An introduction to Koopman operator theory and its applications

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1 The Koopman operator
   - General idea
   - Koopman for discrete dynamical systems
   - Koopman for continuous dynamical systems

2 Example: optimal control

3 Perspectives
   - Nonlinear optimal control
   - Invariant subspaces of the Koopman operator
   - Stability

4 References
1 The Koopman operator
   - General idea
   - Koopman for discrete dynamical systems
   - Koopman for continuous dynamical systems

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3 Perspectives
   - Nonlinear optimal control
   - Invariant subspaces of the Koopman operator
   - Stability

4 References
General idea

- Introduced by Koopman in 1931 to study Hamiltonian systems
- Relevant in Ergodic Theory (Mean Ergodic Theorem)
- Re-introduced in the Dynamical Systems community in 2004 by Mézic

(Image extracted from [9])
Koopman for discrete dynamical systems

Discrete Dynamical System (DDS):
- \((X, \mathcal{F}, \mu)\) measure space
- \(T : X \to X\) measure-preserving map

**Definition**

For each \(1 \leq p \leq \infty\), the Koopman operator associated to the DDS \((X, T)\) is the composition operator

\[
K_p : L^p(X) \to L^p(X) \\
g \mapsto K(g) := g \circ T
\]

If \(x_n = T^n(x_0)\) and \(g \in L^p(X)\) we have:

\[
K(g)(x_n) = g(x_{n+1}) \text{ for each } n
\]
Many properties of \((X, T)\) are codified in the Koopman operator.

**Example (Ergodicity via Koopman)**

The DDS \((X, T)\) is ergodic if and only if 1 is a simple eigenvalue of \(K_p\) for some \(1 \leq p \leq \infty\).

\[
t \mapsto (e^{2\pi i \cdot at}, e^{2\pi i \cdot bt}) \text{ dense } \iff \frac{b}{a} \not\in \mathbb{Q}
\]
Koopman for continuous dynamical systems

Continuous Dynamical System (CDS):
- $\Omega \subseteq \mathbb{R}^n$ compact domain
- $\dot{x} = F(x)$ autonomous system on $\Omega$
- $\{\tau_t\}_{t \geq 0}$ one-parameter flow:
  $$\tau_t(x_0) = x(t)$$
  where $x$ is such that $x(0) = x_0$

Definition

For each $t \geq 0$, define

$$K_t : C^0(\Omega) \to C^0(\Omega)$$

$$f \mapsto K_t(f) := f \circ \tau_t$$

$\{K_t\}_{t \geq 0}$ is called the Koopman semi-group

- $K_{t+s} = K_t \circ K_s = K_s \circ K_t$ for every $t, s \geq 0$
- $K_0 = \text{Id}_{C^0(\Omega)}$
Lemma

\( \{K_t\}_{t \geq 0} \) is a strongly continuous semigroup of linear contracting operators in \((C^0(\Omega), \|\cdot\|_{\infty})\).

Definition

The Koopman operator associated to \( \dot{x} = F(x) \) is the infinitesimal generator of the semigroup \( \{K_t\}_{t \geq 0} \), i.e., is the linear operator given by

\[
K : D(K) \to C^0(\Omega)
\]

\[
f \mapsto K(f) := \lim_{t \to 0^+} \frac{K_t(f) - f}{t}
\]
Koopman for continuous dynamical systems

Observation

- $\mathcal{D}(K)$ is dense in $C^0(\Omega)$
- For each $k \geq 1$ we have $C^k(\Omega) \subseteq \mathcal{D}(K)$, and for every $g \in C^k(\Omega)$:

$$K(g) = f_1 \frac{\partial g}{\partial x_1} + \ldots + f_n \frac{\partial g}{\partial x_n}$$

where $F = (f_1, \ldots, f_n)$

For example, the energy of a Hamiltonian system is an eigenfunction of the Koopman operator ($K(H) = 0$).

