CONTROLLABILITY OF THE 1D FRACTIONAL HEAT EQUATION UNDER POSITIVITY CONSTRAINTS

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We are interested in analyzing controllability properties under positivity state/control constraints.
The fractional Laplacian

\[ (-d_x^2)^s z = C_s \text{ P.V.} \int_{\mathbb{R}} \frac{z(x) - z(y)}{|x - y|^{1+2s}} dy. \]

\[ C_s = \left( \int_{\mathbb{R}} \frac{1 - \cos(\zeta_1)}{|\zeta|^{1+2s}} d\zeta \right)^{-1} = \frac{s2^{2s}\Gamma\left(\frac{1+2s}{2}\right)}{\pi^{\frac{1}{2}}\Gamma(1 - s)}. \]
Fractional Sobolev spaces

Fractional Sobolev space

\[ H^s(-1, 1) := \left\{ z \in L^2(-1, 1) : \int_{-1}^{1} \int_{-1}^{1} \frac{|z(x) - z(y)|^2}{|x - y|^{1+2s}} \, dx \, dy < \infty \right\} \]

- Intermediate Hilbert space between \( L^2(-1, 1) \) and \( H^1(-1, 1) \).

Scalar product

\[ \langle z, w \rangle_{H^s(-1, 1)} := \int_{-1}^{1} zw \, dx + \int_{-1}^{1} \int_{-1}^{1} \frac{(z(x) - z(y))(w(x) - w(y))}{|x - y|^{1+2s}} \, dx \, dy \]

Fractional Sobolev norm

\[ \|z\|_{H^s(-1, 1)} := \left( \int_{-1}^{1} |z|^2 \, dx + \int_{-1}^{1} \int_{-1}^{1} \frac{|z(x) - z(y)|^2}{|x - y|^{1+2s}} \, dx \, dy \right)^{\frac{1}{2}} \]
Known controllability results

**Theorem (B. and Hernández-Santamaría, IMA J. Math. Control Inf., 2018)**

There exists \( u \in L^2(\omega \times (0, T)) \) such that the fractional heat equation \((\mathcal{FH})\) is

- **null-controllable** at time \( T > 0 \) if and only if \( s > 1/2 \).
- **approximately controllable** at time \( T > 0 \) for all \( s \in (0, 1) \).

**Remark**: the equation being linear, by translation if \( s > 1/2 \) we have controllability to trajectories \( \hat{z} \).
**Proof**

**Null controllability**: through the moment method, based on the following behavior of the spectrum.

**Eigenvalues** *(Kwaśnicki, J. Funct. Anal., 2012)*

\[
\lambda_k = \left( \frac{k \pi}{2} - \frac{(1 - s) \pi}{4} \right)^{2s} + O\left( \frac{1}{k} \right).
\]

**Approximate controllability**: The result follows from the following property.

**Parabolic unique continuation**

Given \( s \in (0, 1) \) and \( p^T \in L^2(-1, 1) \), let \( p \) be the unique solution to the adjoint equation. Let \( \omega \subset (-1, 1) \) be an arbitrary open set. If \( p = 0 \) on \( \omega \times (0, T) \), then \( p = 0 \) on \( (-1, 1) \times (0, T) \).

This, in turn, is a consequence of the **unique continuation** property for the Fractional Laplacian.

• The fractional heat equation **preserves positivity**: if $z_0$ is a given non-negative initial datum in $L^2(-1, 1)$ and $u$ is a non-negative function, then so it is for the solution $z$ of ($\mathcal{FH}$).

B., Warma and Zuazua, 2019

**Question**

Can we control the fractional heat dynamics ($\mathcal{FH}$) from any initial datum $z_0 \in L^2(-1, 1)$ to any positive trajectory $\hat{z}$, under positivity constraints on the control and/or the state?
Let $s > 1/2$, $z_0 \in L^2(-1, 1)$ and let $\hat{z}$ be a positive trajectory, i.e., a solution of $(\mathcal{F} \mathcal{H})$ with initial datum $0 < \hat{z}_0 \in L^2(-1, 1)$ and right hand side $\hat{u} \in L^\infty(\omega \times (0, T))$. Assume that there exists $\nu > 0$ such that $\hat{u} \geq \nu$ a.e in $\omega \times (0, T)$. Then, the following assertions hold.

