CONTROLLABILITY OF ONE-DIMENSIONAL VISCOUS FREE BOUNDARY FLOWS

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ABSTRACT. In this work, we address the local controllability of a one-dimensional free boundary problem for a fluid governed by the viscous Burgers equation. The free boundary manifests itself as one moving end of the interval, and its evolution is given by the value of the fluid velocity at this endpoint. We prove that, by means of a control actuating along the fixed boundary, we may steer the fluid to constant velocity in addition to prescribing the free boundary's position, provided the initial velocities and interface positions are close enough.

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1. INTRODUCTION AND MAIN RESULT

Let $T > 0$ be a given positive time. We consider the following problem for the viscous Burgers equation:

$$\begin{cases} v_t - v_{zz} + v v_z = 0 & \text{in } (0,T) \times (0, \ell(t)) \\ v(t,0) = u(t), \quad v_z(t,\ell(t)) = 0 & \text{in } (0,T) \\ \ell'(t) = v(t,\ell(t)) & \text{in } (0,T) \\ v(0,z) = v_0(z), \quad \ell(0) = \ell_0 & \text{in } (0,\ell_0). \end{cases} \tag{1.1}$$

System (1.1) is a free boundary-value problem, where the unknown is the pair (v, ℓ), with ℓ representing the free boundary. Here $\ell_0 > 0$, and $u = u(t)$ is a control actuating along the fixed boundary $z = 0$. Henceforth and in the above, we use the notation $(0,T) \times (0, \ell(t))$ for the set $\{(t,z) \in (0,T) \times \mathbb{R} \colon 0 < z < \ell(t)\}$, with an analog notation for the closure of the latter.

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Model (1.1) is presented and studied by Caboussat & Rappaz in [5, 6], where local-in-time existence and uniqueness of strong solutions is shown, supplemented by numerical studies. It may be seen as a simplification in one space dimension of the incompressible Navier-Stokes equations with a free surface, as encountered in the works of Beale [2, 3], and Maronnier, Picasso & Rappaz [20], where particular emphasis is given on the application to *mould filling*. The state of System (1.1) involves the velocity $v(t,z)$ of the one-dimensional fluid and the free boundary $\ell(t)$, whose analog in dimension ≥ 2 would represent the position of the free surface. The fluid velocity is governed by the viscous Burgers equation, while the dynamics of the free boundary follow the fluid velocity, as per the equation $\ell'(t) = v(t, \ell(t))$.

As the state of the system (1.1) consists of two components (v, ℓ), the natural exact-controllability problem, which is the main goal of this work, is to steer *both components* to a priori defined targets. Formulated as such, this question has not been accurately addressed in the literature for problems of a similar nature as (1.1). It is thus worth mentioning what could constitute a feasible target to which one may control both components of (1.1).

Aside from the trivial solution $(0, \ell_*)$ where $\ell_* > 0$, we may also look to compute the *stationary solutions* of (1.1), namely, time-independent solutions. In other words, given $\ell_* > 0$ and $\bar{v} \in \mathbb{R}$ we seek to compute the solutions to

$$\begin{cases} -v_{zz} + v v_z = 0 & \text{in } (0, \ell_*) \\ v(0) = \bar{v}, \quad v(\ell_*) = 0, \quad v_z(\ell_*) = 0. \end{cases} \tag{1.2}$$

It may be checked that the only solution to the second order differential equation in (1.2) is $v \equiv 0$. Thus, the sole stationary solution $(0, \ell_*)$ to (1.1) corresponds to the *null-controllability* case. The general targets would be *time-dependent trajectories* of (1.1), namely free solutions to (1.1).

The question of controllability to non-trivial trajectories is however not straightforward. This is observed on the level of the system linearized around the target trajectory, which contains several trace terms (see (2.3)). Consequently, in terms of the adjoint problem one obtains non-standard boundary conditions (see (5.3)) for which, up to the best of our knowledge, observability inequalities are lacking. This is discussed in more detail in Section 5.1, and the general problem of controllability to arbitrary trajectories is still open.

At this point, we observe that for any $\ell_* > 0$, the pair $(\bar{v}, \bar{\ell})$ with

$$\bar{v} \in \mathbb{R}, \quad \bar{\ell}(t) = \ell_* + \bar{v}t > 0 \quad \text{in } [0, T], \tag{1.3}$$

is an explicit, non-trivial solution to System (1.1) with $u \equiv \bar{v}$. As discussed in Section 2, the system linearized around this trajectory does not manifest the issues appearing in the general trajectory case. The main goal of this work is to prove the local exact-controllability for (1.1) to this particular trajectory. To be more precise, given an arbitrary constant velocity $\bar{v}$ and an initial position ℓ_*, we want to show that whenever (v_0, ℓ_0) are sufficiently close to $(\bar{v}, \ell_*)$ (see Figure 1), one can find a control $u(t)$ such that the corresponding trajectory (v, ℓ) to (1.1) connects (v_0, ℓ_0) to the target $(\bar{v}, \ell_* + \bar{v}T)$ at time T. This is reflected in our main result.

