Dynamics and control for the "Guidance by repulsion" model

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Motivation: Shepherd dogs and sheep

The number of individuals is small, yet the interaction dynamics and control strategies is complex

We consider the "guidance by repulsion" model based on the two-agents framework: the driver tries to drive the evader.

The drivers want to control the evaders:

1. Gathering of the evaders,
2. Driving the evaders into a desired area.

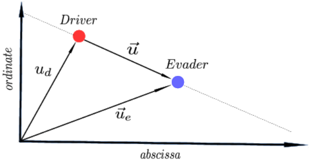


Figure: Picture of Border Collie [from Wikipedia] and the diagram of the model

Motivation: "Guidance by repulsion" model

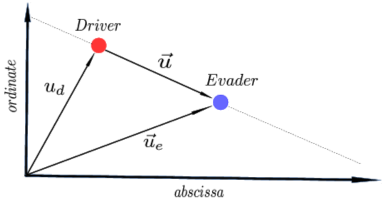
R. Escobedo, A. Ibañez and E.Zuazua, Optimal strategies for driving a mobile agent in a "guidance by repulsion" model, Communications in Nonlinear Science and Numerical Simulation, 39 (2016), 58-72.

[R. Escobedo, A. Ibañez, E. Zuazua, 2016] suggested a **guidance by repulsion** model based on the two-agents framework: *the driver*, which tries to drive the *evader*.

1. The driver follows the evader but cannot be arbitrarily close to it (because of chemical reactions, animal conflict, etc).
2. The evader moves away from the driver but doesn't try to escape beyond a not so large distance.
3. The driver is faster than the evader.
4. At a critical short distance, the driver can display a **circumvention maneuver** around the evader, forcing it to change the direction of its motion.
5. By adjusting the circumvention maneuver, the evader can be driven towards a desired target or along a given trajectory.

One sheep + one dog + Circumvention control

The control $k(t)$ is chosen in feedback form to align the gate, the sheep and the dog.



In short, the model for $\mathbf{u}_d, \mathbf{u}_e \in \mathbf{R}^2$ can be written with nonlinear interaction kernels $f_d(\cdot)$ and $f_e(\cdot)$:

$$\begin{cases} \dot{\mathbf{u}}_d = \mathbf{v}_d, \quad \dot{\mathbf{u}}_e = \mathbf{v}_e \\ m_d \dot{\mathbf{v}}_d = -f_d(|\mathbf{u}_d - \mathbf{u}_e|)(\mathbf{u}_d - \mathbf{u}_e) - \nu_d \mathbf{v}_d + \kappa(t)(\mathbf{u}_d - \mathbf{u}_e)^\perp \\ m_e \dot{\mathbf{v}}_e = -f_e(|\mathbf{u}_e - \mathbf{u}_d|)(\mathbf{u}_d - \mathbf{u}_e) - \nu_e \mathbf{v}_e \\ \mathbf{u}_d(0) = \mathbf{u}_d^0, \ \mathbf{u}_e(0) = \mathbf{u}_e^0, \ \mathbf{v}_d(0) = 0, \ \mathbf{v}_e(0) = 0 \end{cases} \tag{1}$$

Studies on the repulsive interactions

In this setting, they considered bang-bang type controls with open-loop and feed-back strategies.

Similar consideration have been addressed with repulsive interactions in control theory:

- Defender-intruder strategy : [Wang, Biegler, 2006],
- Hunting strategy model : [Muro, Escobedo, Spector, Coppinger, 2011 and 2014],
- Dog-sheep gathering problem : Well-posedness of optimal control problems [Burger, Pinnau, Roth, Totzeck, Tse, 2016] and its simulations [Pinnau, Totzeck, 2018].