1. There exist $T > 0$ and a non-negative control $u \in L^\infty(\omega \times (0, T))$ such that the corresponding solution $z$ of $(\mathcal{F} \mathcal{H})$ satisfies $z(x, T) = \hat{z}(x, T)$ a.e. in $(-1, 1)$. Moreover, if $z_0 \geq 0$, we also have $z(x, t) \geq 0$ for every $(x, t) \in (-1, 1) \times (0, T)$.

2. Define the minimal controllability time by

$$T_{\min}(z_0, \hat{z}) := \inf \left\{ T > 0 : \exists \ 0 \leq u \in L^\infty(\omega \times (0, T)) \text{ s.t.} \right. $$

$$z(\cdot, 0) = z_0 \text{ and } z(\cdot, T) = \hat{z}(\cdot, T) \right\}.$$

Then, $T_{\min} > 0$.

3. For $T = T_{\min}$, there exists a non-negative control $u \in \mathcal{M}(\omega \times (0, T_{\min}))$, the space of Radon measures on $\omega \times (0, T_{\min})$, such that the corresponding solution of $(\mathcal{F} \mathcal{H})$ satisfies $z(x, T) = \hat{z}(x, T)$ a.e. in $(-1, 1)$. 

Theorem (B., Warma and Zuazua, 2019)
Proof - main ingredients

The proof requires two main ingredients:

1. **Controllability** through $L^\infty$ **controls**, consequence of the following observability inequality

   \[ \| p(\cdot, 0) \|^2_{L^2((-1, 1))} \leq C \left( \int_0^T \int_\omega |p(x, t)| \, dx \, dt \right)^2 \]

2. **Dissipativity** of the fractional heat semi-group.
Some preliminary results

Theorem

Let \( \{\mu_k\}_{k \geq 1} \) be a sequence of real numbers satisfying the conditions:

1. There exists \( \gamma > 0 \) such that \( \mu_{k+1} - \mu_k \geq \gamma \) for all \( k \geq 1 \).
2. \( \sum_{k \geq 1} \mu_k^{-1} < +\infty \).

Then, for any \( T > 0 \), there is a positive constant \( C = C(T) > 0 \) such that, for any finite sequence \( \{c_k\}_{k \geq 1} \) it holds the inequality

\[
\sum_{k \geq 1} |c_k|^2 e^{-2\mu_k T} \leq C \left\| \sum_{k \geq 1} c_k e^{-\mu_k t} \right\|_{L^1(0,T)}^2.
\]

PROOF:

Under the above hypothesis on \( \{\mu_k\}_{k \geq 1} \), the function \( F(t) := \sum_{k \geq 1} c_k e^{-\mu_k t} \) satisfies

\[
|c_k| \leq C \|F\|_{L^1(0,T)} \quad \text{and} \quad \sum_{k \geq 1} |c_k| e^{(\mu_1 - \mu_k)t} \leq C(t) \|F\|_{L^1(0,T)} \quad \text{for all } t > 0.
\]

Then

\[
\sum_{k \geq 1} |c_k|^2 e^{-2\mu_k T} = \sum_{k \geq 1} |c_k|^2 e^{(\mu_1 - \mu_k)t} \left( |c_k| e^{(\mu_k - \mu_1)t} e^{-2\mu_k T} \right) \leq C \|F\|_{L^1(0,T)}^2.
\]
Lemma

Consider the eigenvalue problem for the Dirichlet fractional Laplacian in $(-1, 1)$:

$$
\begin{cases}
(-d_x^2)^s \phi_k = \lambda_k \phi_k, & x \in (-1, 1) \\
\phi_k = 0, & x \in (-1, 1)^c.
\end{cases}
$$

Then for any open subset $\omega \subset (-1, 1)$, there is a positive constant $\beta > 0$ such that $\|\phi_k\|_{L^1(\omega)} \geq \beta > 0$.

