The turnpike propety in nonlinear optimal control — A geometric approach

Noboru Sakamoto[1,*], Dario Pighin[2,3,§] and Enrique Zuazua[2,3,4,†]

Abstract— **This paper presents, using dynamical system theory, a framework for investigating the turnpike property in nonlinear optimal control. First, it is shown that a turnpike-like property appears in general dynamical systems with hyperbolic equilibrium and then, apply it to optimal control problems to obtain sufficient conditions for the turnpike occurs. The approach taken is geometric and gives insights for the behaviors of controlled trajectories, allowing us to find simpler proofs for existing results on the turnpike properties.**

I. Introduction

The turnpike property was first recognized in the context of optimal growth by economists (see, e.g., [1]). The turnpike theorems say that for a long-run growth, regardless of starting and ending points, it will pay to get into a growth phase, called *von Neumann path*, in the most of intermediate stages. It is exactly like a turnpike and a network of minor roads; "if origin and destination are far enough apart, it will always pay to get on the turnpike and cover distance at the best rate of travel ..." (quoted from [2]).

In control theory, independently of the turnpike theorems in econometrics, this property was investigated as *dichotomy* in linear optimal control [3], [4] and later extended to nonlinear systems [5]. In optimal control, the turnpike property essentially means that the solution of an optimal control problem is determined by the system and cost function and independent of time intervals, initial and terminal conditions except in the thin layers at the both ends of the time interval (see, e.g., [6], [7], [8]). In the last decades, much progress has been made in the theory of turnpike for finite or infinite dimensional and linear or nonlinear control systems. In [9], the authors study the turnpike for linear finite and infinite dimensional systems and derive a simple but meaningful inequality, for which we term *turnpike inequality*. Their works are extended to finite-dimensional nonlinear systems [10], the semi-linear heat equation [11], the wave equation [12], [13] and periodic turnpike for systems in Hilbert spaces[14]. The authors in [15], [16], [17], [18], [19], [20] investigate the turnpike property from the viewpoints of model predictive control and dissipative systems (see, e.g., [21], [22]).

In this paper, we first show that turnpike-like behaviors naturally appear in general dynamical systems with hyperbolic equilibrium. The main technique we use is the λ-lemma which describes trajectory behaviors near invariant manifolds such as stable and unstable manifolds. That the turnpike-like inequality holds implies that if one fixes two ends of a trajectory close to stable and unstable manifolds and designates the time duration sufficiently large, then the trajectory necessarily converges to these manifolds to spend the most of time near the equilibrium. This is exactly the turnpike property. It should be noted that the two ends, as long as they are close to the manifold, do not need to lie in the vicinity of the equilibrium.

We, then, apply this inequality to an optimal control problems in which terminal states are not specified and the steady optimal solutions are not the origin as in [9], [11], [13], [14] and to an optimal control problem in which two terminal states are specified and the steady optimal point is the origin as in [3], [5], [10]. For both classes of problems, we employ a Dynamic Programming approach with Hamilton-Jacobi equations (HJEs). The characteristic equations for HJEs are Hamiltonian systems and the stabilizability (controllability for the second class) and detectability conditions assure that the equilibriums of the Hamiltonian systems are hyperbolic. The *controlled trajectories* appear in these Hamiltonian systems and we apply the turnpike result for dynamical system. Then, the existence of the

1: Faculty of Science and Engineering, Nanzan University, Yamazato-cho 18, Showa-ku, Nagoya, 464-8673, Japan

2: Chair of Computational Mathematics, Fundación Deusto, Avda Universidades, 24, 48007, Bilbao, Basque Country, Spain

3: Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain

4: Chair in Applied Analysis, Alexander von Humboldt-Professorship, Department of Mathematics, Friedrich-Alexander-Universität Erlangen-Nürnberg 91058 Erlangen, Germany

∗: Supported, in part, by JSPS KAKENHI Grant Numbers JP26289128, JP19K04446 and by Nanzan University Pache Research Subsidy I-A-2 for 2019 academic year. E-mail: noboru.sakamoto@nanzan-u.ac.jp

§: This work has been partially funded by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 694126-DyCon), grant MTM2017-82996-C2-1-R of MINECO (Spain), the Marie Curie Training Network ConFlex", the ELKARTEK project KK-2018/00083 ROAD2DC of the Basque Government, and Nonlocal PDEs: Analysis, Control and Beyond", AFOSR Grant FA9550-18-1-0242.