In particular, [Pinnau, Totzeck, 2018] uses a transport equation:

$$\begin{cases} \partial_t \mu + \nabla_x \cdot (v\mu) = \nabla_v \cdot (G(t,x,v,\mu)\mu), \\ G(t,x,v,\mu) = \displaystyle\int_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla_x P_1(x-y) d\mu(t,y,v) \\ \qquad\qquad + \displaystyle\frac{1}{M} \sum_{j=1}^{M} \nabla_x P_2(x - \mathbf{u}_{dj}(t)) - \nu v, \\ \dot{\mathbf{u}}_{dj} = \kappa_j(t) = (\kappa_j^1(t), \kappa_j^2(t)), \quad j = 1, \dots, M, \\ \mathbf{u}_{dj}(0) = \mathbf{u}_{dj}^0 \in \mathbb{R}^2, \quad \mu(0,\cdot,\cdot) = \mu_0. \end{cases}$$

Our objective is to consider many drivers and many evaders in the Guidance-by-repulsion model, however, based on the interaction between one driver and one evader.

This is a common strategy on the study of collective behavior model, the emergent dynamics should rely on the model with small number of individuals.

Guidance-by-repulsion model with many individuals

Let $\mathbf{u}_{dj}, \mathbf{u}_{ei} \in \mathbb{R}^2$ are positions of drivers and evaders for $i = 1, \cdots, N$ and $j = 1, \cdots, M$.

When there are many evaders, we need to suggest a representative position of evaders which the drivers follow. We set the barycenter of evaders,

$$\mathbf{u}_{ec} := \frac{1}{N} \sum_{k=1}^{N} \mathbf{u}_{ek},$$

then the dynamics can be described by

$$\begin{cases} \ddot{\mathbf{u}}_{dj} = -f_d(|\mathbf{u}_{dj} - \mathbf{u}_{ec}|)(\mathbf{u}_{dj} - \mathbf{u}_{ec}) - \nu \dot{\mathbf{u}}_{dj} + \kappa_j(t)(\mathbf{u}_{dj} - \mathbf{u}_{ec})^{\perp}, \\ \ddot{\mathbf{u}}_{ei} = -\frac{1}{M} \sum_{j=1}^{M} f_e(|\mathbf{u}_{dj} - \mathbf{u}_{ei}|)(\mathbf{u}_{dj} - \mathbf{u}_{ei}) \\ \qquad - \frac{1}{N} \sum_{k=1}^{N} f_g(|\mathbf{u}_{ek} - \mathbf{u}_{ei}|)(\mathbf{u}_{ek} - \mathbf{u}_{ei}) - \nu \dot{\mathbf{u}}_{ei}, \\ \mathbf{u}_{dj}(0) = \mathbf{u}_{dj}^0, \ \mathbf{u}_{ei}(0) = \mathbf{u}_{ei}^0, \ \dot{\mathbf{u}}_{dj}(0) = \mathbf{v}_{dj}^0, \ \dot{\mathbf{u}}_{ei}(0) = \mathbf{v}_{ei}^0. \end{cases}$$

A simulation

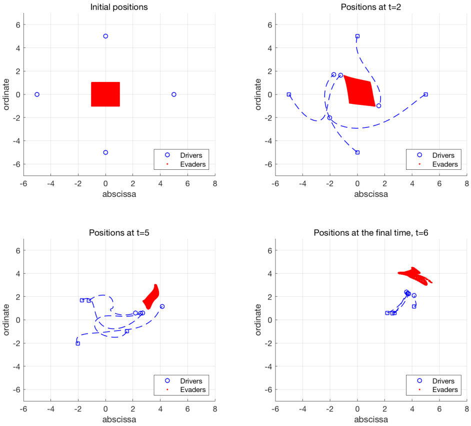


Figure: Trajectories of 4 drivers and 1024 evaders towards the point $(4, 4)$

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First order reduced model with one driver and one evader

From now on, we consider one driver and one evader model for analytic results.