PROOF (main idea):

The proof is based on the fact that

$$
\int_{\omega} |\phi_k(x)| \, dx \geq \int_{\omega} \left| \sin \left( \mu_k (1 + x) + \frac{(1 - s)\pi}{4} \right) - \frac{c(1 - s)}{\sqrt{s}} \mu_k^{1-2s} \right| \, dx
$$

$$
\mu_k := \frac{k\pi}{2} - \frac{(1 - s)\pi}{4}
$$
L^1 observability

Proposition

For any \( T > 0 \) and \( p_T \in L^2(-1, 1) \), let \( p \in L^2((0, T); H_0^s(-1, 1)) \cap C([0, T]; L^2(-1, 1)) \) with \( p_t \in L^2((0, T); H^{-s}(-1, 1)) \) be the weak solution of the adjoint system

\[
\begin{cases}
-\partial_t p + (-\partial_x^2)^s p = 0, & (x, t) \in (-1, 1) \times (0, T) \\
p = 0, & (x, t) \in (-1, 1)^c \times (0, T) \\
p(\cdot, T) = p_T(\cdot), & x \in (-1, 1).
\end{cases}
\]

Then, for any \( s > 1/2 \), there is a constant \( C = C(T) > 0 \) such that

\[
\|p(\cdot, 0)\|_{L^2(-1, 1)}^2 \leq C \left( \int_0^T \int_{-1}^1 |p(x, t)| \, dx \, dt \right)^2.
\]
For any $z_0 \in L^2(-1,1)$, $s > 1/2$ and $T > 0$, there exists a control function $u \in L^\infty(\omega \times (0, T))$ such that the corresponding unique weak solution $z$ of $(\mathcal{F} \mathcal{H})$ with initial datum $z(x, 0) = z_0(x)$ satisfies $z(x, T) = 0$ a.e. in $(-1, 1)$. Moreover, there is a constant $C > 0$ (depending only on $T$) such that

$$\|u\|_{L^\infty(\omega \times (0,T))} \leq C \|z_0\|_{L^2(-1,1)}.$$ 

**PROOF:**

Classical duality argument.
Proof of the main result - 1: constrained controllability

- Subtracting $\hat{z}$ in the equation, it is enough to show that there exist a time $T > 0$ and a control $v \in L^\infty(\omega \times (0, T))$, $v > -\nu$ a.e. in $\omega \times (0, T)$, such that

\[
\begin{cases}
\xi_t + (-d_x^2)^s \xi = v\chi_\omega, & (x, t) \in (-1, 1) \times (0, T) \\
\xi = 0, & (x, t) \in (-1, 1)^c \times (0, T) \\
\xi(\cdot, 0) = z_0(\cdot) - \hat{z}_0(\cdot), & x \in (-1, 1)
\end{cases}
\Rightarrow \xi(x, T) = 0.
\]

- This is equivalent to the observability inequality

\[
\|p(\cdot, \tau)\|_{L^2(-1, 1)}^2 \leq C(T - \tau) \left( \int_0^T \int_\omega |p(x, t)| \, dx \, dt \right)^2.
\]

- Using that the eigenvalues $\{\lambda_k\}_{k \geq 1}$ form a non-decreasing sequence, and the dissipativity of the fractional heat semi-group:

\[
\|p(\cdot, 0)\|_{L^2(-1, 1)}^2 \leq e^{-2\lambda_1 \tau} \|p(\cdot, \tau)\|_{L^2(-1, 1)}^2 
\leq e^{-2\lambda_1 \tau} C(T - \tau) \left( \int_0^T \int_\omega |p(x, t)| \, dx \, dt \right)^2.
\]
By duality, the control $v$ can be chosen such that
\[ \|v\|_{L^\infty(\omega \times (0, T))}^2 \leq e^{-2\lambda_1 \tau} C(T - \tau) \|z_0 - \hat{z}_0\|_{L^2(-1, 1)}^2. \]

Taking $\tau = T/2$, we obtain
\[ \|v\|_{L^\infty(\omega \times (0, T))}^2 \leq e^{-\lambda_1 T} C(T) \|z_0 - \hat{z}_0\|_{L^2(-1, 1)}^2. \]

The observability constant $C(T)$ is uniformly bounded for any $T > 0$. Hence, for $T$ large enough we have
\[ \|v\|_{L^\infty(\omega \times (0, T))}^2 < \nu. \]

This implies that $v > -\nu$. Therefore, the control $v > -\nu$ steers $\xi$ from $z_0 - \hat{z}_0$ to zero in time $T > 0$, provided $T$ is large enough. Consequently, $z$ is controllable to the trajectory $\hat{z}$ in time $T$.