†: Supported, in part, by the Alexander von Humboldt-Professorship program, by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 694126-DyCon), by grant MTM2017-92996 of MINECO (Spain), by ELKARTEK project KK-2018/00083 ROAD2DC of the Basque Government, by ICON of the French ANR and Nonlocal PDEs: Analysis, Control and Beyond, and by AFOSR Grant FA9550-18-1-0242.

trajectory satisfying initial and boundary conditions is guaranteed. In this paper, we derive *sufficient conditions* for optimality by using the Dynamic Programming and HJEs and by imposing a condition that guarantees the existence of the solution to the HJEs (Lagrangian submanifold property, see, e.g., [23]).

The organization of the paper is as follows. In Section II we review some of key tools from dynamical system theory and derive the turnpike inequality. In Section III, we apply it to optimal control problems. Section III-A handles the problem where boundary state is free and Section III-B handles the problem where initial and boundary states are fixed.

II. Turnpike in dynamical systems

Let us consider a nonlinear dynamical system of the form

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is of C^r class ($r \geqslant 1$). We assume that $f(0) = 0$ and *the hyperbolicity of f at* 0, namely, assume that $Df(0) \in \mathbb{R}^{n\times n}$ has k eigenvalues with strictly negative real parts and $n - k$ eigenvalues with strictly positive real parts.

It is known, as *the stable manifold theorem*, that there exist C^r manifolds S and U, called *stable manifold* and *unstable manifold* of (1) at 0, respectively, defined by

$$S := \{x \in \mathbb{R}^n \,|\, \varphi(t, x) \to 0 \text{ as } t \to \infty\},$$
$$U := \{x \in \mathbb{R}^n \,|\, \varphi(t, x) \to 0 \text{ as } t \to -\infty\},$$

where $\varphi(t, x)$ is the solution of (1) starting x at $t = 0$. Let E^s, E^u be stable and unstable subspaces in $\mathbb{R}^n$ of $Df(0)$ with dimension k, $n - k$, respectively. It is known that S, U are invariant under the flow of f and are tangent to E^s, E^u, respectively, at $x = 0$. See, e.g., [24], [25] for more details on the theory of stable manifold.

We will consider limiting behavior of submanifolds under the flow of f and need to introduce topology for maps and manifolds. Let M be a compact manifold of dimension m and the space $C^r(M, \mathbb{R}^l)$ of C^r maps, $0 \leqslant r < \infty$, defined on M. There exists a natural vector space structure on $C^r(M, \mathbb{R}^l)$. Since M is compact, we take a finite cover of M by open sets $V_1, \ldots, V_k$ and take a local chart (x_i, U_i) for M with $x_i(U_i) = B(2)$ such that $x_i(V_i) = B(1)$, $i = 1, \ldots, k$, where $B(1)$ and $B(2)$ are the balls of radius 1 and 2 at the origin of $\mathbb{R}^m$. For a map $g \in C^r(M, \mathbb{R}^l)$, we define a norm by

$$\|g\|_r := \max_i \sup\{|g(u)|, \|Dg^i(u)\|, \ldots,$$
$$\|D^r g^i(u)\| \,|\, u \in B(1)\},$$

where $g^i = g \circ {x_i}^{-1}$, local representation of g, and $\|\cdot\|$ is a norm for linear maps. It is known that $\|\cdot\|_r$ does not depend on the choice of finite cover and we call it C^r norm. For maps in $C^r(M, N)$ where N is a manifold, we embed N in a Euclidean space with sufficiently high dimension. Let L, L' be C^r submanifolds of M and let $\varepsilon > 0$. We say that *L and L' are ε C^r-close* if there exists a C^r diffeomorphism $h : L \to L'$ such that $\|i' \circ h - i\|_r < \varepsilon$, where $i : L \to M$ and $i' : L' \to M$ are inclusion maps. We use the notation $d^1_L(L') := \|i' \circ h - i\|_1$.