For simplicity, we first observe the dynamics of its reduced limit, $m_e, m_d \to 0$. This singular limit removes the effect of inertia, hence, we get the long-time behavior monotonically.

$$\begin{cases} \dot{\mathbf{u}}_d = \mathbf{v}_d, \quad \dot{\mathbf{u}}_e = \mathbf{v}_e \\ \nu_d \dot{\mathbf{u}}_d = -f_d(|\mathbf{u}_d - \mathbf{u}_e|)(\mathbf{u}_d - \mathbf{u}_e) + \kappa(t)(\mathbf{u}_d - \mathbf{u}_e)^\perp \\ \nu_e \dot{\mathbf{u}}_e = -f_e(|\mathbf{u}_e - \mathbf{u}_d|)(\mathbf{u}_d - \mathbf{u}_e) \\ \mathbf{u}_d(0) = \mathbf{u}_d^0, \ \mathbf{u}_e(0) = \mathbf{u}_e^0, \end{cases}$$

where the relative position $\mathbf{u} := \mathbf{u}_d - \mathbf{u}_e$ satisfies a closed equation,

$$\dot{\mathbf{u}} = -f(|\mathbf{u}|)\mathbf{u} + \kappa(t)\mathbf{u}^\perp.$$

From this relation, we can separately treat central velocity $-f(|\mathbf{u}|)\mathbf{u}$ and its perpendicular velocity $\kappa(t)\mathbf{u}^\perp$.

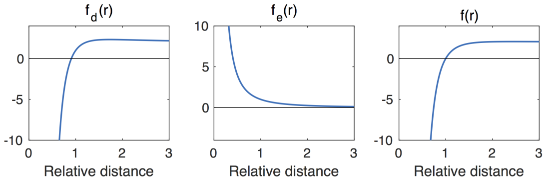
Interaction functions

Since we want the relative position $\mathbf{u}$ to satisfy the regulation between the driver and evader, we assume $f(r) = f_d(r) - f_e(r)$ satisfy

$$f(r) = \begin{cases} \geq 0 & \text{for } \; r \geq r_c, \\ < 0 & \text{for } \; 0 < r < r_c \end{cases} \quad \text{with} \quad f'(r_c) > 0,$$

which implies that $|\mathbf{u}|$ tends to r_c in the absence of control $\kappa(t)$.
As an example, we suggest

$$f_d(r) = \frac{2}{r^2} - \frac{3}{r^4} + 2 \quad \text{and} \quad f_e(r) = \frac{1}{r^2},$$



Potential function as a Lyapunov function

For the potential function

$$P(r) := \int_{r_c}^{r} sf(s)ds,$$

we may describe its gradient property:

$$\dot{\mathbf{u}} = -\nabla P(|\mathbf{u}|) + \kappa(t)\mathbf{u}^{\perp},$$

The potential function plays the role of Lyapunov function.

$$\dot{P}(|\mathbf{u}|) = \frac{dP}{d|\mathbf{u}|} \cdot \frac{d|\mathbf{u}|}{dt} = |\mathbf{u}|f(|\mathbf{u}|)\frac{\langle \mathbf{u}, \dot{\mathbf{u}} \rangle}{|\mathbf{u}|}$$
$$= f(|\mathbf{u}|)\langle \mathbf{u}, -f(|\mathbf{u}|)\mathbf{u} + \kappa(t)\mathbf{u}^{\perp} \rangle = -f(|\mathbf{u}|)^2|\mathbf{u}|^2 \leq 0.$$

Therefore, if we assume proper conditions on $f(r)$,

$$\int_0^{r_c} rf(r) = -\infty \quad \text{and} \quad \gamma_m := \liminf_{r\to\infty} f(r) > 0,$$

so that P is smooth, coercive, and blow-up at $r = 0$. Then, from the time derivative,

$$\dot{P}(|\mathbf{u}|) = -f(|\mathbf{u}|)^2 |\mathbf{u}|^2 \leq 0.$$

we obtain dynamical properties.

