On the emergence of local flocking phenomena in Cucker-Smale ensemble

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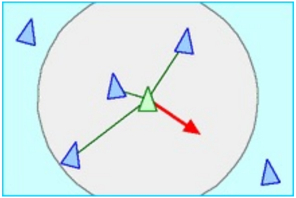
Overview

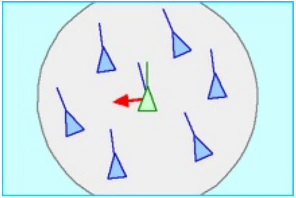
1. Introduction
   - Cucker-Smale model
   - Previous flocking results

2. Local flocking results
   - Existence of bi-cluster flocking on the particle model
   - Bi-cluster flocking on hydrodynamic C-S model

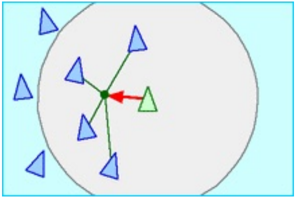
Flocking Rules by Craig Raynolds (1987)



Separation: steer to avoid crowding local flockmates



Alignment: steer towards the average heading of local flockmates



Cohesion: steer to move toward the average position of local flockmates

Dynamics of Cucker-Smale model

Phase space : $\mathbb{R}^{2dN}$, Configuration $\mathcal{C} := \{(\boldsymbol{x}_i, \boldsymbol{v}_i)\}_{i=1}^N$
$(\boldsymbol{x}_i, \boldsymbol{v}_i) \in \mathbb{R}^{2d}$ be i-th Cucker-Smale (C-S) flocking agent.
The dynamics of Cucker-Smale model (2007) :

$$\dot{\boldsymbol{x}}_i = \boldsymbol{v}_i, \quad t > 0, \quad i = 1, \cdots, N,$$
$$\dot{\boldsymbol{v}}_i = \frac{K}{N} \sum_{j=1}^{N} \psi(\|\boldsymbol{x}_j - \boldsymbol{x}_i\|)(\boldsymbol{v}_j - \boldsymbol{v}_i),$$
$$(\boldsymbol{x}_i, \boldsymbol{v}_i)(0) = (\boldsymbol{x}_{i0}, \boldsymbol{v}_{i0}),$$

where K is the positive coupling strength, and
the communication weight $\psi : \mathbb{R}_+ \to \mathbb{R}$ is usually considered as a nonincreasing nonnegative analytic function.
For simplicity, we set

$$\psi(s) = \frac{1}{(1 + s^2)^{\beta/2}}, \quad \text{for } \beta > 1.$$

Kinetic description of Cucker-Smale model

When we apply mean-field limit ($N \to \infty$) on the Cucker-Smale particle model, we have the following equation of distribution function $f = f(x, v, t)$;

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (\xi(f) f) = 0,$$
$$\xi(f)(x, v, t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(|x - y|)(w - v) f(y, w) dy dw,$$
$$f(x, v, 0) = f_0(x, v).$$

This limit process is rigorously presented in [Ha, Tadmor (2008)].

Hydrodynamic description of Cucker-Smale model

Let $\rho = \rho(x,t)$ and $u = u(x,t)$ be the mass density and bulk velocity of the C-S ensemble at position x and time t, then the temporal-spatial evolution of (ρ, u) is governed by

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d,\ t > 0, \\
&\rho \partial_t u + \rho u \cdot \nabla u = -K\rho \int_{\mathbb{R}} \psi(|y-x|)(u(x) - u(y))\rho(y)dy, \\
&(\rho, u)(x, 0) = (\rho_0, u_0),
\end{aligned}$$

where K is the coupling strength and ψ is the communication weight.

This equation comes from the mono-kinetic ansatz
$f(x, v, t) = \rho(x, t)\delta_{v=u(x,t)}(x, v, t)$.

Definition of flocking

Suppose that $\mathcal{C} := \{(\boldsymbol{x}_i, \boldsymbol{v}_i)\}_{i=1}^N$ is an interaction particle system.

Definition

1. $\mathcal{C}$ tends to a mono-cluster flocking state if and only if

$$\sup_{t\geq 0} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| < \infty, \qquad \lim_{t\to\infty} \|\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)\| = 0.$$

2. $\mathcal{C}$ tends to a multi-cluster flocking state if and only if there exist $\mathcal{G}_\alpha = \{(\boldsymbol{x}_{\alpha i}, \boldsymbol{v}_{\alpha i})\}_{i=1}^{N_\alpha}$ for $\alpha = 1, \cdots, m$ such that $\sqcup_{\alpha=1}^m \mathcal{G}_\alpha = \mathcal{C}$,

   (i) Each $\mathcal{G}_\alpha$ tends to a flocking state,

$$\sup_{t\geq 0} \|\boldsymbol{x}_{\alpha i} - \boldsymbol{x}_{\alpha j}\| < \infty, \quad \lim_{t\to\infty} \|\boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha j}\| = 0, \text{ for all } i, j, \alpha.$$

   (ii) All $\mathcal{G}_\alpha$ are separating each other,

$$\sup_{t\geq 0} \|\boldsymbol{x}_{\alpha i} - \boldsymbol{x}_{\beta j}\| = \infty, \qquad \text{for all } i, j, \alpha \neq \beta.$$

   We call it bi-cluster flocking if $m = 2$.

Comparison with other models

1. Heat equation : consider 1D space discretize $x_j = jh$ for some $h > 0$.

$$u_t = u_{xx} \quad \Rightarrow \quad \frac{du_j(t)}{dt} = \frac{1}{h}\left(\frac{u_{j+1} - u_j}{h} + \frac{u_{j-1} - u_j}{h}\right).$$

   Velocities in Cucker-Smale model follows basic rules of heat dissipation on velocities.

2. Coulomb force : Force $= -K\frac{q_1 q_2}{r^2}\hat{r}$,

$$\dot{v}_j = K\frac{q_j}{m_j}\sum_{k=1}^{N}\frac{1}{\|x_k - x_j\|^2}\frac{(x_k - x_j)}{\|x_k - x_j\|}.$$

   Interactions are related to the relative positions, then it becomes the Hamiltonian system.

Basic properties of Cucker-Smale model

- all velocities of particles are attracted to the mean value.
  The mono-cluster flocking state is the equilibrium state.
- The multi-cluster flocking state is not an equilibrium.
  It only exists as an asymptotic behavior as $t \to \infty$.
- The interaction between two particles is always attractive:
  The second momentum is not increasing.

Global flocking condition

Flocking condition was first suggested by Cucker and Smale.

Define a functional $\|\boldsymbol{x}\| = \left(\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_i - \boldsymbol{x}_c)^2\right)^{\frac{1}{2}}$ where $\boldsymbol{x}_c = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{x}_i$.

Theorem (*Ha-Liu, 2009*)

Let $(\boldsymbol{x}, \boldsymbol{v})$ be a solution with initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ satisfying the following condition:

$$\|\boldsymbol{v}_0\| < 2K \int_{\|\boldsymbol{x}_0\|}^{\infty} \psi(2\sqrt{N}r)dr.$$

Then there exists a positive number x_M such that

$$\sup_{t\geq 0}\|\boldsymbol{x}(t)\| \leq x_M, \quad \|\boldsymbol{v}(t)\| \leq \|\boldsymbol{v}_0\|e^{-\psi(2x_M)t}, \quad t \geq 0.$$

Note that if $\int_0^\infty \psi(r)dr = \infty$, then mono-cluster flocking always hold.