The following lemma is known as *the λ-lemma* or *inclination lemma* and plays a crucial role in the theory of dynamical systems (see [25], [26]).

Lemma 2.1 (The λ-lemma): Suppose that $x = 0$ is a hyperbolic equilibrium for (1). Suppose also that S and U are k, $(n - k)$-dimensional stable and unstable manifolds of f at 0, respectively. For any $(n - k)$-dimensional disc B in U, any point $x \in S$, any $(n - k)$-dimensional disc D transversal to S at x and any $\varepsilon > 0$, there exists a $T > 0$ such that if $t > T$, $\varphi(t, D)$ contains an $(n - k)$-dimensional disc $\tilde{D}$ with $d^1_B(\tilde{D}) < \varepsilon$.

Next, we show that *the turnpike behavior* appears in the transition of points near the stable manifold to points near the unstable manifold if the transition duration is designated large. Let $x_0 \in S$ and $x_1 \in U$ be given points. From the stable manifold theorem, it holds that

$$|\varphi(t, x_0)| \leqslant K e^{-\mu t} \text{ for } t \geqslant 0 \tag{2}$$
$$|\varphi(t, x_1)| \leqslant K e^{\mu t} \text{ for } t \leqslant 0, \tag{3}$$

where $K > 0$ is a constant dependent on x_0 and x_1 and $\mu > 0$ is a constant independent of x_0 and x_1.

Proposition 2.2: (i) There exists a $T_0 > 0$ such that for every $T > T_0$ there exists a $\rho > 0$ such that

$$|\varphi(t, y)| \leqslant K e^{-\mu t} \text{ for } t \in [0, T],\ y \in B(x_0, \rho),$$

where $B(x_0, \rho)$ is the n-dimensional ball centered at x_0 with radius ρ. Moreover, $\rho \to 0$ when $T \to \infty$.

(ii) There exist a $T_0 < 0$ such that for every $T < T_0$ there exists a $\rho > 0$ such that

$$|\varphi(t, y)| \leqslant K e^{\mu t} \text{ for } t \in [T, 0],\ y \in B(x_1, \rho),$$

Moreover, $\rho \to 0$ when $T \to -\infty$.

(iii) For any $(n - k)$-dimensional disc $\bar{D}$ transversal to S at x_0 and any k-dimensional disc $\bar{E}$ transversal to U at x_1, there exists a $T_0 > 0$ such that for any $T > T_0$ there exist an $(n-k)$-dimensional disc $D \subset \bar{D}$ transversal to S at x_0 and a k-dimensional disc $E \subset \bar{E}$ transversal to U at x_1 such that $\varphi(T, D)$ intersects $\varphi(-T, E)$ at a single point.

Proof: (iii) First we take an $(n - k)$-dimensional disc U_0 in U passing through 0, a k-dimensional disc S_0 in S passing through 0 and an $\varepsilon > 0$ arbitrarily. From the λ-lemma, there exists a $T_0 > 0$ such that for any $T > T_0$ there exists an $(n - k)$-dimensional disc $D \subset \bar{D}$ transversal to S at x_0 and a k-dimensional disc $E \subset \bar{E}$ transversal to U at x_1 such that $d^1_{U_0}(\varphi(T, D)) < \varepsilon$, $d^1_{S_0}(\varphi(-T, E)) < \varepsilon$. Since $E^s \cap E^u = \{0\}$, it is possible to take ε, S_0 and U_0 so that $\varphi(T, D)$ intersects $\varphi(-T, E)$ at a single point. ∎

Remark 2.1: It should be noted that the above statements, especially (i) and (ii), are only on finite interval $[0, T]$. This is the major difference from the trajectories on the stable and unstable manifolds.

