

Numerical simulations



Proof on the half circle

Idea of proof) Estimate diameter [Choi, Ha, Jung, Kim, 2012] with stochastic collision time and do estimates in a similar way as the two particle case.

The main idea comes from the following inequality,

$$\sin(\theta_M - \theta_m) + \sin(\theta_l - \theta_M) + \sin(\theta_m - \theta_l) \leq 0,$$

where θ_M and θ_m represent the maximal and minimal phase.

In the noiseless case, [Choi, Ha, Jung, Kim, 2012] showed that

$$|\theta_M - \theta_m| \text{ decreases exponentially.}$$

Proof on the half circle

Using the inequality, the relative phase dynamics

$$d(\theta_t^i - \theta_t^j)$$
$$= \Big[\Omega^i - \Omega^j + \frac{K}{N}\sum_{k=1}^{N}\Big(\sin(\theta_t^k - \theta_t^i) - \sin(\theta_t^k - \theta_t^j)\Big)\Big]dt + \sigma d(B_t^i - B_t^j)$$

leads to the differential inequality on the diameter,

$$d(\theta_t^M - \theta_t^m)$$
$$= \Big[\Omega^M - \Omega^m + \frac{K}{N}\sum_{k=1}^{N}\Big(\sin(\theta_t^k - \theta_t^M) - \sin(\theta_t^k - \theta_t^m)\Big)\Big]dt$$
$$+ \sigma d(B_t^M - B_t^m)$$
$$\leq (D(\Omega) - KR_\infty(\theta_t^M - \theta_t^m))dt + \sqrt{2}\sigma dW_t.$$

Proof on the half circle

However, we cannot control the diameter $(\theta_t^M - \theta_t^m)$ as in the deterministic Kuramoto model.

In the Kuramoto model, the equation is on the analytic function, hence the collision of oscillators does not occur infinitely many in a finite time.

In the stochastic model, Brownian motion can hit 0 infinitely in a finite time.

Proof on the half circle

Therefore, the proof consists of following steps,

1. Choose the maximum θ_t^M and minimum θ_t^m at time 0.
2. Define $\tau = \min\{\ t \geq 0\ :\ \theta_t^j > \theta_t^M + \delta$ or $\theta_t^j < \theta_t^m - \delta\ \}$ for some δ.
3. Now we can estimate $\mathbb{P}\{\tau \geq T^*\}$ for $T^* = 1/K$.
4. Construct differential inequality of $(\theta_t^M - \theta_t^m)$ for $t < \tau$.
5. $D(\theta) \leq (\theta_t^M - \theta_t^m) + 2\delta$. Iterate this procedure on each $[nT^*, (n+1)T^*]$.

Summary

Result ($K/\sigma^2 \gg 1$)

1. Synchronization transition : from potential analysis.
   The synchronization keeps until $T \sim \exp\{K/\sigma^2\}$, and after that, do transition to other stable points.
2. Emergence of synchronization : from phase diameter analysis.
   If the phase are aranged in half circle, then it monotonically goes to a stable point until $T \sim \exp\{K/\sigma^2\}$.

Side results

1. The number of positive eigenvalues for each equilibrium points.
   (The number of equilibriums is unknown)
2. Extend the ordering problems to stochastic case when $D(\Theta) < \pi/2$.
   (If $D(\Theta) > \pi$, the emergence of synchronization is open)

Thank you