An introduction to Koopman operator theory and its applications

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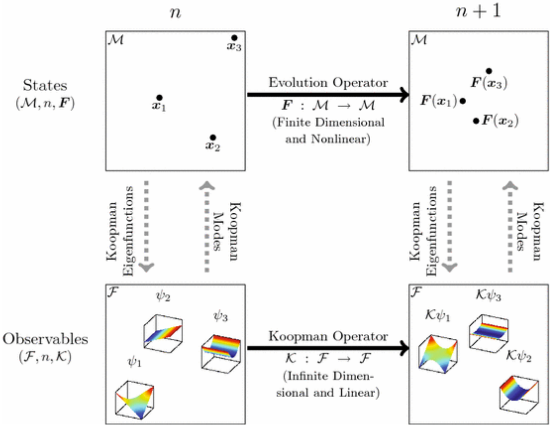
General idea

- Introduced by Koopman in 1931 to study Hamiltonian systems
- Relevant in Ergodic Theory (Mean Ergodic Theorem)
- Re-introduced in the Dynamical Systems community in 2004 by Mézic



(Image extracted from [9])

Koopman for discrete dynamical systems

Discrete Dynamical System (DDS):

- $(X, \mathcal{F}, \mu)$ measure space
- $T : X \to X$ measure-preserving map

Definition

For each $1 \leq p \leq \infty$, the Koopman operator associated to the DDS (X, T) is the composition operator

$$\begin{aligned} K_p : L^p(X) &\to L^p(X) \\ g &\mapsto K(g) := g \circ T \end{aligned}$$

If $x_n = T^n(x_0)$ and $g \in L^p(X)$ we have:

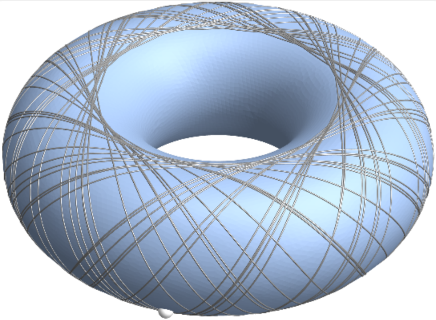
$$K(g)(x_n) = g(x_{n+1}) \text{ for each } n$$

Koopman for discrete dynamical systems

Many properties of (X, T) are codified in the Koopman operator

Example (Ergodicity via Koopman)

The DDS (X, T) is ergodic if and only 1 is a simple eigenvalue of K_p for some $1 \leq p \leq \infty$



$$t \mapsto (e^{2\pi i \cdot at}, e^{2\pi i \cdot bt}) \text{ dense} \Leftrightarrow \frac{b}{a} \notin \mathbb{Q}$$

Image extracted from https://mathematica.stackexchange.com/questions/103624/geodesics-on-a-torus

Koopman for continuous dynamical systems

Continuous Dynamical System (CDS):

- $\Omega \subseteq \mathbb{R}^n$ compact domain
- $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$ autonomous system on Ω
- $\{\tau_t\}_{t \geq 0}$ one-parameter flow:

$$\tau_t(\boldsymbol{x}_0) = \boldsymbol{x}(t) \text{ where } \boldsymbol{x} \text{ is such that } \boldsymbol{x}(0) = \boldsymbol{x}_0$$

Definition

For each $t \geq 0$, define

$$\begin{aligned} K_t : \mathcal{C}^0(\Omega) &\to \mathcal{C}^0(\Omega) \\ f &\mapsto K_t(f) := f \circ \tau_t \end{aligned}$$

$\{K_t\}_{t \geq 0}$ is called the Koopman semi-group

- $K_{t+s} = K_t \circ K_s = K_s \circ K_t$ for every $t, s \geq 0$
- $K_0 = \mathrm{Id}_{\mathcal{C}^0(\Omega)}$

Koopman for continuous dynamical systems

Lemma

$\{K_t\}_{t\geq 0}$ is a strongly continuous semigroup of linear contracting operators in $(\mathcal{C}^0(\Omega), \|\cdot\|_\infty)$