Theorem 2.3: For any $x_0 \in S$, any $x_1 \in U$, any $(n-k)$-dimensional disc $\bar{D}$ transversal to S at x_0 and any k-dimensional disc $\bar{E}$ transversal to U at x_1, there exists a $T_0 > 0$ such that for every $T > T_0$ there exist a $\rho > 0$, $y_0 \in B(x_0, \rho) \cap \bar{D}$ and $y_1 \in B(x_1, \rho) \cap \bar{E}$ such that $\varphi(T, y_0) = y_1$ and

$$|\varphi(t, y_0)| \leqslant K \left[e^{-\mu t} + e^{-\mu(T-t)} \right] \text{ for } t \in [0, T].$$

Moreover, $\rho \to 0$ when $T \to \infty$.

Proof: Take the largest T_0 and the smallest ρ in Proposition 2.2. We rename this T_0 as $T_0/2$. Take arbitrary $T > T_0$ and use Proposition 2.2-(iii) to get a disc D which is $(n-k)$-dimensional and transversal to S at x_0 and a disc E which is k-dimensional and transversal to U at x_1 satisfying $D \subset B(x_0, \rho)$ and $E \subset B(x_1, \rho)$. This is possible by taking smaller S_0 and U_0 in the proof of Proposition 2.2-(iii). Then, there exists a single point z such that $\varphi(T/2, D) \cap \varphi(-T/2, E) = \{z\}$ (see Fig 1). Let $y_0 := \varphi(-T/2, z)$. Then, $y_0 \in D \subset B(x_0, \rho)$ and by Proposition 2.2-(i), we have

$$|\varphi(t, y_0)| \leqslant K e^{-\mu t} \text{ for } 0 \leqslant t \leqslant T/2. \quad (4)$$

Let $y_1 := \varphi(T/2, z)$. Then, $y_1 \in E \subset B(x_1, \rho)$ and

$$|\varphi(t, y_1)| \leqslant K e^{\mu t} \text{ for } -T/2 \leqslant t \leqslant 0.$$

This shows that

$$|\varphi(t+T, y_0)| \leqslant K e^{\mu t} \text{ for } -T/2 \leqslant t \leqslant 0,$$

and consequently,

$$|\varphi(t, y_0)| \leqslant K e^{-\mu(T-t)} \text{ for } T/2 \leqslant t \leqslant T. \quad (5)$$

Combining (4) and (5), we get the inequality in the theorem. The last assertion follows from Proposition 2.2-(i) and (ii). ■

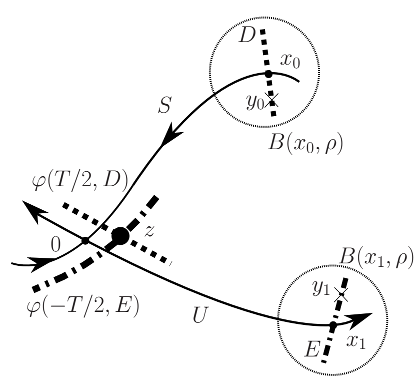

Fig. 1. A scheme of the proof of Theorem 2.3

III. Turnpike in nonlinear optimal control

Let us consider a nonlinear control system

$$\dot{x} = f(x) + g(x)u, \; x(t_0) = x_0, \quad (6)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are of C^2 class, $x(t) \in \mathbb{R}^n$ is state variables and $u(t) \in \mathbb{R}^m$ is control input. The optimal control problem or OPC is to find a control input for (6) such that the cost functional

$$J = \int_0^T L(x(t), u(t))\, dt$$

is minimized. There are several types in OCPs depending on whether or not the terminal time T is specified and whether or not the state variables are specified at the terminal time. In this paper, we consider OCPs where the terminal time T is specified and two types of OCPs; one in which the state variables are free at $t = T$ and another in which they are fixed at $t = T$. For both types of OCPs, we are interested in the relationship between the solution u^* and corresponding trajectory x^* of an OCP and steady-state optimum pair $(\bar{u}, \bar{x})$, which will be defined more precisely later on.