Relative distance of the reduced model

- The relative distance $|\mathbf{u}|$ cannot be 0 from nonzero initial data, and uniformly bounded along time.
- $\mathbf{u}$ tends to the steady solution $\bar{\mathbf{u}}(t)$ which satisfies $f(|\bar{\mathbf{u}}|)|\bar{\mathbf{u}}| = 0$, that is,

$$|\mathbf{u}| \to r_c \quad \text{if} \quad |\mathbf{u}_0| \neq 0.$$

Note that the convergence is exponential since $f'(r_c) \neq 0$ so that $P(r)$ and $\dot{P}(r)$ are both quadratic on $f(r)$ locally.

Steady states and controllability

Finally, we may classify the steady states of $\mathbf{u}_d$ and $\mathbf{u}_e$.

- If $\kappa(t) \equiv 0$, then the dynamics is in a one-dimensional line including $\mathbf{u}_d^0$ and $\mathbf{u}_e^0$. Eventually, two agents tend to uniform linear motions.
- If $\kappa(t) \equiv 1$, then they converge to circular motions, where the relative distance is r_c and angular velocities are 1.

From these two states, we can control $\mathbf{u}_e$ into a desired position:

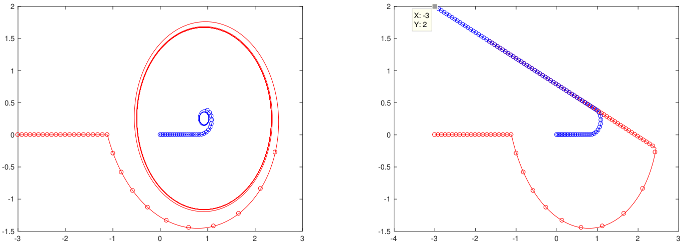


Figure: Rotational states (left) and off-bang-off control using it (right)

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The Guidance-by-repulsion model

Next, we go back to the second order Guidance-by-repulsion model.

$$\ddot{\mathbf{u}} + f(|\mathbf{u}|)\mathbf{u} + \nu\dot{\mathbf{u}} = \kappa(t)\mathbf{u}^{\perp}.$$

For the interaction coefficient $f(r)$, we assume the same condition: for

$$P(r) := \int_{r_c}^{r} sf(s)ds \geq 0,$$

$P(0) = \infty$ and P grows quadratically $\left(\sim \frac{\gamma_m}{2}|\mathbf{u}|^2\right)$ as $r \to \infty$.

- The equation now follows the motion of damped oscillator under a central potential $P(|\mathbf{u}|)$ with an additional control term.
- The negativity/positivity of f makes the relative distance $\mathbf{u} \sim r_c$.

Two main regimes arise: Pursuit $\kappa(t) = 0$ / Circumvention $\kappa(t) \neq 0$.

Steady states

For each mode, we have the following steady states which characterize the dynamics:

- Pursuit mode: $\kappa(t) \equiv 0$:

$$\mathbf{u}(t) = \mathbf{u}_* \in \mathbb{R}^2 \quad \text{and} \quad \mathbf{v}(t) = (0,0) \quad \text{with} \quad |\mathbf{u}_*| = r_c,$$

  where the driver and evader behave uniform linear motions,

$$\mathbf{u}_\ell(t) = -\frac{f_d(\mathbf{u}_*)\mathbf{u}_*}{\nu} t + \mathbf{u}_\ell(0), \quad \ell = d, e.$$

- Circumvention mode, $\kappa(t) \equiv \kappa$:

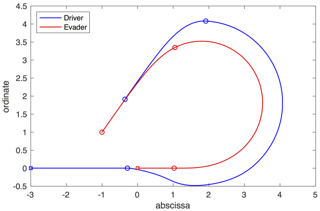
$$\mathbf{u}(t) = r_p \left(\cos\left(\frac{\kappa}{\nu} t\right), \sin\left(\frac{\kappa}{\nu} t\right) \right),$$

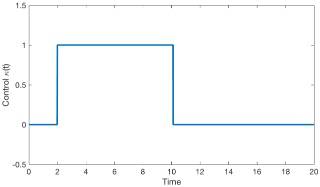
  where the driver and evader have rotational motions on circles centered at the same point,

$$\mathbf{u}_\ell(t) = r_\ell \left(\cos\left(\frac{\kappa}{\nu} t + \phi_\ell\right), \sin\left(\frac{\kappa}{\nu} t + \phi_\ell\right) \right) + \mathbf{u}^*, \quad \mathbf{u}^* \in \mathbb{R}^2,\ \ell = d, e.$$

Off-Bang-Off control of the evader

Combining these two modes, we can construct an Off-Bang-Off control: choose the direction by rotations in the circumvention mode, and drive the evaders to the target in the pursuit mode.