Two particle result

Initial condition for global flocking

Let $(x, v) \in \mathbb{R}^2$ be the relative position and velocity between two particles, with initial data (x_0, v_0) satisfying $v_0 > 0$ and

$$v_0 < K \int_{x_0}^{\infty} \psi(|y|)dy.$$

Then there it tends to the global flocking state. Converse is also true.

Two particle result; proof

Let $x := x_1 - x_2$ and $v := v_1 - v_2$ in $\mathbb{R}^1$. We assume $v_0 > 0$.
Then the differences of x and v satisfy

$$\dot{x} = v, \quad \dot{v} = -K\psi(|x|)v,$$

or equivalently,

$$dv = -K\psi(|x|)dx.$$

Integrating the above relation yields

$$v(t) = v_0 - K\int_{x_0}^{x(t)} \psi(|y|)dy.$$

Suppose

$$v_0 \geq K\int_{x_0}^{\infty} \psi(|y|)dy.$$

Then, it follows that $v(t) \geq K\int_{x(t)}^{\infty} \psi(|y|)dy > 0$ for all t.

Proof on global flocking [Ha, Liu (2009)]

Lemma : differential inequalities

$$\left|\frac{d\|\boldsymbol{x}\|}{dt}\right| \le \|\boldsymbol{v}\|, \quad \frac{d\|\boldsymbol{v}\|}{dt} \le -2K\psi(2\sqrt{N}\|\boldsymbol{x}\|)\|\boldsymbol{v}\|.$$

proof of Lemma

$$\frac{d\|\boldsymbol{x}\|^2}{dt} = \frac{2}{N}\sum_{j=1}^{N}\langle \boldsymbol{x}_j, \boldsymbol{v}_j\rangle,$$

$$\left|\frac{d\|\boldsymbol{x}\|^2}{dt}\right| \le 2\left(\frac{1}{N}\sum_{j=1}^{N}|\boldsymbol{x}_j|^2\right)^{\frac{1}{2}}\left(\frac{1}{N}\sum_{j=1}^{N}|\boldsymbol{v}_j|^2\right)^{\frac{1}{2}} = 2\|\boldsymbol{x}\|\|\boldsymbol{v}\|.$$

Proof on global flocking

proof of Lemma (Continued)

$$
\begin{aligned}
\frac{d\|\boldsymbol{v}\|^2}{dt} &= \frac{2}{N}\sum_{j=1}^{N}\langle \boldsymbol{v}_j, \dot{\boldsymbol{v}}_j\rangle, \\
&= \frac{2}{N}\sum_{j=1}^{N}\left\langle \boldsymbol{v}_j, \frac{K}{N}\sum_{k=1}^{N}\psi(|\boldsymbol{x}_k - \boldsymbol{x}_j|)(\boldsymbol{v}_k - \boldsymbol{v}_j)\right\rangle, \\
&= \frac{2K}{N^2}\sum_{j,k=1}^{N}\langle \boldsymbol{v}_j, \boldsymbol{v}_k - \boldsymbol{v}_j\rangle\,\psi(|\boldsymbol{x}_k - \boldsymbol{x}_j|), \\
&= -\frac{K}{N^2}\sum_{j,k=1}^{N}\psi(|\boldsymbol{x}_k - \boldsymbol{x}_j|)|\boldsymbol{v}_k - \boldsymbol{v}_j|^2,
\end{aligned}
$$

since ψ is symmetric to the pair $\{k, j\}$.

Proof on global flocking

proof of Lemma (Continued)

On each term of x and v,

$$
\begin{aligned}
|\boldsymbol{x}_k - \boldsymbol{x}_j| &\le 2\|\boldsymbol{x}\|_\infty \le 2\sqrt{N}\|\boldsymbol{x}\|, \\
\sum_{j,k} |\boldsymbol{v}_k - \boldsymbol{v}_j|^2 &= \sum_{j,k} \left(|\boldsymbol{v}_k|^2 + |\boldsymbol{v}_j|^2 - 2\langle \boldsymbol{v}_k, \boldsymbol{v}_j\rangle\right) \\
&= 2\sum_{j,k} |\boldsymbol{v}_j|^2 - 2\left\langle \sum_{j,k} \boldsymbol{v}_k, \sum_{j,k} \boldsymbol{v}_j \right\rangle = 2\sum_{j,k} |\boldsymbol{v}_j|^2,
\end{aligned}
$$

since we assumed zero mean velocity.

Therefore,

$$
\frac{d\|\boldsymbol{v}\|}{dt} \le -2K\psi(2\sqrt{N}\|\boldsymbol{x}\|)\|\boldsymbol{v}\|.
$$

Proof on global flocking

proof of Theorem (Ha, Liu)

Define a Lyapunov functional,

$$L(t) = \|\boldsymbol{v}(t)\| + 2K \int_0^{\|\boldsymbol{x}(t)\|} \psi(2\sqrt{N}s)ds, \quad \Rightarrow \quad \frac{dL(t)}{dt} \le 0.$$

$$L(t) \le L(0) \Rightarrow \|\boldsymbol{v}(t)\| + 2K \int_{\|\boldsymbol{x}(0)\|}^{\|\boldsymbol{x}(t)\|} \psi(2\sqrt{N}s)ds \le \|\boldsymbol{v}(0)\|.$$

Under the condition of

$$\|\boldsymbol{v}(0)\| \le 2K \int_{\|\boldsymbol{x}(0)\|}^{\infty} \psi(2\sqrt{N}s)ds,$$

there should be $x_M > 0$ such that $\|\boldsymbol{x}(t)\| < x_M$.

Proof on global flocking

Hence

$$\frac{d\|\boldsymbol{v}\|}{dt} \leq -2K\psi(2\sqrt{N}x_M)\|\boldsymbol{v}\|,$$

and we get result from Gronwall's lemma.

$$\|\boldsymbol{v}(t)\| \leq \|\boldsymbol{v}(0)\| \exp\left(-2K\psi(2\sqrt{N}x_M)\right).$$

Remark

- $\|x\|_\infty$ plays a key role to set a lower bound of ψ term.
- We first should get a bound for $\|\boldsymbol{x}(t)\|$ to use Gronwall-type inequalities.

Global flocking condition

Remark

- If $K > K_0$(initial data), than it flocks.

- This result shows a sufficient condition for flocking.

Question

What can we say if there is no global flocking.

Goal of this talk

- Construct sufficient conditions on configurations, which tend to the multi-cluster flocking.