Definition 3.1: [6] An OPC problem has the turnpike property if for any $\varepsilon > 0$, there exists an $\eta_\varepsilon > 0$ such that

$$|\{t \geqslant 0 \,|\, |u^*(t) - \bar{u}| + |x^*(t, x_0) - \bar{x}| > \varepsilon\}| < \eta_\varepsilon$$

for all $T > 0$, where η_ε depends only on ε, f, g, x_0, and L and $|\cdot|$ denotes length (Lebesgue measure) of interval.

Remark 3.1: Turnpike inequality is to require x^* and u^* to satisfy

$$|u^*(t) - \bar{u}| + |x^*(t, x_0) - \bar{x}| \leqslant K \left[e^{-\mu t} + e^{-\mu(T-t)} \right] \quad (7)$$

for some constants $K > 0$ and $\mu > 0$ independent of T, which is a sufficient condition for the turnpike property in Definition 3.1. Also, it should be noted that requiring (7) limits ourselves to *the exponential input-state turnpike* defined in [17].

A. The OCP with state variables unspecified at the terminal time

For system (6), we consider the following cost functional

$$J(u) = \frac{1}{2} \int_0^T |Cx(t) - z|^2 + |u(t)|^2\, dt,$$

where $C \in \mathbb{R}^{r \times n}$ and $z \in \mathbb{R}^r$ is a given vector. We call this problem $(\text{OCP}_1)_T$;

$(\text{OCP}_1)_T$: Find a control $u \in L^\infty(0, T; \mathbb{R}^m)$ such that $J(u)$ along (6) is minimized over all $u \in L^\infty(0, T, \mathbb{R}^m)$.

Associated with $(\mathrm{OCP}_1)_T$, we consider a steady optimization problem

$$\text{(SOP)}: \quad \text{Minimize } J_s(x,u) = \frac{1}{2}(|Cx - z|^2 + |u|^2) \quad \text{over all } (x,u) \in \mathbb{R}^n \times \mathbb{R}^m \text{such that} \quad f(x) + g(x)u = 0.$$

We assume the following.

Assumption 1: (SOP) has a solution $(\bar{x}, \bar{u})$.

Also, associated with $(\mathrm{OCP}_1)_T$, we can derive a Hamilton-Jacobi equation

$$V_t(t,x) + V_x(t,x)f(x) - \frac{1}{2}V_x(t,x)g(x)g(x)^\top V_x(t,x)^\top + \frac{1}{2}|Cx - z|^2 = 0, \tag{8}$$

$$V_x(T,x) = 0, \tag{9}$$

for $V(t,x)$, where $V_t = D_t V$, $V_x = D_x V$. Defining a Hamiltonian

$$H(x,p) = p^\top f(x) - \frac{1}{2}p^\top g(x)g(x)^\top p + \frac{1}{2}|Cx - z|^2,$$

we consider the corresponding characteristic equation for (8)-(9)

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \ \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, \dots, n \tag{10}$$

with $p_i(T) = 0$, $i = 1, \dots, n$. Note that since the system (6) is time-invariant, the equation corresponding to V_t is not necessary. The following fact is readily verified.

Fact. A solution $(\bar{x}, \bar{u})$ of (SOP) corresponds to an equilibrium point $(\bar{x}, \bar{p})$ of (10) with $\bar{u} = -g(\bar{x})^\top \bar{p}$.

Assumption 2: Let $A = D_x D_p H(\bar{x}, \bar{p})$, $B = g(\bar{x})$. The triplet (C, A, B) is stabilizable and detectable.

Under Assumption 2, the equilibrium $(\bar{x}, \bar{p})$ is hyperbolic equilibrium for the Hamiltonian system (10) and there exist stable and unstable manifolds for (10) at $(\bar{x}, \bar{p})$ which are expressed as

$$S = \tilde{S} + \{(\bar{x}, \bar{p})\}, \quad U = \tilde{U} + \{(\bar{x}, \bar{p})\}. \tag{11}$$

Here, $\tilde{S}$, $\tilde{U}$ are the stable and unstable manifold of (10) in the coordinates $(\tilde{x}, \tilde{p})$, where $\tilde{x} = x - \bar{x}$, $\tilde{p} = p - \bar{p}$, which is re-written as

$$\frac{d}{dt}\begin{bmatrix} \tilde{x} \\ \tilde{p} \end{bmatrix} = \begin{bmatrix} A & -BB^\top \\ -C^\top C & -A^\top \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{p} \end{bmatrix} + o(|\tilde{x}| + |\tilde{p}|).$$

We can now state the main theorem of this subsection. Let $\pi_1 : (x,p) \mapsto x$, $\pi_2 : (x,p) \mapsto p$ be canonical projections.