Theorem [K.-Zuazua (preprint)]

Let $f(r)$ be as before. Then, for a given destination $\mathbf{u}_f \in \mathbb{R}^2$ and $\mathbf{u}_0 \neq (0,0)$, there exist t_1, t_2, t_f and κ such that the control function

$$\kappa(t) = \begin{cases} \kappa & \text{if} \quad t \in [t_1, t_2], \\ 0 & \text{if} \quad t \in [0, t_1) \cup (t_2, t_f], \end{cases} \qquad \text{satisfies} \quad \mathbf{u}_e(t_f) = \mathbf{u}_f.$$

Stability to the steady states

In order to analyze the off-bang-off control, we need to show the asymptotic stability to the steady states on each constant $\kappa(t)$.

The equation of the relative position $\mathbf{u}$ with constant control $\kappa(t) \equiv \kappa$,

$$\ddot{\mathbf{u}} + f(|\mathbf{u}|)\mathbf{u} + \nu\dot{\mathbf{u}} = \kappa\mathbf{u}^{\perp}, \quad \mathbf{u} \in \mathbf{R}^2,$$

which is the damped potential oscillator with an external source term.

However, the standard energy,

$$E(t) := \frac{1}{2}|\mathbf{v}|^2 + P(|\mathbf{u}|),$$

is no more non-increasing from the perpendicular term $\kappa(t)\mathbf{u}^{\perp}$.

$$\begin{aligned}\dot{E}(t) &= \mathbf{v} \cdot \dot{\mathbf{v}} + f(|\mathbf{u}|)\mathbf{u} \cdot \dot{\mathbf{u}} \\ &= \mathbf{v} \cdot (-f(|\mathbf{u}|)\mathbf{u} - \nu\mathbf{v} + \kappa(t)\mathbf{u}^{\perp}) + f(|\mathbf{u}|)\mathbf{u} \cdot \mathbf{v} \\ &= -\nu|\mathbf{v}|^2 + \kappa(t)\mathbf{u}^{\perp} \cdot \mathbf{v}.\end{aligned}$$

To fix it, we use hypocoercivity theory[1], and construct a perturbed energy using inner product terms:

$$L_{\pm}(t) = E(t) \pm \frac{\nu}{2}\left(\frac{\nu}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{v}\right).$$

Then, its time derivative is

$$\dot{L}_{\pm}(t) \leq -\frac{\nu}{2}|\mathbf{v}|^2 + \frac{1}{2}(\nu f(|\mathbf{u}|) + \kappa(t))|\mathbf{u}|^2,$$

which is nonpositive if $|\mathbf{u}|$ is close to 0 or ∞.

On the other hand, if $\kappa(t)$ is constant, we may define κ dependent functions,

$$L_{\kappa}(t) = E(t) - \frac{\kappa}{\nu}\mathbf{u}^{\perp}\cdot\mathbf{v} \quad \text{and} \quad \dot{L}_{\kappa}(t) = -\nu\left|\mathbf{v} - \frac{\kappa}{\nu}\mathbf{u}^{\perp}\right|^2 \leq 0,$$

which is always nonpositive.

[1] [C. Villani, 2009] and [K. Beauchard, E. Zuazua, 2011]

Therefore, we have the following dynamical properties.

Boundedness of relative distance

Suppose that the control $\kappa(t)$ is bounded: $\limsup_{t\to\infty} |\kappa(t)| < \nu\sqrt{\gamma_m}$.