In general, the eigenfunctions of the Koopman operator contain important information about the underlying dynamical system.
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- For example, the energy of a Hamiltonian system is an eigenfunction of the Koopman operator ($K(H) = 0$)
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Perspectives

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References
Example: optimal control

Nonlinear system $\dot{x} = F(x)$ given by:

\[
\begin{align*}
\dot{x}_1 &= \mu x_1 \\
\dot{x}_2 &= \lambda (x_2 - x_1^2), \text{ } \lambda \text{ and } \mu \text{ parameters}
\end{align*}
\]

- Koopman operator $K = \mu x_1 \frac{\partial}{\partial x_1} + \lambda (x_2 - x_1^2) \frac{\partial}{\partial x_2}$

- Observable $g = x_1^2$

\[
\begin{align*}
K(x_1) &= \mu x_1 \\
K(x_2) &= \lambda (x_2 - g) \\
K(g) &= 2x_1 K(x_1) = 2\mu x_1^2 = 2\mu g
\end{align*}
\]

- $\langle x_1, x_2, g \rangle$ is an invariant subspace for $K$
Example: optimal control

Global linearization of the system $\dot{x} = F(x)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{g} \end{pmatrix} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & 2\mu \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ g \end{pmatrix}$$

Idea: control the nonlinear system using this global linearization
Example: optimal control

Global linearization of the system $\dot{x} = F(x)$

$$
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{g}
\end{pmatrix} = 
\begin{pmatrix}
\mu & 0 & 0 \\
0 & \lambda & -\lambda \\
0 & 0 & 2\mu
\end{pmatrix} \cdot 
\begin{pmatrix}
x_1 \\
x_2 \\
g
\end{pmatrix}
$$

Idea: control the nonlinear system using this global linearization

$$
\begin{aligned}
\dot{x}_1 &= \mu x_1 \\
\dot{x}_2 &= \lambda (x_2 - x_1^2) + u
\end{aligned}
$$

Koopman linear control system:

$$
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{g}
\end{pmatrix} = 
\begin{pmatrix}
\mu & 0 & 0 \\
0 & \lambda & -\lambda \\
0 & 0 & 2\mu
\end{pmatrix} \cdot 
\begin{pmatrix}
x_1 \\
x_2 \\
g
\end{pmatrix} + 
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} \cdot u
$$
Example: optimal control

Quadratic cost function:

\[ J[u] := \int_{0}^{\infty} \left( \langle x(t), x(t) \rangle + \langle u(t), u(t) \rangle \right) dt \]

Optimal control:

\[ u(t) = -C \cdot \begin{pmatrix} x_1 \\ x_2 \\ g \end{pmatrix} \]

Optimal nonlinear controller:

\[ u(t) = -2.41x_2(t) + 1.49g(t) = -2.41x_2(t) + 1.49x_1(t)^2 \]
Example: optimal control

(Image extracted from [1])
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   • General idea
   • Koopman for discrete dynamical systems
   • Koopman for continuous dynamical systems

2 Example: optimal control

3 Perspectives
   • Nonlinear optimal control
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4 References
Nonlinear optimal control

General idea:

Nonlinear control system $\implies$ Linear optimization on the Koopman linear control system $\implies$ Nonlinear optimal controller
General idea:

Nonlinear control system $\Rightarrow$ Linear optimization on the Koopman linear control system $\Rightarrow$ Nonlinear optimal controller

**Problem:** in general we can’t find a closed finite-dimensional representation of a nonlinear system
Nonlinear optimal control

Example:

\[
\dot{x} = x^2
\]

Koopman operator \( K = x^2 \frac{\partial}{\partial x} \)

\[
K(x) = x^2
\]

\[
K(x^2) = 2x \cdot K(x) = 2x^3
\]

\[
K(x^3) = 3x^2 \cdot K(x) = 3x^4
\]

And so on...

Restriction of \( K \) to \( < x, x^2, x^3, \ldots > \)

\[
K = \begin{pmatrix}
0 & 0 & 0 & \cdots \\
1 & 0 & 0 & \cdots \\
0 & 2 & 0 & \cdots \\
0 & 0 & 3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
Invariant subspaces of the Koopman operator

Koopman operator associated to $\dot{x} = F(x)$ and $F = (f_1, \ldots, f_n)$

Finite-dimensional invariant subspaces of the Koopman operator = Finite-dimensional representations of the Koopman operator

Eigenfunction equation for smooth $\phi$:

$K(\phi) = \lambda \cdot \phi$ \iff \n\sum_{i=1}^{n} f_i \partial \phi / \partial x_i = \lambda \cdot \phi$
In the Koopman operator associated to \( \dot{x} = F(x) \) and \( F = (f_1, \ldots, f_n) \)

Finite-dimensional invariant subspaces of the Koopman operator = Finite-dimensional representations of the Koopman operator

- A natural candidate: eigenfunctions of \( K \)
- Eigenfunction equation for smooth \( \varphi \): 

\[
K(\varphi) = \lambda \cdot \varphi \iff \sum_{i=1}^{n} f_i \cdot \frac{\partial \varphi}{\partial x_i} = \lambda \cdot \varphi
\]
Invariant subspaces of the Koopman operator