Theorem 3.2: Suppose that $x_0 \in \mathrm{Int}(\pi_1(S))$, where $\mathrm{Int}(\cdot)$ is the interior of a set, and that U intersects $p = 0$ transversally. If T is taken sufficiently large, then there exists a solution $(x(t,x_0), p(t,x_0))$ to (10) satisfying $x(0,x_0) = x_0$ and $p(T,x_0) = 0$. If, moreover,

$$\det D_{x_0} x(t,x_0) \neq 0 \text{ for } t \in [0,T], \tag{12}$$

then

$$u^*(t) = -g(x(t,x_0))^\top p(t,x_0)$$

is the local optimal solution for $(\mathrm{OCP}_1)_T$ and turnpike inequality (7) holds for some constants $k > 0$ and $\mu > 0$ which are independent of T.

Proof: Take a sufficiently large $T > 0$. Then, Theorem 2.3 implies that there exist suitable $p_0 \in \mathbb{R}^n$ and $x_1 \in \mathbb{R}^n$ such that there exist a solution to (10) connecting (x_0, p_0) and $(x_1, 0)$. Also, there exist $K' > 0$ and $\mu > 0$ such that

$$|x(t,x_0) - \bar{x}| + |p(t,x_0) - \bar{p}| \leqslant K'[e^{-\mu t} + e^{-\mu(T-t)}] \quad \text{for } 0 \leqslant t \leqslant T.$$

Since $|u^*(t) - \bar{u}| \leqslant \sup \|g(x)\| |p(t,x_0) - \bar{p}|$, (7) holds with $K = 2K'(1 + \sup \|g(x)\|)$, where supremum is taken along the trajectory. The condition (12) guarantees that there exists a Lagrangian submanifold in a neighborhood of this trajectory and this implies the existence of solution $V(t,x)$ to (8)-(9) in the neighborhood. Then, the verification theorem in Dynamic Programming (see, e.g., [27]) shows that the control u^* is locally optimal. ■

Remark 3.2: The condition (12) guarantees that the solution V to (8) exists in a neighborhood of the trajectory $(x(t,x_0), p(t,x_0))$, $0 \leqslant t \leqslant T$. The optimality of u^* is valid only in the neighborhood. This existence theory is described using the notion of *Lagrangian submanifold* (see, e.g., [23]) and when one seeks for larger domain of existence, the non-uniqueness issue of solution arises. We refer to [28] for general analysis of non-unique solutions and [29], [30], [31] for non-unique optimal controls for mechanical systems.

Next Corollary has been proved in [9], [14] in the study of the turnpike property for infinite dimensional systems. Their proofs are based on the estimates on adjoint variables in the linear Hamiltonian system (10) which is derived as a necessary condition of optimality. Here, we give an alternative proof using the geometric picture in Theorem 3.2.

Corollary 3.3: Suppose that the system (6) is linear, that is, $f(x) = Ax$ and $g(x) = B$ with real constant matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Under Assumption 2, $(\mathrm{OCP}_1)_T$ has the global solution $u^*(t)$, $0 \leqslant t \leqslant T$ for any $z \in \mathbb{R}^r$. Moreover, turnpike inequality (7) holds.