Three particle case

The three particles, however, we cannot expect simple conditions.
In $\mathbb{R}^1$, let (x_i, v_i) for $i = 1, 2, 3$ represents three particles. In this dynamics, we consider the initial conditions

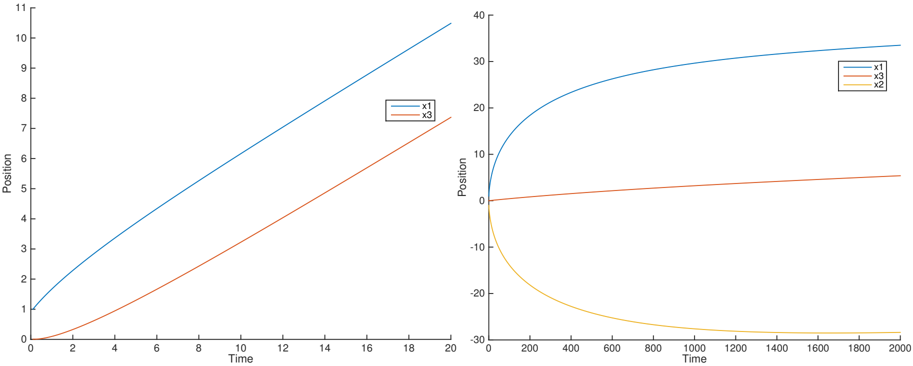
$$x_1(0) = x_0 > 0, \; x_2(0) = -x_0, \; x_3(0) = X \in [0, x_0],$$
$$v_1(0) = v_0 > 0, \; v_2(0) = -v_0, \; v_3(0) = V \in [0, v_0].$$

The difficulty comes from the flocking condition of 1 and 3. The dynamics of $(v_1 - v_3)$ contains the interaction with particle 2,

$$\frac{d(v_1 - v_3)}{dt} = -(\text{flocking term}) + \frac{K}{3}(\psi(x_2 - x_3) - \psi(x_1 - x_2))(v_1 - v_2),$$

but it is not negative. Hence, for the dynamics between particles 1 and 3, the particle 2 want to make them separated.

Three particle case; simulation



Two time-position graph with $x_0 = 1, v_0 = 0.8, X = 0.005$, and $V = 0.004$.
Time from 0 to 20 and 0 to 2000, respectively.
Left : 2 particles, x_1 and x_3, they get really closer in a short time. (Scale : 20)
Right: 3 particles, which tend to the global flocking eventually. (Scale : 2000)

Local flocking results

1. Existence of bi-cluster flocking

   (with J. Choi, S.-Y. Ha, F. Huang, C. Jin)

2. Multi-cluster flocking in terms of coupling strength

   (with S.-Y. Ha, Y. Zhang)

3. Bi-cluster flocking on hydrodynamic C-S model

   (with S.-Y. Ha, X. Zhang, Y. Zhang)

Existence of bi-cluster flocking

To measure the asymptotic state of local flocking, it is convenient to use variations with respect to each group's central value.

$$\mathcal{X} := \sum_{\alpha=1}^{2} \|\hat{\boldsymbol{x}}_\alpha\| := \sum_{\alpha=1}^{2} \left(\sum_{i=1}^{N_\alpha} \|\boldsymbol{x}_{\alpha i} - \boldsymbol{x}_\alpha^c\|^2 \right)^{1/2}, \quad \mathcal{V} := \sum_{\alpha=1}^{2} \|\hat{\boldsymbol{v}}_\alpha\|,$$

where $||\hat{\boldsymbol{x}}_\alpha||, ||\hat{\boldsymbol{v}}_\alpha||$ are the spatial and velocity L^2-norms with respect to the mean values $\boldsymbol{x}_\alpha^c, \boldsymbol{v}_\alpha^c$ of group α.

To see the separation of two groups, let Δ_x and Δ_v be the average velocity difference between the two local groups. For some reference unit direction $\boldsymbol{e} \in \mathbb{R}^d$, we consider

$$\Delta_x \cdot \boldsymbol{e} := (\boldsymbol{x}_2^c - \boldsymbol{x}_1^c) \cdot \boldsymbol{e}, \quad \Delta_v \cdot \boldsymbol{e} := (\boldsymbol{v}_2^c - \boldsymbol{v}_1^c) \cdot \boldsymbol{e}.$$

In terms of Lyapunov L^2 - functionals...

Goal : Existence of bi-cluster flocking

Definition of bi-cluster flocking needs two conditions;

1. Local flocking of each group,
2. Separation between two groups.

Hence we get bi-cluster flocking configurations if

1. $\mathcal{X} < \infty$, $\mathcal{V} \to 0$
2. $\Delta_v \cdot \boldsymbol{e} > c > 0$, for some c.

If (1) is satisfied, then (2) is similar to the two particle result.
If (2) is true, we can estimate (1) using Gronwall type inequalities.

Existence of bi-cluster flocking

Theorem 1; [Choi-Ha-Huang-Jin-K. 2016]

A well-prepared initial data which satisfies the following conditions $(\mathcal{C}_0 0) - (\mathcal{C}_0 2)$ tends to a bi-cluster flocking.

- $(\mathcal{C}_0 0)$ (Parameters): For some fixed unit vector $\boldsymbol{e}$,

$$N \geq 3, \quad 2\lambda_0 := \Delta_v(0) \cdot \boldsymbol{e}, \quad \mathcal{V}_0 := \mathcal{V}(0),$$
$$\mathcal{X}_0 := \mathcal{X}(0), \quad r_0 := \min_{i,j} \{(\boldsymbol{x}_{1i}(0) - \boldsymbol{x}_{2j}(0)) \cdot \boldsymbol{e}\}.$$

Existence of bi-cluster flocking

- $(\mathcal{C}_0 1)$ (Small perturbations):

$$\mathcal{V}_0 < \frac{K \min\{N_1, N_2\}}{2N} \int_{\mathcal{X}_0}^{\infty} \psi(\sqrt{2}x)dx.$$

- $(\mathcal{C}_0 2)$ (Close to bi-cluster): For the vector $\boldsymbol{e}$ from $(\mathcal{C}_0 0)$,

$$\lambda_0 > 4\sqrt{2}\mathcal{V}_0, \quad \int_{r_0}^{\infty} \psi(x)dx < \frac{\lambda_0 \mathcal{V}_0}{2\sqrt{2}K\sqrt{M_2(0)}},$$

where $M_2(t)$ is the second momentum,

$$M_2(t) := \sum_{i=1}^{N} \|\boldsymbol{v}_i\|^2.$$

Sketch of proof

Lemma 0; Statistical inequlities

Let $\{(\boldsymbol{x}_i, \boldsymbol{v}_i)\}_{i=1}^N$ be the solution to the C-S model with $\mathcal{V}(0) > 0$. Then the functionals $(\mathcal{X}, \mathcal{V}, \Delta_v)$ satisfy the following coupled system of dissipative differential inequalities (SDDI):

$$
\begin{aligned}
&(i)\ |\dot{\mathcal{X}}| \le \mathcal{V},\\
&(ii)\ \dot{\mathcal{V}} \le -\frac{K\min\{N_1, N_2\}}{N}\psi(\sqrt{2}\mathcal{X})\mathcal{V} + 2\sqrt{2}K\sqrt{M_2(0)}\psi_M,\\
&(iii)\dot{\Delta}_v \ge -K\sqrt{\frac{2M_2(0)}{\min\{N_1, N_2\}}}\psi_M,
\end{aligned}
$$

where ψ_M is the maximal interaction strength between diffent groups,

$$\psi_M(t) := \max_{i,j}\psi(\|\boldsymbol{x}_{2j} - \boldsymbol{x}_{1i}\|).$$

Sketch of proof

Step 1 : Continuity argument over the separating velocity

At $t = 0$, the following inequlity holds from the definition of λ_0 and r_0.

$$\min_{i,j} \|\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)\| > \lambda_0 t + r_0, \qquad \text{for } t \in [0, T). \tag{1}$$

We now define T^* to be the supremum among all T satisfying (1).
We will show that $T^* = \infty$ by assuming $T^* < \infty$.