If $z_0 \geq 0$, thanks to the maximum principle, we also have $z(x, t) \geq 0$ for every $(x, t) \in (-1, 1) \times (0, T)$.
Proof of the main result - 2: positivity of $T_{\min}$

- Solution of $(\mathcal{F}\mathcal{H})$ in the basis of the eigenfunctions $\{\phi_k\}_{k \geq 1}$:

\[
z(x, t) = \sum_{k \geq 1} z_k(t) \phi_k(x).
\]

\[
z_k(t) = z_k^0 e^{-\lambda_k t} + \int_0^t e^{-\lambda_k(t-\tau)} u_k(\tau) \, d\tau,
\]

\[
u_k(t) := \int_\omega u(x, t) \phi_k(x) \, dx.
\]

- $z(x, T) = \hat{z}(x, T)$ a.e. in $(-1, 1)$:

\[
z_k(T) = \int_{-1}^1 \hat{z}(x, T) \phi_k(x) \, dx =: \zeta_k \quad \Rightarrow \quad \zeta_k - z_k^0 e^{-\lambda_k T} = \int_0^T e^{-\lambda_k(T-\tau)} u_k(\tau) \, d\tau.
\]

- For every $0 \leq \tau \leq T$:

\[
\zeta_k - z_k^0 e^{-\lambda_k T} \leq \int_0^T u_k(\tau) \, d\tau \leq \zeta_k e^{\lambda_k T} - z_k^0.
\]

- Assume by contradiction that, for every $T > 0$, there exists a non-negative control function $u^T$ steering $z_0$ to $\hat{z}(\cdot, T)$ in time $T$, and that $\hat{z}(\cdot, T) \neq z_0$. Then:

\[
\lim_{T \to 0^+} \int_0^T u_k^T(\tau) \, d\tau = \zeta_k - z_k^0 =: \gamma \quad \Rightarrow \quad z_k^0 = \zeta_k - \gamma.
Proof of the main result - 2: positivity of $T_{\text{min}}$

- $z_0 \in L^2(-1, 1)$:
  \[
  \sum_{k \geq 1} |z^0_k|^2 = \sum_{k \geq 1} \left( \zeta_k^2 - 2\gamma \zeta_k + \gamma^2 \right) < +\infty \Rightarrow \lim_{k \to +\infty} \left( \zeta_k^2 - 2\gamma \zeta_k + \gamma^2 \right) = 0.
  \]

- Since $\{\phi_k\}_{k \geq 1}$ is an orthonormal complete system in $L^2(-1, 1)$, $\phi_k \to 0$ in $L^2(-1, 1)$ as $k \to +\infty$. Hence:
  \[
  \lim_{k \to +\infty} (\hat{z}(. , T), \phi_k)_{L^2(-1, 1)} = \lim_{k \to +\infty} \int_{-1}^{1} \hat{z}(x, T) \phi_k(x) \, dx = \lim_{k \to +\infty} \zeta_k = 0 \Rightarrow \gamma = 0.
  \]

Consequently
\[
0 = z_k^0 - \zeta_k = \int_{-1}^{1} (z_0(x) - \hat{z}(x, T)) \phi_k(x) \, dx, \text{ for all } k \geq 1.
\]

- This is possible if and only if $z_0(x) = \hat{z}(x, T)$ a.e. in $(-1, 1)$, which contradicts our previous assumption.
Proof of the main result - 3: minimal-time control