Proof: Let us first denote $\mathrm{Ham} = \left[\begin{smallmatrix} A & -BB^\top \\ -C^\top C & -A^\top \end{smallmatrix}\right]$ and employ the eigen structure analysis in [32] using a symplectic transform in [33], [32].

$$\mathrm{Ham}\begin{bmatrix} I & L \\ P & PL + I \end{bmatrix} = \begin{bmatrix} I & L \\ P & PL + I \end{bmatrix} \begin{bmatrix} A_c & 0 \\ 0 & -A_c{}^\top \end{bmatrix}, \tag{13}$$

where $A_c := A - BB^\top P$ and $P \geqslant 0$, $L \leqslant 0$ are the solutions for the following Riccati and Lyapunov equations:

$$PA + A^\top P - PBB^\top P + C^\top C = 0,$$
$$L{A_c}^\top A_c L = BB^\top.$$

It should be noted that A_c is an asymptotically stable matrix and that Ham has no eigenvalues on the imaginary axis. The unique solution $(\bar{x}, \bar{p})$ to (SOC) is expressed as $\left[\begin{smallmatrix}\bar{x}\\ \bar{p}\end{smallmatrix}\right] = -\mathrm{Ham}^{-1}\left[\begin{smallmatrix}0\\ C^\top z\end{smallmatrix}\right]$. Then, U and S in (11) can be written as

$$S = \{(u, Pu) \,|\, u \in \mathbb{R}^n\} + \{(\bar{x}, \bar{p})\},$$
$$U = \{(Lu, (PL + I)u) \,|\, u \in \mathbb{R}^n\} + \{(\bar{x}, \bar{p})\}.$$

It is readily seen that $x_0 \in \mathrm{Int}(\pi_1(S))$ for any $x_0 \in \mathbb{R}^n$ since $\pi_1(S) = \mathbb{R}^n$. Moreover, under Assumption 2, it can be shown that $PL+I$ is nonsingular and therefore, U intersects $p = 0$ transversally for any $z \in \mathbb{R}^r$. As for the condition (12), we study the property of $\Phi_{11}(t)$ in the transition matrix of Ham:

$$\exp[t\mathrm{Ham}] = \begin{bmatrix}\Phi_{11}(t) & \Phi_{12}(t)\\ \Phi_{21}(t) & \Phi_{22}(t)\end{bmatrix}.$$

It can be readily seen, using (13) that

$$\Phi_{11}(t) = \exp[tA_c] \times \left\{I + \left(L - \exp[-tA_c]\, L \exp\left[-t{A_c}^\top\right]\right) P\right\}.$$

It is possible to prove that $\Phi_{11}(t)$ is nonsingular for $t \geqslant 0$. ■

Remark 3.3: Although the problem in Corollary 3.3 is linear, it is not an easy task to explicitly write down the solution for (8)-(9) except for $z = 0$. This corollary, however, says that the solution globally exists.

B. The OCP with state variables specified at the terminal time

In this subsection, we consider an OCP for (6) with arbitrarily specified terminal states. Let $x_1 \in \mathbb{R}^n$ be given. Let us define cost functional

$$J(u) = \frac{1}{2}\int_0^T x(t)^\top C^\top C x(t) + |u|^2\, dt,$$

and consider

$(\mathrm{OCP}_2)_T$: Find a control $u \in L^\infty(0, T; \mathbb{R}^m)$ such that $J(u)$ along (6) is minimized over all $u \in L^\infty(0, T; \mathbb{R}^m)$ such that $x(T) = x_1$.

We impose an additional assumption on f.

Assumption 3: $f(0) = 0$.

Then, the corresponding steady optimization problem has a unique solution $(x, u) = (0, 0)$. The Hamilton-Jacobi equation associated with $(\mathrm{OCP}_2)_T$ is

$$V_t(t,x) + V_x(t,x) f(x) - \frac{1}{2}V_x(t,x) g(x) g(x)^\top V_x(t,x)^\top + \frac{1}{2}x^\top C^\top C x = 0. \tag{14}$$

The Hamiltonian in this case is

$$H(x,p) = p^\top f(x) - \frac{1}{2}p^\top g(x)g(x)^\top p + \frac{1}{2}x^\top C^\top C x,$$

and the corresponding characteristic equation for (14) is

$$\dot{x}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n \tag{15}$$

with $x(0) = x_0$ and $x(T) = 0$.