Then, the relative position $\mathbf{u}(t)$ does not hit $(0,0)$ or blow-up in a finite time. Moreover, if $\kappa(t)$ is constant, then $\mathbf{u}(t)$ is uniformly bounded.

Global stability of steady states

The positions $\mathbf{u}_d(t)$ and $\mathbf{u}_e(t)$ converge to the steady states asymptotically if $\kappa(t) \equiv \kappa$ and $\kappa < \nu\sqrt{\gamma_m}$:

- If $\kappa = 0$, then $\mathbf{u}_d(t)$ and $\mathbf{u}_e(t)$ tend to linear motions.
- If $0 < |\kappa| < \nu\sqrt{\gamma_m}$, then $\mathbf{u}_d(t)$ and $\mathbf{u}_e(t)$ tend to rotational motions.

By combining these asymptotic steady states, we may prove the controllability of the evader's position to any desired point.

Since we can apply the Off-Bang-Off controls to any initial data, we may use it to pass multiple target points:

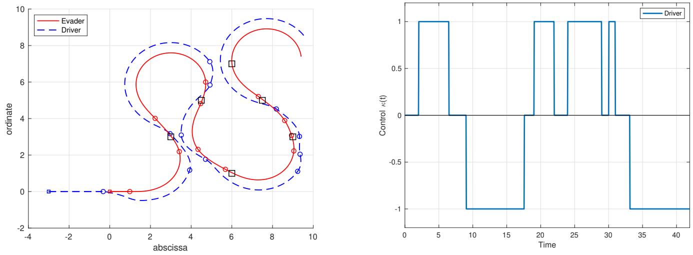


Figure: A trajectory of the evader which passes near points $(3, 3)$, $(4.5, 5)$, $(6, 1)$, $(9, 3)$, $(7.5, 5)$ and $(6, 7)$ denoted by black boxes.

This can be done by turning on and off $\kappa(t)$ using two control modes, where the dynamics converges to the corresponding steady state ('rotational motion' and 'linear motion') in a short time.

The effect of the number of evaders

If the evaders are gathered initially, the dynamics are similar to the one evader case, as we have one fat evader.

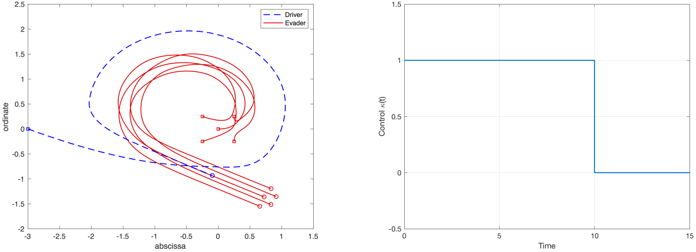


Figure: Trajectories of five evaders with a bang-off control $\kappa(t)$.

$$f_d(r) = \frac{2}{r^2} - \frac{3}{r^4} + 2, \quad f_e(r) = \frac{1}{r^2}, \quad \text{and} \quad f_g(r) = 10\left(\frac{(0.2)^2}{r^2} - \frac{(0.2)^4}{r^4}\right).$$

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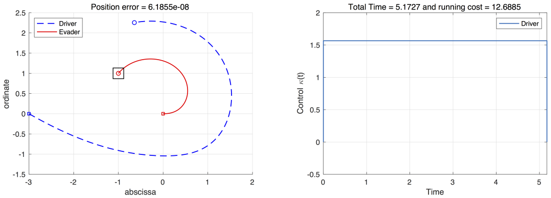
Optimal control strategies

While Off-Bang-Off controls can drive the evaders properly, it is natural to find an optimal control strategy which minimizes a given cost.