Constrained controllability of the system \((\mathcal{FH})\) holds in the minimal time \(T_{\text{min}}\) with controls in the (Banach) space of the Radon measures \(\mathcal{M}(\omega \times (0, T_{\text{min}}))\) endowed with the norm

\[
\|\mu\|_{\mathcal{M}(\omega \times (0, T_{\text{min}}))} = \sup \left\{ \int_{\omega \times (0, T_{\text{min}})} \varphi(x, t) \, d\mu(x, t) : \varphi \in C(\overline{\omega} \times [0, T_{\text{min}}], \mathbb{R}), \max_{\overline{\omega} \times [0, T_{\text{min}}]} |\varphi| = 1 \right\}.
\]

Solutions of \((\mathcal{FH})\) with controls in \(\mathcal{M}(\omega \times (0, T_{\text{min}}))\) are defined by transposition

**Transposition solution**

Given \(z_0 \in L^2(-1, 1), T > 0,\) and \(u \in \mathcal{M}(\omega \times (0, T))\), the function \(z \in L^1((-1, 1) \times (0, T))\) is a solution of \((\mathcal{FH})\) defined by transposition if

\[
\int_{\omega \times (0, T)} p(x, t) \, du(x, t) = \langle z(\cdot, T), p_T \rangle - \int_{-1}^1 z_0(x)p(x, 0) \, dx,
\]

where, for every \(p_T \in L^\infty(-1, 1)\), the function \(p \in L^\infty(Q)\) is the unique solution of

\[
\begin{cases}
-p_t + (-d_x^2)^s p = 0, & (x, t) \in (-1, 1) \times (0, T) \\
p = 0, & (x, t) \in (-1, 1)^c \times (0, T) \\
p(\cdot, T) = p_T, & x \in (-1, 1).
\end{cases}
\]
Proof of the main result - 3: minimal-time control

- Denote \( T_k := T_{\text{min}} + \frac{1}{k}, k \geq 1 \).

  There exists a sequence of non-negative controls \( \{u^T_k\}_{k \geq 1} \subset L^\infty(\omega \times (0, T_k)) \) such that the corresponding solution \( z^k \) of \((\mathcal{FH})\) with \( z^k(x, 0) = z_0(x) \) a.e. in \((-1, 1)\) satisfies \( z^k(x, T_k) = \tilde{z}(x, T_k) \) a.e. in \((-1, 1)\).

- Extend these controls by \( \tilde{u} \) on \((T_k, T_{\text{min}} + 1)\) to get a new sequence in \( L^\infty(\omega \times (0, T_{\text{min}} + 1)) \).

- \( p \geq 0 \Rightarrow p(x, t) \geq \theta > 0 \) for all \((x, t) \in (-1, 1) \times (0, T_{\text{min}} + 1)\). Then,

  \[
  \theta \left\| u^T_k \right\|_{L^1(\omega \times (0, T_{\text{min}} + 1))} = \theta \int_0^{T_{\text{min}} + 1} \int_{\omega} u^T_k(x, t) \, dx \, dt \\
  \leq \int_0^{T_{\text{min}} + 1} \int_{-1}^{1} p(x, t) u^T_k(x, t) \, dx \, dt \\
  = \langle z(\cdot, T), p \rangle - \int_{-1}^{1} z_0(x) p(x, 0) \, dx \leq M.
  \]

- \( \{u^T_k\}_{k \geq 1} \) is bounded in \( L^1(\omega \times (0, T_{\text{min}} + 1)) \), hence, it is bounded in the space \( \mathcal{M}(\omega \times (0, T_{\text{min}} + 1)) \). Thus, extracting a sub-sequence, we have:

  \( u^T_k \rightharpoonup \tilde{u} \quad \text{weakly} -* \quad \text{in} \quad \mathcal{M}(\omega \times (0, T_{\text{min}} + 1)) \quad \text{as} \quad k \to +\infty. \)

The limit control \( \tilde{u} \) satisfies the non-negativity constraint.
For any \( k \) large enough and \( T_{\text{min}} < T_0 < T_{\text{min}+1} \), we have

\[
\int_{\omega \times (0, T_0)} p(x, t) \, du^{T_k}(x, t) = \langle \hat{z}(\cdot, T_0), p_T \rangle - \int_{-1}^{1} z_0(x) p(x, 0) \, dx.
\]