The Dynamic Mode Decomposition (DMD) algorithm:

- Collect samples of the system’s state $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_N)$
- $A =$ best-fit linear operator such that

$$
\mathbf{x}(t_{i+1}) = A\mathbf{x}(t_i) \text{ for every } i = 1, \ldots, N
$$

- The DMD algorithm computes the eigenvalues and eigenvectors of $A$ without explicitly computing $A$
- $A$ can be seen as a finite-dimensional linear approximation of the Koopman operator, so its eigendecomposition will provide approximations of Koopman-invariant subspaces
- The DMD algorithm can be extended (eDMD) by considering a dictionary of nonlinear observations of the state samples $g(\mathbf{x}(t_i))$ and asking that

$$
g(\mathbf{x}(t_{i+1})) = Ag(\mathbf{x}(t_i)) \text{ for every } i = 1, \ldots, N
$$

- For a dictionary of $L^2$-linearly independent orthogonal observables, the matrix $A$ in the eDMD algorithm always converges to $K$
Invariant subspaces of the Koopman operator

- Novel algorithms to compute eigenfunctions? (Statistical learning)
- Theoretical properties of Koopman eigenfunctions? ($\lambda = 0$ are conservation laws)
- Extensions of the DMD algorithm?
- Optimal choice of dictionaries in eDMD?
- Reduced Order Modelling via DMD?
- DMD for systems with continuous spectrum?
Stability

$K$ Koopman operator associated to $\dot{x} = F(x)$

Eigenfunctions of $K$ provide eigenfunctions of $K_t$:

$$K(\varphi) = \lambda \varphi \iff K_t \varphi = e^{\lambda t} \varphi \text{ for every } t \geq 0$$

Proposition

If $\varphi \in C^0(\Omega)$ satisfies

$$K(\varphi) = \lambda \cdot \varphi \text{ with } \Re(\lambda) < 0$$

then $\{x \in \Omega : \varphi(x) = 0\}$ is a global attractor of $\dot{x} = F(x)$

Jorge Mallo (Deusto)
Introduction to Koopman theory
Novel results relating stability of dynamical systems and the Koopman framework?

Numerical methods taking advantage of these results?

Relation with Lyapunov functions?
Thank you for your attention!
Contents

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   • Koopman for discrete dynamical systems
   • Koopman for continuous dynamical systems

2 Example: optimal control

3 Perspectives
   • Nonlinear optimal control
   • Invariant subspaces of the Koopman operator
   • Stability

4 References
Brunton, S. L., Brunton, B. W., Proctor, J. L., and Kutz, J. N.
Koopman invariant subspaces and finite linear representations of nonlinear dynamical systems for control.
*PloS one* 11, 2 (2016), e0150171.
In this article was presented the construction of a non-linear optimal controller using the Koopman framework given in this talk.

Eisner, T., Farkas, B., Haase, M., and Nagel, R.
*Operator theoretic aspects of ergodic theory*, vol. 272.
This book contains a rigorous exposition of the Koopman framework and its relation with Ergodic Theory.
Koopman, B. O.
Hamiltonian systems and transformation in hilbert space.
*Proceedings of the National Academy of Sciences of the United States of America* 17, 5 (1931), 315.
This is the original article in which Koopman introduced the operator that now bears his name.

Korda, M., and Mezić, I.
On convergence of extended dynamic mode decomposition to the koopman operator.
In this article the convergence of the eDMD algorithm with a dictionary of linearly independent orthogonal square-integrable functions is proved.
Mauroy, A., and Mezić, I.
Global stability analysis using the eigenfunctions of the koopman operator.
In this article some general results relating stability of dynamical systems and the Koopman framework are proved.

Rowley, C. W., Mezić, I., Bagheri, S., Schlatter, P., and Henningson, D. S.
Spectral analysis of nonlinear flows.
This article first related Dynamic Mode Decomposition with the Koopman framework, in a fluid mechanics context.
Schmid, P. J.  
Dynamic mode decomposition of numerical and experimental data.  
In this article the Dynamic Mode Decomposition algorithm was first introduced.

Susuki, Y., and Mezić, I.  
Nonlinear koopman modes and power system stability assessment without models.  
This article applies the Koopman framework to study power system stability.
Williams, M. O., Kevrekidis, I. G., and Rowley, C. W.
This article introduced the extended Dynamic Mode Decomposition algorithm.