Definition

The Koopman operator associated to $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$ is the infinitesimal generator of the semigroup $\{K_t\}_{t\geq 0}$, i.e., is the linear operator given by

$$K : \mathcal{D}(K) \to \mathcal{C}^0(\Omega)$$

$$f \mapsto K(f) := \lim_{t\to 0^+} \frac{K_t(f) - f}{t}$$

Koopman for continuous dynamical systems

Observation

- *$\mathcal{D}(K)$ is dense in $\mathcal{C}^0(\Omega)$*
- *For each $k \geq 1$ we have $\mathcal{C}^k(\Omega) \subseteq \mathcal{D}(K)$, and for every $g \in \mathcal{C}^k(\Omega)$:*

$$K(g) = f_1 \frac{\partial g}{\partial x_1} + \ldots + f_n \frac{\partial g}{\partial x_n}$$

where $F = (f_1, \ldots, f_n)$

Koopman for continuous dynamical systems

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Observation

- *For example, the energy of a Hamiltonian system is an eigenfunction of the Koopman operator ($K(H) = 0$)*
- *In general, the eigenfunctions of the Koopman operator contain important information about the underlying dynamical system*

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Example: optimal control

Nonlinear system $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$ given by:

$$\begin{cases} \dot{x}_1 & = \mu x_1 \\ \dot{x}_2 & = \lambda(x_2 - x_1^2) \end{cases}, \lambda \text{ and } \mu \text{ parameters}$$

- Koopman operator $K = \mu x_1 \frac{\partial}{\partial x_1} + \lambda(x_2 - x_1^2)\frac{\partial}{\partial x_2}$
- Observable $g = x_1^2$

$$\begin{aligned} K(x_1) &= \mu x_1 \\ K(x_2) &= \lambda(x_2 - g) \\ K(g) &= 2x_1 K(x_1) = 2\mu x_1^2 = 2\mu g \end{aligned}$$

- $< x_1, x_2, g >$ is an invariant subspace for K

Example: optimal control

Global linearization of the system $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{g} \end{pmatrix} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & 2\mu \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ g \end{pmatrix}$$

Idea: control the nonlinear system using this global linearization

Global linearization of the system $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$

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Idea: control the nonlinear system using this global linearization

$$\begin{cases} \dot{x}_1 & = \mu x_1 \\ \dot{x}_2 & = \lambda(x_2 - x_1^2) + u \end{cases}$$

Koopman linear control system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{g} \end{pmatrix} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & 2\mu \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ g \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot u$$

Example: optimal control

Quadratic cost function:

$$J[u] := \int_0^\infty (< \boldsymbol{x}(t), \boldsymbol{x}(t) > + < u(t), u(t) >)\mathrm{d}t$$

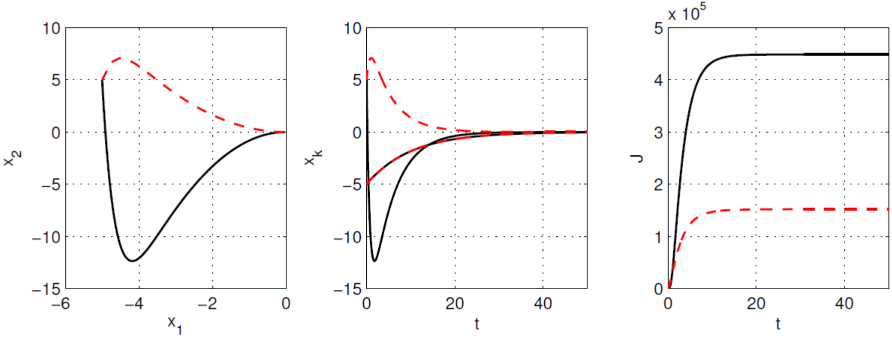
Optimal control:

$$u(t) = -C \cdot \begin{pmatrix} x_1 \\ x_2 \\ g \end{pmatrix}$$

Optimal nonlinear controller:

$$u(t) = -2.41x_2(t) + 1.49g(t) = -2.41x_2(t) + 1.49x_1(t)^2$$

Example: optimal control



(Image extracted from [1])