Step 2 : Lyapunov function approach from global flocking

If the separation inequlity (1) holds,
then for large α_0 and r_0, we can get $\mathcal{X}(t) < \infty$ using the Lyapunov functional method;

$$\Phi(\mathcal{X}, \mathcal{V}) := \mathcal{V} + \frac{K \min\{N_1, N_2\}}{N} \int_0^{\mathcal{X}} \psi(\sqrt{2}x)dx.$$

Sketch of proof; Step 1

Lemma 1

Suppose that conditions $(\mathcal{C}_0)$ hold. Then

$$\mathcal{V}(t) < 2\mathcal{V}_0 \quad \text{for } t \in [0, T^*).$$

Lemma 2

Suppose that conditions $(\mathcal{C}_0)$ hold. Then

$$(v_{1i}(t) - v_{2j}(t)) \cdot \boldsymbol{e} > \lambda_0, \quad \text{for} \quad \forall i, j \text{ and } t \in [0, T^*).$$

This implies $T^* = \infty$, which ends the continuity argument.

Sketch of proof; Step 1

Proof of lemma 1

From Lemma 0, we have

$$\begin{aligned}\frac{d\Phi}{dt} &\le \frac{d\mathcal{V}}{dt} + \frac{K\min\{N_1, N_2\}}{N}\psi(\sqrt{2}\mathcal{X})\mathcal{V} \\ &\le 2\sqrt{2}K\sqrt{M_2(0)}\psi(\lambda_0 t + r_0) \quad \text{for } t \in [0, T^*).\end{aligned}$$

Integrating the above inequality, we obtain

$$\begin{aligned}&\mathcal{V}(t) + \frac{K\min\{N_1, N_2\}}{N}\int_0^t \psi(\sqrt{2}\mathcal{X}(t))\mathcal{V}(t)dt \\ &\quad\le \mathcal{V}(0) + 2\sqrt{2}K\sqrt{M_2(0)}\int_0^\infty \psi(\lambda_0 t + r_0)dt\end{aligned}$$

for $t \in [0, T^*]$. By condition $(\mathcal{C}_0 1)$, we have $\mathcal{V}(t) < 2\mathcal{V}_0$ for $t \in [0, T^*]$.

Sketch of proof; Step 1

Proof of lemma 2

From lemma 0,

$$\dot{\Delta}_v \geq -K\sqrt{\frac{2M_2(0)}{\min\{N_1, N_2\}}}\psi(\lambda_0 t + r_0) \quad \text{for } t \in [0, T^*).$$

This yields

$$\Delta_v(t) \geq 2\alpha_0 - K\sqrt{\frac{2M_2(0)}{\min\{N_1, N_2\}}}\int_0^\infty \psi(\lambda_0 t + r_0)dt \quad \text{for } t \in [0, T^*).$$

We now use the condition $(\mathcal{C}_0 1)$ to obtain $\Delta_v(t) \geq \frac{3}{2}\alpha_0$. Then,

$$\begin{aligned}(v_{1i}(t) - v_{2j}(t)) \cdot \boldsymbol{e} &= \Delta_v(t) + (\hat{\boldsymbol{v}}_{1i} - \hat{\boldsymbol{v}}_{2j}) \cdot \boldsymbol{e} \\ &> \frac{3}{2}\alpha_0 - \sqrt{2}\mathcal{V}(t) > \lambda_0, \text{for } t \in [0, T^*].\end{aligned}$$

Sketch of proof; Step 2

Lemma 3

We can prove the theorem by showing following properties.

$$(i)\ \exists X_M > 0 \quad s.t. \quad \mathcal{X}(t) \leq X_M, \quad t \in [0, \infty),$$
$$(ii)\ \mathcal{V}(t) < C\psi(t) \quad \text{as} \quad t \to \infty, \text{ for some constant } C > 0.$$

Sketch of proof; Step 2

Proof of lemma 3 (i)

(i) Using the Lyapunov functional $\Phi(\mathcal{X}, \mathcal{V})$, we get

$$\begin{aligned} &\mathcal{V}(t) + \frac{K \min\{N_1, N_2\}}{N} \int_{\mathcal{X}_0}^{\mathcal{X}(t)} \psi(\sqrt{2}x)dx \\ &\quad \le \mathcal{V}_0 + 2\sqrt{2}K\sqrt{M_2(0)} \int_0^{\infty} \psi(\lambda_0 t + r_0)dt, \ \text{for } t \in [0, \infty). \end{aligned} \tag{2}$$

Since condition $(\mathcal{C}_0 2)$ means that there exist $X_M > 0$ such that

$$2\mathcal{V}_0 = \frac{K \min\{N_1, N_2\}}{N} \int_{\mathcal{X}_0}^{X_M} \psi(\sqrt{2}x)dx.$$

Using condition $(\mathcal{C}_0 1)$, we obtain $\mathcal{X}(t) \le X_M$ for $t \in [0, \infty)$.

Sketch of proof; Step 2

Proof of lemma 3 (ii)

(ii) Define $\beta_0 := \frac{K \min\{N_1, N_2\}}{N}\psi(\sqrt{2}X_M)$. From Lemma 0, we get

$$\dot{\mathcal{V}} \le -\beta_0\mathcal{V} + 2\sqrt{2}K\sqrt{M_2(0)}\psi(\lambda_0 t + r_0) \text{ for } t \in [0,\infty).$$

We use Gronwall's inequality to obtain

$$\begin{aligned}\mathcal{V}(t) &\le \mathcal{V}_0 e^{-\beta_0 t} + \frac{2\sqrt{2}K\sqrt{M_2(0)}\psi(r_0)}{\beta_0}e^{-\frac{\beta_0}{2}t} + \frac{2\sqrt{2}K\sqrt{M_2(0)}}{\beta_0}\psi(\frac{\lambda_0}{2}t + r_0)\\ &\le \; C\psi(t) \quad \text{as} \quad t \to \infty.\end{aligned}$$

Decay rate of velocity fluctuations

Corollary; [Choi-Ha-Huang-Jin-K. 2016]

Suppose that conditions ($\mathcal{C}_0$0)-($\mathcal{C}_0$2) hold. Then there exist $\boldsymbol{v}^c_{1\infty}$, $\boldsymbol{v}^c_{2\infty}$ and positive constants C_1 and C_2 such that for any i, j, s.t.

$$C_1 \int_0^t \psi(\tau) d\tau \leq \|\boldsymbol{v}_{1i}(t) - \boldsymbol{v}^c_{1\infty}\| + \|\boldsymbol{v}_{2j}(t) - \boldsymbol{v}^c_{2\infty}\| \leq C_2 \int_0^t \psi(\tau) d\tau.$$

Remark

- The velocity fluctuation is decaying as in the order of $\int \psi(\tau) d\tau$, while it decays exponentially in the emergence of the mono-cluster flocking.