For the cont function, we suggest two optimal control problems: On one hand, we want to minimize the running cost of controls:

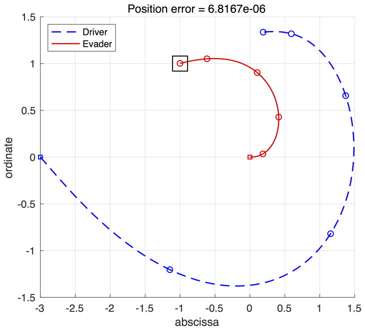
$$J(\kappa(\cdot)) = \frac{1}{N}\sum_{i=1}^{N} |\mathbf{u}_{ei}(t_f) - \mathbf{u}_f|^2 + \frac{0.001}{M}\sum_{k=1}^{M}\int_0^{t_f} |\kappa_k(t)|^2 dt.$$

The simulations are done by gradient descent methods with flexible final time t_f, where the initial guess is given by constant control functions. For example, $\kappa(t) \equiv 1.5662$ and $t_f = 5.1727$ to make $\mathbf{u}(t_f) = (-1, 1)$:



One driver and one evader

We can observe that the optimal strategy is not an Off-Bang-Off control, but it shares the main idea: 'rotate and then drive'.



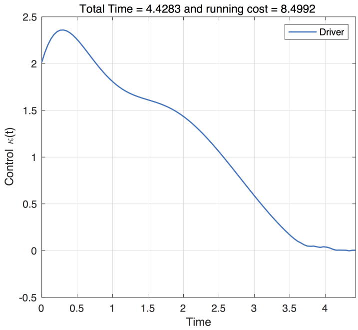


Figure: Diagrams for the optimal control leading to $\mathbf{u}_e(t_f) \simeq (-1, 1)$ when initially $\mathbf{u}_e^0 = (0, 0)$, $\mathbf{u}_d^0 = (-3, 0)$ and zero velocities.

Two drivers and one evader

This 'rotate and then drive' strategy also works with two drivers. In a similar initial data from the previous simulation, we can observe that two drivers act like one driver.

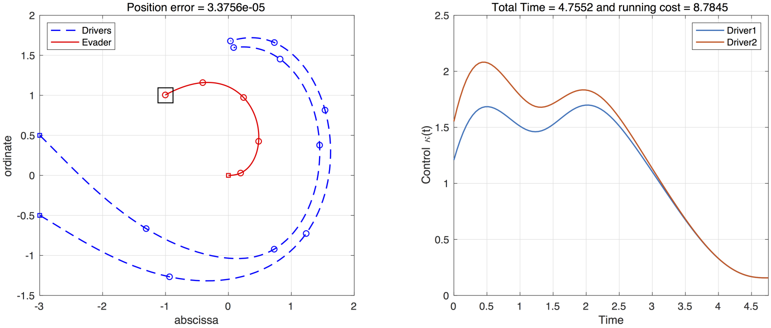


Figure: Diagrams for the control leading to $\mathbf{u}_{e1}(t_f) \simeq (-1, 1)$ when initially $\mathbf{u}_e^0 = (0, 0)$, $\mathbf{u}_{d1}^0 = (-3, 0.5)$, $\mathbf{u}_{d2}^0 = (-3, -0.5)$ and zero velocities.

Two drivers and one evader: Minimizing the driving time

It is not changed much even if we want to minimize the driving time,

$$J(\kappa(\cdot)) = \frac{1}{N}\sum_{i=1}^{N} |\mathbf{u}_{ei}(t_f) - \mathbf{u}_f|^2 + \frac{0.001}{M}\sum_{k=1}^{M}\int_0^{t_f} |\kappa_k(t)|^2 dt + 0.1 t_f.$$