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Nonlinear optimal control

General idea:

Nonlinear control system $\Longrightarrow$ Linear optimization on the Koopman linear control system $\Longrightarrow$ Nonlinear optimal controller

Nonlinear optimal control

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Nonlinear control system $\Longrightarrow$ Linear optimization on the Koopman linear control system $\Longrightarrow$ Nonlinear optimal controller

Problem: in general we can't find a closed finite-dimensional representation of a nonlinear system

Nonlinear optimal control

Example:

$$\dot{x} = x^2$$

Koopman operator $K = x^2 \frac{\partial}{\partial x}$

$$K(x) = x^2$$
$$K(x^2) = 2x \cdot K(x) = 2x^3$$
$$K(x^3) = 3x^2 \cdot K(x) = 3x^4$$

And so on...

Restriction of K to $< x, x^2, x^3, \dots >$

$$K = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 2 & 0 & \dots \\ 0 & 0 & 3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Invariant subspaces of the Koopman operator

K Koopman operator associated to $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$ and $F = (f_1, \ldots, f_n)$

Finite-dimensional invariant subspaces of the Koopman operator = Finite-dimensional representations of the Koopman operator

Invariant subspaces of the Koopman operator

K Koopman operator associated to $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$ and $F = (f_1, \ldots, f_n)$

Finite-dimensional invariant subspaces of the Koopman operator	=	Finite-dimensional representations of the Koopman operator

- A natural candidate: eigenfunctions of K
- Eigenfunction equation for smooth φ:

$$K(\varphi) = \lambda \cdot \varphi \Leftrightarrow \sum_{i=1}^{n} f_i \cdot \frac{\partial \varphi}{\partial x_i} = \lambda \cdot \varphi$$

Invariant subspaces of the Koopman operator

The Dynamic Mode Decomposition (DMD) algorithm:

- Collect samples of the system's state $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_N)$
- $A =$ best-fit linear operator such that

$$\mathbf{x}(t_{i+1}) = A\mathbf{x}(t_i) \text{ for every } i = 1, \ldots, N$$

- The *DMD* algorithm computes the eigenvalues and eigenvectors of A without explicitly computing A
- A can be seen as a finite-dimensional linear approximation of the Koopman operator, so its eigendecomposition will provide approximations of Koopman-invariant subspaces
- The DMD algorithm can be extended (eDMD) by considering a dictionary of nonlinear observations of the state samples $g(\mathbf{x}(t_i))$ and asking that $g(\mathbf{x}(t_{i+1})) = Ag(\mathbf{x}(t_i))$ for every $i = 1, \ldots, N$
- For a dictionary of L^2-linearly independent orthogonal observables, the matrix A in the eDMD algorithm always converges to K

Invariant subspaces of the Koopman operator

- Novel algorithms to compute eigenfunctions? (Statistical learning)
- Theoretical properties of Koopman eigenfunctions? ($\lambda = 0$ are conservation laws)
- Extensions of the DMD algorithm?
- Optimal choice of dictionaries in eDMD?
- Reduced Order Modelling via DMD?
- DMD for systems with continuous spectrum?

Stability

K Koopman operator associated to $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$
Eigenfunctions of K provide eigenfunctions of K_t:

$$K(\varphi) = \lambda\varphi \Leftrightarrow K_t\varphi = e^{\lambda t}\varphi \text{ for every } t \geq 0$$

Proposition

If $\varphi \in \mathcal{C}^0(\Omega)$ satisfies

$$K(\varphi) = \lambda \cdot \varphi \textit{ with } \mathrm{Re}(\lambda) < 0$$

then $\{x \in \Omega : \varphi(x) = 0\}$ is a global attractor of $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$

Stability

- Novel results relating stability of dynamical systems and the Koopman framework?
- Numerical methods taking advantage of these results?
- Relation with Lyapunov functions?

Thank you for your attention!

Contents

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