Decay rate of velocity fluctuations

Proof of algebraic decay

In the proof of lemma 0, assume $N_1 \leq N_2$, then

$$\boldsymbol{v}_1^c(t) = \boldsymbol{v}_1^c(0) + \frac{K}{N_1 N} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \int_0^t \psi(\|\boldsymbol{x}_{2j}(s) - \boldsymbol{x}_{1i}(s)\|)(\boldsymbol{v}_{2j}(s) - \boldsymbol{v}_{1i}(s))ds.$$

If we multiply both sides by $\boldsymbol{e}$, then the velocity can be estimated using the result of Theorem 1.

Lyapunov (L^∞,L^2) - functionals

To measure the asymptotic state of local flocking, it is convenient to use variations with respect to each group's central value.

$$\mathcal{X}^\alpha := \max_i \|\boldsymbol{x}_{\alpha i} - \boldsymbol{x}_\alpha^c\|, \quad \mathcal{V}^\alpha := \max_i \|\boldsymbol{v}_{\alpha i} - \boldsymbol{v}_\alpha^c\|,$$

where $\boldsymbol{x}_\alpha^c, \boldsymbol{v}_\alpha^c$ are the mean values of group α.

To see the separation of two groups, we define Δ_x and Δ_v as the followings in order to measure the average difference between two groups.

$$\Delta_x := \min_{\alpha \neq \beta} \left\| (\boldsymbol{x}_\beta^c - \boldsymbol{x}_\alpha^c) \cdot \frac{(\boldsymbol{v}_\beta^c(0) - \boldsymbol{v}_\alpha^c(0))}{\|\boldsymbol{v}_\beta^c(0) - \boldsymbol{v}_\alpha^c(0)\|} \right\|, \quad \Delta_v := \min_{\alpha \neq \beta} \|\boldsymbol{v}_\beta^c - \boldsymbol{v}_\alpha^c\|.$$

Note : By using L^∞-type norm, we can sort out the flocking condition in terms of coupling strength K.

Multi-cluster flocking in terms of K

Theorem 2; [Ha-K.-Zhang 2016]

A well-prepared initial data which satisfies the following conditions $(\mathcal{C}_1 0) - (\mathcal{C}_1 2)$ tends to a multi-cluster flocking.

- $(\mathcal{C}_1 0)$ (Parameters):

$$\lambda_0 := \frac{1}{2} \min_{\beta \neq \alpha} \| \boldsymbol{v}^c_\beta(0) - \boldsymbol{v}^c_\alpha(0) \|, \quad \text{and}$$

$$\int_{r_0}^{\infty} \psi(s) ds := \frac{\gamma_N \lambda_0}{8(1-\gamma_N)\sqrt{2M_2(0)}} \min_\alpha \int_{\mathcal{X}^\alpha(0)}^{\infty} \psi(2x) dx,$$

where $\gamma_N = \min_\alpha N_\alpha / N$.

Multi-cluster flocking in terms of K

- $(\mathcal{C}_1 1)$ (restriction on initial configurations):

$$\mathcal{V}^\alpha(0) < \frac{\lambda_0}{4}, \qquad \min_{\beta\neq\alpha,i,k} \big(\boldsymbol{x}_{\beta k0} - \boldsymbol{x}_{\alpha i0}\big) \cdot \frac{\boldsymbol{v}^c_\beta(0) - \boldsymbol{v}^c_\alpha(0)}{\|\boldsymbol{v}^c_\beta(0) - \boldsymbol{v}^c_\alpha(0)\|} \geq r_0.$$

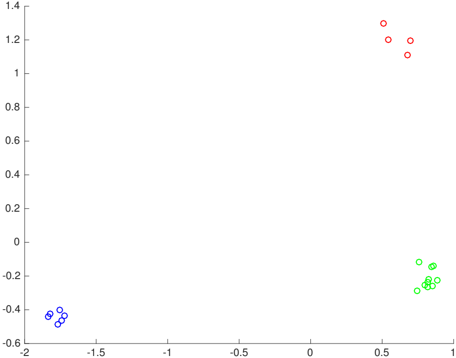
- $(\mathcal{C}_1 2)$ (restriction on coupling strengths):

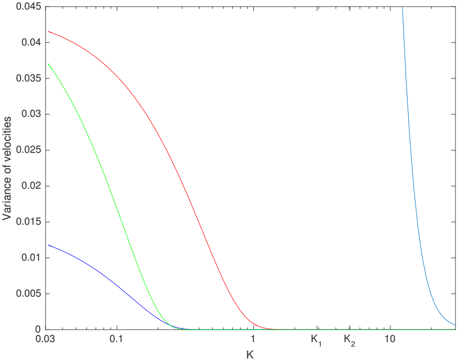
$$K_1 < K < K_2, \quad \text{for}$$

$$K_1 = \max_\alpha \left\{ \frac{2\mathcal{V}^\alpha(0)}{\gamma_N \int_{\mathcal{X}^\alpha(0)}^\infty \psi(2x)dx} \right\},$$

$$K_2 = \min_\alpha \left\{ \frac{2\lambda_0}{3\gamma_N \int_{\mathcal{X}^\alpha(0)}^\infty \psi(2x)dx} \right\}.$$

Numerical simulation on coupling strength





Velocity fluctuation changes along the parameter K, coupling sterngth.

Red, green, blue lines are for each group, and the bright blue line is for the global.

Left : Initial velocity distributions = Initial position distributions. $N = 20$.

Right : Velocity fluctuation evaluated at time $t = 100K$.

Remark

Remark

The conclusion from the analysis on K

- L^1 estimate of $\psi(\|\mathbf{x}_{\beta j}(t) - \mathbf{x}_{\alpha i}(t)\|)$ is essential to control $\mathcal{X}^\alpha(t)$
  Therefore, initial space distribution critically affects local flocking.

- K_1 and K_2 are in a similar order, in particular,
  $K_1 : K_2 \sim \min \|\mathbf{v}^c_\beta - \mathbf{v}^c_\alpha\| : \max \mathcal{V}^\alpha$.
  This is the main difficulty to the local flocking problem when we use Lyapunov functional approach.

Hydrodynamic description of Cucker-Smale model

Recall that the temporal-spatial evolution of (ρ, u) is governed by

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d,\ t > 0, \\
&\rho \partial_t u + \rho u \cdot \nabla u = -K\rho \int_{\mathbb{R}} \psi(|y-x|)(u(x) - u(y))\rho(y)dy, \\
&(\rho, u)(x, 0) = (\rho_0, u_0),
\end{aligned}$$

where K is the coupling strength and ψ is the communication weight.

In order to analyze the bi-cluster flocking phenomena, we consider a coupled system of hydrodynamic equations.