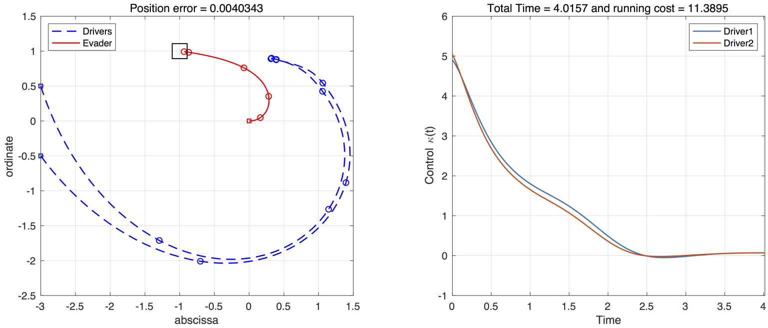


Figure: Diagrams for the control leading to $\mathbf{u}_{e1}(t_f) \simeq (-1, 1)$ when initially $\mathbf{u}_e^0 = (0, 0)$, $\mathbf{u}_{d1}^0 = (-3, 0.5)$, $\mathbf{u}_{d2}^0 = (-3, -0.5)$ and zero velocities.

The trajectories can be significantly different if initial positions are not well-ordered, in terms of the initial velocity of the evader. However, for any case, the drivers want the evader to get the right direction in a short time.

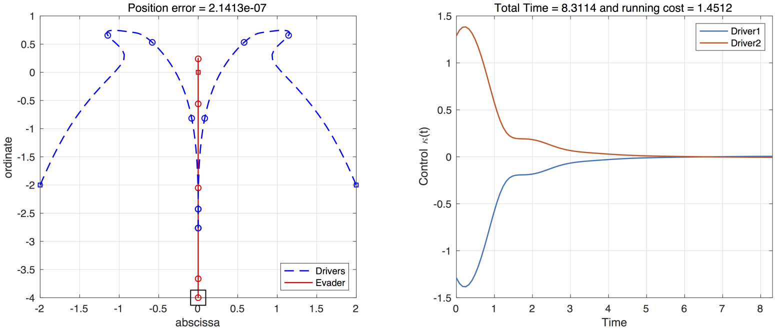


Figure: Diagrams for the control leading to $\mathbf{u}_{e1}(t_f) \simeq (0, -4)$ when initially $\mathbf{u}_e^0 = (0,0)$, $\mathbf{u}_{d1}^0 = (-2,-2)$, $\mathbf{u}_{d2}^0 = (-2,2)$ and zero velocities.

In the same way, the minimal time optimal strategy contains strong control functions and wants to decrease the relative position.

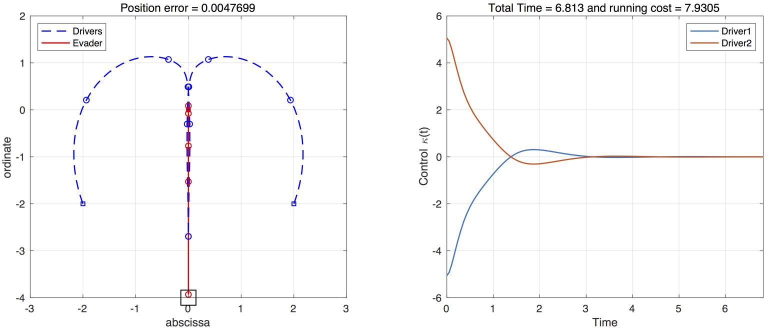


Figure: Diagrams for the control leading to $\mathbf{u}_{e1}(t_f) \simeq (0, -4)$ when initially $\mathbf{u}_e^0 = (0, 0)$, $\mathbf{u}_{d1}^0 = (-2, -2)$, $\mathbf{u}_{d2}^0 = (-2, 2)$ and zero velocities.

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Conclusions

- Results
  - The guidance-by-repulsion is difficult since it is a bi-linear control problem which deals with partial controllability. (Null-controllability is trivially false)
  - In summary, one-driver and one-evader model with special assumptions (same frictions, constant control, good potential) leads to the controllability of the evader's position.
- Future objectives
  - It is natural to extend this model to measure-valued distributions (transport equation) or stochastic particles (Fokker-Planck equation) by adding diffusive effects.
  - Numerical calculation costs too much in gradient methods since it has a lot of possible controls which may have locally optimal costs.

THANK YOU FOR YOUR INVITATION AND ATTENTION!