Under Assumptions 2 and 3, the Hamiltonian system (15) can be written as

$$\frac{d}{dt}\begin{bmatrix}x\\ p\end{bmatrix} = \mathrm{Ham}\begin{bmatrix}x\\ p\end{bmatrix} + \text{higher oder terms},$$

where $\mathrm{Ham} = \left[\begin{smallmatrix}A & -BB^\top\\ -C^\top C & -A^\top\end{smallmatrix}\right]$ with $A = Df(0)$, $B = g(0)$ and the origin is a hyperbolic equilibrium with n stable and n unstable eigenvalues. Let S and U be the stable and unstable manifolds of (15) at the origin.

Theorem 3.4: Suppose that $x_0 \in \mathrm{Int}(\pi_1(S))$ and $x_1 \in \mathrm{Int}(\pi_1(U))$. If $T > 0$ is taken sufficiently large, there exists a solution $(x(t, x_0), p(t, x_0))$ to (15) satisfying $x(0) = x_0$ and $x(T) = x_1$. If, moreover,

$$\det D_{x_0} x(t, x_0) \neq 0 \text{ for } t \in [0, T), \tag{16}$$

then

$$u^*(t) = -g(x(t, x_0))^\top p(t, x_0)$$

is the local optimal solution for $(\mathrm{OCP}_2)_T$ and turnpike inequality (7) hols for some $K > 0$, $\mu > 0$ independent of T.

Proof: For $T > 0$ sufficiently large, from Theorem 2.3, there exist $p_0, p_1 \in \mathbb{R}^n$ such that a solution to (15) connecting (x_0, p_0) and (x_1, p_1) exists. The rest of the proof is almost the same as Theorem 3.2. ■

Corollary 3.5: Let us impose the controllability of (A, B) in Assumption 2 instead of the stabilizability. Then, for sufficiently small $|x_0|$ and $|x_1|$ and sufficiently large T, the local optimal control exists and turnpike inequality (7) holds.

Proof: We again employ the eigen structure analysis (13). The tangent spaces of S and U at the origin are written as

$$T_0 S = \{(u, Pu) \,|\, u \in \mathbb{R}^n\},$$
$$T_0 U = \{(u, (PL + I)L^{-1}u) \,|\, u \in \mathbb{R}^n\}.$$

The latter is obtained by showing, using the controllability of (A, B), that L is strictly negative definite. Therefore, $x_0 \in \mathrm{Int}(\pi_1(S))$ and $x_1 \in \mathrm{Int}(\pi_1(U))$ for sufficiently small $|x_0|$, $|x_1|$. It is seen that the condition (16) holds for these $|x_0|$, $|x_1|$ (making them smaller if necessary) from the analysis on $\Phi_{11}(t)$ in the proof of Theorem 3.2. ■

Remark 3.4: The linear counterpart of Corollary 3.5 is in [3] where they use anti-stabilizing solution P_u for the Riccati equation. In this case, the turnpike holds for all x_0 and x_1. It can be shown that $P_u = (PL + I)L^{-1}$. Note that we do not need the observability condition.

Corollary 3.5 is obtained in [5] using the Hamilton-Jacobi theory. Compared with their conditions, we use only the *linear* controllability and detectability which can be easily checked. The authors of [10] obtain similar results to Corollary 3.5 with more generalized terminal conditions.

IV. Conclusions

In this paper, using techniques from dynamical system theory such as invariant manifolds and the λ-lemma, we showed that turnpike-like behavior naturally appears in hyperbolic dynamical systems. This is then applied to analyze Hamiltonian systems describing controlled trajectories to obtain sufficient conditions for optimal controls yielding the turnpike to exist. The framework proposed in the paper is geometric and an alternative to existing ones. Since our interests were to discover geometric nature in turnpike, we focused on OCPs without constraints and exponential turnpike. Future works include applications of this approach to specific problems and considering OCPs with constraints.

Acknowledgement. The authors would like to thank Emmanuel Trélat and Lars Grüne for their comments on the manuscript.

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