Hydrodynamic description of Cucker-Smale model

For two homogeneous ensemble, we use the following equations, which we only use for non-overlapping ρ_1 and ρ_2. Set $(\rho_1, \rho_2, u_1, u_2)$ be the phase,

$$
\begin{aligned}
&\partial_t \rho_1 + \nabla \cdot (\rho_1 u_1) = 0, \quad \partial_t \rho_2 + \nabla \cdot (\rho_2 u_2) = 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+, \\
&\rho_1 \partial_t u_1 + \rho_1 u_1 \cdot \nabla u_1 \\
&\qquad = \kappa_{11} \int_{\Omega_1(t)} \rho_1(x)\rho_1(y)\psi(|y-x|)(u_1(y) - u_1(x))dy \\
&\qquad + \kappa_{12} \int_{\Omega_2(t)} \rho_1(x)\rho_2(y)\psi(|y-x|)(u_2(y) - u_1(x))dy, \\
&\rho_2 \partial_t u_2 + \rho_2 u_2 \cdot \nabla u_2 \\
&\qquad = \kappa_{22} \int_{\Omega_2(t)} \rho_2(x)\rho_2(y)\psi(|y-x|)(u_2(y) - u_2(x))dy \\
&\qquad + \kappa_{21} \int_{\Omega_1(t)} \rho_2(x)\rho_1(y)\psi(|y-x|)(u_1(y) - u_2(x))dy,
\end{aligned}
$$

A Lagrangian formulation

In order to use the techniques of particle model, we use a Lagrangian formulation to describe the particle-path.

First, we introduce the triplet (η_i, q_i, v_i) for Lagrangian variables, consisting of the forward particle path $\eta_i = \eta_i(x,t)$, Lagrangian mass $q_i = q_i(x,t)$, and velocity densities $v_i = v_i(x,t)$.

We set $\Omega_1 := \mathrm{spt}(\rho_1)(0)$ and $\Omega_2 := \mathrm{spt}(\rho_2)(0)$ for notational convenience. Then, for a fixed $x \in \Omega_i$,

$$\begin{cases} \dfrac{d\eta_i(x,t)}{dt} = u_i(\eta_i(x,t),t), \quad t > 0, \quad i = 1, 2, \\ \eta_i(x,0) = x. \end{cases}$$

and

$$q_i(x,t) := \rho_i(\eta_i(x,t),t), \quad v_i(x,t) := u_i(\eta_i(x,t),t).$$

A Lagrangian formulation

Using the Lagrangian formulation, we define Lyapunov functionals as in the particle model:

$$\mathcal{V}_i(t) := \|v_i(t) - v_{ic}(t)\|_{L^\infty(\Omega_i)}, \quad i = 1, 2,$$
$$\mathcal{X}_i(t) := \|\eta_i(t) - \eta_{ic}(t)\|_{L^\infty(\Omega_i)},$$
$$\mathcal{V}(t) := \max\left\{\|v_1(\cdot, t) - v_c(t)\|_{L^\infty(\Omega_1)}, \|v_2(\cdot, t) - v_c(t)\|_{L^\infty(\Omega_2)}\right\},$$
$$\mathcal{X}(t) := \max\left\{\|\eta_1(\cdot, t) - \eta_c(t)\|_{L^\infty(\Omega_1)}, \|\eta_2(\cdot, t) - \eta_c(t)\|_{L^\infty(\Omega_2)}\right\},$$
$$\mathcal{V}_d(t) := \min_{x\in\Omega_1, y\in\Omega_2} |v_1(x, t) - v_2(y, t)|,$$
$$\mathcal{X}_d(t) := \min_{x\in\Omega_1, y\in\Omega_2} |x_1(x, t) - x_2(y, t)|.$$

Note that $(\mathcal{X}_i, \mathcal{X})$ and $(\mathcal{V}_i, \mathcal{V})$ measure the velocity and spatial fluctuations around the local averages and global averages.

A Lagrangian formulation

Definition of flocking

Let $Z = \{(\eta_i, q_i, v_i)\}_{i=1}^{2}$ be a classical global solution to the system.

1. The Lagrangian configuration Z exhibits an asymptotic "mono-cluster flocking" if and only if the functionals $\mathcal{X}$ and $\mathcal{V}$ satisfy

$$\sup_{0 \le t < \infty} \mathcal{X}(t) < \infty \qquad \lim_{t \to \infty} \mathcal{V}(t) = 0.$$

2. The Lagrangian configuration Z exhibits an asymptotic "bi-cluster flocking" if and only if the functionals $\mathcal{X}_i, \mathcal{V}_i$ and $\mathcal{V}_d$ satisfy

$$\sup_{0 \le t < \infty} \mathcal{X}_i(t) < \infty, \quad \lim_{t \to \infty} \mathcal{V}_i(t) = 0, \quad 1 \le i \le 2, \quad \inf_{0 \le t < \infty} \mathcal{V}_d(t) > 0.$$

Mono-cluster flocking on hydrodynamic C-S model

Theorem 3; mono-cluster flocking [Ha-K.-Zhang-Zhang, 2017, the corresponding result of Ha-Kang-Kwon(2014)]

Suppose that the following conditions $(\mathcal{C}_8 1) - (\mathcal{C}_8 3)$ hold.

- $(\mathcal{C}_8 1)$: Initial supports of ρ_{i0} are bounded and disjoint:

$$\mathcal{L}^d(\mathrm{spt}(\rho_{i0})) < \infty, \quad \rho_{i0}(x) > 0, \quad x \in \Omega_i^0, \quad i = 1, 2, \quad \mathcal{X}(0) > 0,$$

  where $\mathcal{L}^d$ is a d-dimensional Lebesgue measure in $\mathbb{R}^d$.

- $(\mathcal{C}_8 2)$: Initial data are sufficiently regular:

$$(q_{i0}, v_{i0}) \in H^s(\Omega_i) \times H^{s+1}(\Omega_i), \quad i = 1, 2, \quad s > \frac{d}{2} + 1.$$

- $(\mathcal{C}_8 3)$: The coupling strengths are symmetric and bounded below:

$$\kappa_{12} = \kappa_{21}, \quad \min_{i,j} \kappa_{ij} \|\rho_{j0}\|_{L^1(\Omega_1)} > \frac{\mathcal{V}_0}{\int_{\mathcal{X}_0}^{\infty} \psi(2x)dx}.$$

Mono-cluster flocking on hydrodynamic C-S model

Then, there exists a positive constant ε_0 depending only on the ρ_{i0} such that if $\|v_{i0}\|_{H^{s+1}(\Omega_i)} < \varepsilon_0, i = 1, 2$, then the initial value problem has a unique classical solution $(\rho_i, u_i), i = 1, 2$ satisfying regularity and flocking estimates:

$$(i)\ (q, v) \in \mathcal{Q}_s(\infty), \quad \eta \in \mathcal{C}^0\big([0,\infty); H^{s+1}\big).$$
$$(ii)\ \sup_{0 \le t < \infty} \mathcal{X}(t) < x_\infty, \quad \mathcal{V}(t) \le \mathcal{V}_0 e^{-C_1 \psi(2x_\infty) t}, \quad t \ge 0,$$

where $C_1 := \min_{1 \le i \le 2} ||\rho_{i0}||_{L^1(\Omega_i)} \times \min_{1 \le i,j \le 2} \kappa_{ij}$ and we the solution space

$$\mathcal{Q}_s(\infty) := \Big\{(q_i, v_i) : q_i \in \mathcal{C}^0([0,T); H^k) \cap \mathcal{C}^1([0,T); H^{k-1}), \\ v_i \in \mathcal{C}^0([0,T); H^{k+1}) \cap \mathcal{C}^1([0,T); H^k)\ \Big\}.$$

Bi-cluster flocking on hydrodynamic C-S model

Theorem 4; bi-cluster flocking [Ha-K.-Zhang-Zhang, 2017]

Suppose that the following conditions $(\mathcal{C}_9 1) - (\mathcal{C}_9 4)$ hold.