- \( p_T \): first non-negative eigenfunction of \((-d_x^2)^s\)

\[
p \in C([0, T]; D((-d_x^2)^s)) \hookrightarrow C([0, T] \times [-1, 1]).
\]

- By definition of weak* limit, letting \( k \to +\infty \), we obtain

\[
\int_{\omega \times (0, T_0)} p(x, t) \, d\tilde{u}(x, t) = \langle \hat{z}(\cdot, T_0), p_T \rangle - \int_{-1}^{1} z_0(x) p(x, 0) \, dx,
\]

which implies that \( z(x, T_0) = \hat{z}(x, T_0) \) a.e. in \((-1, 1)\).

- Taking the limit as \( T_0 \to T_{\text{min}} \) and using the fact that

\[
|\tilde{u}|(\omega \times (T_{\text{min}}, T_0)) = |\hat{u}|(\omega \times (T_{\text{min}}, T_0)) = 0, \quad \text{as } T_0 \to T_{\text{min}}
\]

we deduce that \( z(x, T_{\text{min}}) = \hat{z}(x, T_{\text{min}}) \) a.e. in \((-1, 1)\).
Numerical simulations

- We consider the problem of steering the initial datum $z_0(x) = \frac{1}{2} \cos \left( \frac{\pi}{2} x \right)$ to the target trajectory $\hat{z}$ solution of $\mathcal{F} \mathcal{H}$ with initial datum $\hat{z}_0(x) = 6 \cos \left( \frac{\pi}{2} x \right)$ and right-hand side $\hat{u} \equiv 1$.

- We choose $s = 0.8$ and $\omega = (-0.3, 0.8) \subset (-1, 1)$ as the control region.

- The approximation of the minimal controllability time is obtained by solving the following constrained minimization problem:

\[
\begin{align*}
\text{minimize } & T \\
\text{subject to } & T > 0 \\
& z_t + (-d_x^2)^s z = u \chi_\omega, \quad \text{a.e. in } (-1, 1) \times (0, T) \\
& z(\cdot, 0) = z_0 \geq 0, \quad \text{a.e. in } (-1, 1) \\
& z \geq 0, \quad \text{a.e. in } (-1, 1) \times (0, T) \\
& u \geq 0, \quad \text{a.e. in } \omega \times (0, T).
\end{align*}
\]
To perform the simulations, we apply a FE method for the space discretization of the fractional Laplacian on a uniform space-grid

\[ x_i = -1 + \frac{2i}{N_x}, \quad i = 1, \ldots, N_x, \]

with \( N_x = 20 \). Moreover, we use an explicit Euler scheme for the time integration on the time-grid

\[ t_j = \frac{Tj}{N_t}, \quad j = 0, \ldots, N_t, \]

with \( N_t \) satisfying the **Courant-Friedrich-Lewy** condition. In particular, we choose here \( N_t = 100 \).
Numerical simulations

- We obtain the minimal time $T_{\text{min}} \simeq 0.2101$.
- In this time horizon, the fractional heat equation $\mathcal{FH}$ is controllable from the initial datum $z_0$ to the desired trajectory $\hat{z}(\cdot, T)$ by maintaining the positivity of the solution.
Numerical simulations

- The impulsional behavior of the control is lost when extending the time horizon beyond $T_{\text{min}}$.

This control has been computed by solving the minimization problem:

$$\min \| z(\cdot, T) - \hat{z}(\cdot, T) \|_{L^2((-1, 1))}$$

$$\begin{align*}
T &> 0 \\
z_t + (-d^2_x)^s z = u \chi, & \quad \text{a. e. in } (-1, 1) \times (0, T) \\
z(\cdot, 0) = z_0 \geq 0, & \quad \text{a. e. in } (-1, 1) \\
z \geq 0, & \quad \text{a. e. in } (-1, 1) \times (0, T) \\
u \geq 0, & \quad \text{a. e. in } \omega \times (0, T).
\end{align*}$$
Finally, when considering a time horizon $T < T_{\text{min}}$, constrained controllability fails.
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