- $(\mathcal{C}_9 1)$: Initial supports of ρ_{i0} are bounded and disjoint:

$$\mathcal{L}^d(\mathrm{spt}(\rho_{i0})) < \infty, \quad \rho_{i0}(x) > 0, \quad x \in \Omega_i^0, \quad \|\rho_{i0}\|_{L^1(\Omega_i)} > 0,$$
$$\mathcal{X}(0) > 0, \quad \lambda_0 := \frac{1}{2}|v_{2c}(0) - v_{1c}(0)| > 0, \quad \mathcal{V}_{i0} := \mathcal{V}_i(0) < \frac{\lambda_0}{2},$$
$$r_0 := \min\left[(\eta_2(x,0) - \eta_1(y,0)) \cdot \frac{(v_{2c}(0) - v_{1c}(0))}{|v_{2c}(0) - v_{1c}(0)|}\right] > 0,$$

  where $\mathcal{L}^d$ is a d-dimensional Lebesgue measure in $\mathbb{R}^d$.

- $(\mathcal{C}_9 2)$: Initial data are sufficiently regular:

$$(q_{i0}, v_{i0}) \in H^s(\Omega_i) \times H^{s+1}(\Omega_i), \quad i = 1, 2, \quad s > \frac{d}{2} + 1.$$

Bi-cluster flocking on hydrodynamic C-S model

- $(\mathcal{C}_9 3)$: The coupling strengths are symmetric and bounded below:

$$\kappa_{ii} > \frac{\mathcal{V}_{i0} + \frac{2C\kappa_{12}\sqrt{2M_2(0)}}{\lambda_0}\int_{r_0}^{\infty}\psi(s)ds}{\int_{\mathcal{X}_{i0}}^{+\infty}\psi(2s)ds}, \quad i = 1, 2,$$

$$0 < \kappa_{21} = \kappa_{12} < \frac{\lambda_0}{12C\sqrt{2M_2(0)}\int_{r_0}^{\infty}\psi(s)ds},$$

where $C := \Pi_{i=1}^{2}\|\rho_{i0}\|_{L^1(\Omega_i)} + \max_{1\le i\le 2}\{\|\rho_{i0}\|_{L^2(\Omega_i)}\|\rho_{i0}\|_{L^1(\Omega_i)}^{1/2}\}$ is a positive constant only depending on ρ_{i0}.

- $(\mathcal{C}_9 4)$: (For simplicity,) The communication weight ψ takes the form given by:

$$\psi(r) = \frac{1}{(1+r^2)^{\frac{\beta}{2}}}, \quad \beta > 1.$$

Bi-cluster flocking on hydrodynamic C-S model

Then, there exists a positive constant ε_0 depending only on the ρ_{i0} such that if

$$\max_{1\le i\le 2}\|\nabla_x v_{i0}\|_{H^s(\Omega_i)} + \kappa_{12}\max\{\lambda_0^{-\beta}, (1+r_0^2)^{-\beta/2}\} < \varepsilon_0.$$

Then, the Cauchy problem has a unique classical solution $(\rho_i, u_i), i = 1, 2$ given by

$$\begin{aligned}&(i)\ (q, v) \in \mathcal{Q}_s(\infty), \quad \eta \in \mathcal{C}^0\big([0,\infty); H^{s+1}\big).\\ &(ii)\ \mathcal{V}_d(t) > \lambda_0, \quad \mathcal{V}_i(t) \le \tilde{C}_{ii}\max\left\{e^{-\frac{\kappa_{ii}\psi(2x_{iL})t}{2}}, \psi\Big(\frac{\lambda_0 t}{4}\Big)\right\},\\ &\quad \mathcal{X}_i(t) < \infty, \quad \text{for} \quad t \ge 0, \quad i = 1, 2.\end{aligned}$$

Remark

By the standard Sobolev embedding theorem,
the solution $(q_i, v_i) \in \mathcal{Q}_s(\infty)$ on $s > \frac{d}{2} + 1$ in Theorem are $\mathcal{C}^1$, that is $(q_i, v_i) \in \mathcal{C}^1(\Omega_i \times [0, \infty))$.

On the other hand, the parameter

$$\mathcal{R}_0 := \max\{\lambda_0^{-\beta}, (1 + r_0^2)^{-\beta/2}\}$$

indicates the initial separation of two groups.
The inter-group action factor $\kappa_{12}\mathcal{R}_0$ is needed to control the global existence since $\kappa_{12}\mathcal{R}_0 \simeq 0$ implies the system is close to the steady state and the bi-cluster flocking situation.

Sketch of proof

Flocking estimates

Let $T_* \in (0, \infty]$ be a positive number, and suppose that $(\mathcal{C}_8)$ holds.
Let (η_i, q_i, v_i) be a classical solution to the system in $[0, T_*)$.

Then, there exists positive constants $\bar{x}_\infty \in (0, \infty)$ and
$\bar{C}_1 = \bar{C}_1(\rho_{i0}, \mathcal{V}_{10}, \mathcal{V}_{20}, \kappa_{11}, \kappa_{12}, \kappa_{22}, M_2(0), \bar{x}_\infty)$ such that

$$\sup_{0 \le t < T_*} \mathcal{X}_i(t) \le \bar{x}_\infty, \quad \mathcal{V}_i(t) \le \bar{C}_1 \left[e^{-\frac{1}{2}\kappa_{ii}\psi(2\bar{x}_\infty)t} + \psi\left(r_0 + \frac{\lambda_0 t}{2} \right) \right], \; t \in (0, T_*),$$

i.e., bi-cluster flocking occurs asymptotically. Moreover, if we let
$\mathcal{R}_0 = \max\{\lambda_0^{-\beta}, (1 + r_0^2)^{-\beta/2}\}$, then we have

$$\mathcal{V}_i(t) \le \left(\mathcal{V}_i(0) + \frac{\kappa_{12}}{\kappa_{ii}} \mathcal{R}_0 \right) \frac{\mathcal{O}(1)}{(1+t)^\beta}.$$

Sketch of proof

A priori estimates

For any positive constant $T \in (0, \infty]$, let σ_i be positive constants satisfying

$$\beta - \sum_{i=1}^{k} \sigma_i > 1,$$

and (η_i, q_i, v_i) be a classical solution to the system in $[0, T)$.

Then, for $t \in [0, T)$, there exists C_1 only depend on initial conditions and parameters such that

$$\|v_1(t)\|_{L^2} \le \|v_{10}\|_{L^2} + \frac{C_1}{(1+t)^{\beta-1}}, \quad \|\nabla_x v_1(t)\|_{L^2} \le \frac{C_1}{(1+t)^{\beta}}, \ \text{ and}$$

$$\|\nabla_x v_1(t)\|_{H^1} \le \frac{C_1}{(1+t)^{(\beta-\sigma_1)}}, \quad \|\nabla_x v_1(t)\|_{H^s} \le \sum_{k=1}^{s+1} \frac{C_1}{(1+t)^{(\beta-\sum_{i=1}^{k}\sigma_i)}}.$$

Sketch of proof

Existence of solutions

Note that C_1 is independent of T. we set

$$\varepsilon_1 > \|v_{10}\|_{L^2} + (s+2)C_1.$$

Then for initial data $u_{i0} \in H^{s+1}(\Omega_1)$ satisfying $\|u_0\|_{H^{s+1}} \leq \varepsilon_0 < \varepsilon_1$, where ε_0 is given in a priori estimates, we define

$$T_e^* := \sup\{T \geq 0 : \sup_{0 \leq t \leq T} \|v(t)\|_{H^{s+1}} < \varepsilon_1\}.$$

Note that we know $T_e^* > 0$. Suppose $T_e^* < \infty$. Then, by definition we have

$$\sup_{0 \leq t \leq T_e^*} \|v(t)\|_{H^{s+1}} = \varepsilon_1. \tag{3}$$

A priori estimates says it cannot be true, hence $T_e^* = \infty$.
The iteration scheme on (η^n, v^n) leads to the global existence of solutions for small ε_1.

Sketch of proof

Proof on a priori estimates (Zeroth-order)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_1} |v_1(x)|^2 dx \\
&\quad = \kappa_{11} \iint_{\Omega_1\times\Omega_1} q_1(y,0)\psi(\eta_1(y)-\eta_1(x))(v_1(y)-v_1(x))\cdot v_1(x)dydx \\
&\quad + \kappa_{12} \iint_{\Omega_2\times\Omega_1} q_2(y,0)\psi(\eta_2(y)-\eta_1(x))(v_2(y)-v_1(x))\cdot v_1(x)dydx \\
&\quad \le 2\kappa_{11} \iint_{\Omega_1\times\Omega_1} q_1(y,0)\mathcal{V}_1(t)|v_1(x)|dydx \\
&\quad + \kappa_{12} \iint_{\Omega_2\times\Omega_1} q_2(y,0)|v_2(y)|\psi(\min(|\eta_2(y)-\eta_1(x)|))|v_1(x)|dydx
\end{aligned}
$$

Sketch of proof

Proof on a priori estimates (Zeroth-order)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_1}|v_1(x)|^2dx \\
&\quad\le \frac{\mathcal{O}(1)(\kappa_{11}\mathcal{V}_1(0)+\kappa_{12}\mathcal{R}_0)}{(1+t)^\beta}\|q_1(\cdot,0)\|_{L^1}\|v_1\|_{L^1} \\
&\quad+ \frac{\mathcal{O}(1)\kappa_{12}}{(1+(\lambda_0 t+r_0)^2)^{\beta/2}}\iint_{\Omega_2\times\Omega_1} q_2(y,0)|v_2(y)||v_1(x)|dydx \\
&\quad\le \frac{\mathcal{O}(1)}{(1+t)^\beta}\cdot(\kappa_{11}\mathcal{V}_1(0)+\kappa_{12}\mathcal{R}_0)\|v_1\|_{L^2}.
\end{aligned}$$

Here $\mathcal{O}(1)$ is a positive constant only depending on $\psi_{2\bar{x}_\infty}$. Then we have

$$\|v_1(t)\|_{L^2}\le\|v_{10}\|_{L^2}+\mathcal{O}(1)\frac{\kappa_{11}\mathcal{V}_1(0)+\kappa_{12}\mathcal{R}_0}{(1+t)^{\beta-1}}.$$

Sketch of proof

How about the first order?

$$
\begin{aligned}
\frac{d}{dt}\frac{\|\nabla_x v_1\|_{L^2}^2}{2} &= \kappa_{11}\int_{\Omega_1} q_1(y,0)(v_1(y)-v_1(x))\nabla_x(\psi(\eta_1(y)-\eta_1(x)))\cdot\nabla_x v_1\,dy \\
&-\kappa_{11}\int_{\Omega_1} q_1(y,0)\nabla_x v_1(\psi(\eta_1(y)-\eta_1(x)))\cdot\nabla_x v_1\,dy \\
&+\kappa_{12}\int_{\Omega_2} q_2(y,0)(v_2(y)-v_1(x))\nabla_x(\psi(\eta_2(y)-\eta_1(x)))\cdot\nabla_x v_1\,dy \\
&-\kappa_{12}\int_{\Omega_2} q_2(y,0)\nabla_x v_1(\psi(\eta_2(y)-\eta_1(x)))\cdot\nabla_x v_1\,dy \\
&\le \kappa_{11}\int_{\Omega_1} q_1(y,0)2|v_1(x)||\psi'(\eta_1(y)-\eta_1(x))||\partial_x\eta_1(x)|dy|\nabla_x v_1| \\
&+\kappa_{12}\int_{\Omega_2} q_2(y,0)|v_2(y)-v_1(x)||\psi'(\eta_2(y)-\eta_1(x))||\partial_x\eta_1(x)|dy|\nabla_x v_1|.
\end{aligned}
$$

Sketch of proof

Proof on a priori estimates (first-order)

For the higher order estimates, we set an ansatz to use flocking estimates. Define

$$T^* := \sup\left\{ t \ \in (0, T] \ \middle| \ \|\nabla_x v_1(t)\|_{H^k} < \frac{C_1}{(1+t)^{(\beta - \sum_{i=1}^{k+1}\sigma_i)}}, \ \forall k \le s \right\},$$

Then we need to show $T^* = T$.
The zeroth-order has faster decay rate than our ansatz in T^*, hence it is good for zeroth-order with proper C_1. For the first order, we use

$$\|\nabla_x v_i\|_{L^\infty} \le \mathcal{O}(1)\|\nabla_x v_i\|_{H^s} \le \mathcal{O}(1)\frac{C_1}{(1+t)^{(\beta - \sum_{i=1}^{s+1}\sigma_i)}}.$$

Thus we have the following, which we can use it in estimates of H^k.

$$|\nabla_x \eta_i(x,t)| \le \mathcal{O}(1)\left(1 + \int_0^{+\infty} |\nabla_x v_i(\tau)| d\tau\right) \le \mathcal{O}(1)C_1.$$

Summary

What we did

1. The existence of the bi-cluster flocking state,
2. Conditions on K which occurs multi-cluster flocking,
3. Local flocking phenomena on hydrodynamic C-S model.

Remarks on the local flocking

1. Initial positions are critical to prove the position separation on $t \to \infty$.
2. We should prove the local flocking of each group and the separating among groups simultaneously.

Thank you