Theorem 1.1. *Let $T > 0$, $\ell_* > 0$ and $\bar{v} \in \mathbb{R}$ be such that $\bar{\ell}(t) = \ell_* + \bar{v}t > 0$ for all $t \in [0, T]$. There exists $r > 0$ such that for all $\ell_0 > 0$ and $v_0 \in H^1(0, \ell_0)$ satisfying*

$$\|v_0 - \bar{v}\|_{H^1(0,\ell_0)} + |\ell_0 - \ell_*| \leq r,$$

there exists a control $u \in H^{\frac{3}{4}}(0, T)$ such that the unique solution

$$\ell \in C^1([0, T]) \quad v \in L^2\big(0, T; H^2(0, \ell(t))\big) \cap C^0\big([0, T]; H^1(0, \ell(t))\big)$$

of (1.1) *satisfies*

$$\inf_{t\in[0,T]} \ell(t) > 0 \quad \textit{and} \quad \ell(T) = \overline{\ell}(T) \quad \textit{and} \quad v(T,\cdot) = \overline{v} \quad \textit{in } (0,\ell(T)).$$

Moreover, one has

$$\|u\|_{H^{\frac{3}{4}}(0,T)} \lesssim_T \|v_0 - \overline{v}\|_{H^1(0,\ell_0)} + |\ell_0 - \ell_*|.$$

The result we prove here is local (a global result is not known also for similar problems such as (1.4), (1.5)), as while the PDE component may possess an inherent dissipative mechanism, the asymptotic position of the free boundary is generally not known for problems of this nature. In addition, it is readily seen that our result also covers the case of null-controllability of the state and prescribing the position of the interface, by considering $(\overline{v}, \overline{\ell}) = (0, \ell_*)$.

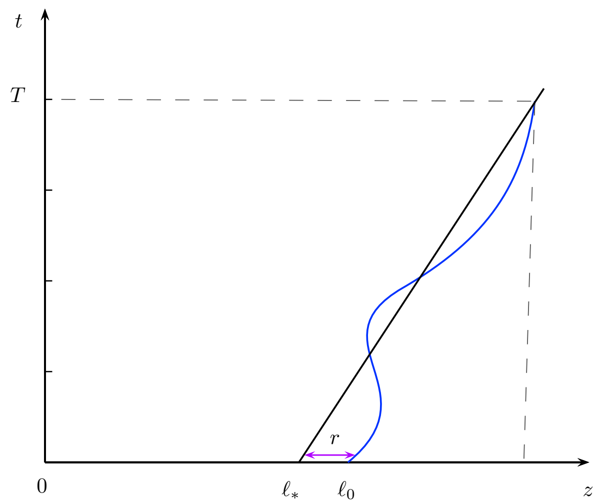

FIGURE 1. Controllability of the position of the free surface ℓ (blue curve) to the reference interface $\overline{\ell}$ (black) at time T, provided the initial positions are close enough.

1.1. **State of the art.** The controllability aspects of one-dimensional, parabolic free-boundary problems similar to (1.1) have been addressed in several recent works. In [9, 13], Fernández-Cara et al. consider the one-phase Stefan problem

$$\begin{cases} v_t - v_{zz} = 0 & \text{in } (0,T)\times(0,\ell(t)) \\ v(t,0) = u(t), \quad v(t,\ell(t)) = 0 & \text{in } (0,T) \\ \ell'(t) = -v_z(t,\ell(t)) & \text{in } (0,T) \\ v(0,z) = v_0(z), \quad \ell(0) = \ell_0 & \text{in } (0,\ell_0). \end{cases} \tag{1.4}$$

We stress that in [9, 13], a null-controllability result where *only the first component* v is controlled is shown, i.e. $v(T,\cdot) = 0$ in $(0,\ell(T))$, for small initial data v_0. The authors' proof relies on fixing the free boundary $\ell \in C^1([0,T])$ (and removing the equation for the velocity ℓ'), and proving an observability inequality for the linear heat equation in the non-cylindrical domain $(0,T)\times(0,\ell(t))$, with a constant uniform in ℓ. The conclusion for (1.4) follows by means of a Schauder fixed-point argument applied to the map $\ell \longmapsto \ell_0 - \int_0^{\cdot} v_x^{\ell}(\tau,\ell(\tau))\,\mathrm{d}\tau$ in an appropriate subspace of $C^1([0,T])$. In [12], the authors obtain the same local controllability result by means of a different technique, which relies on transformation to fixed domain, a linear controllability test and an inverse function argument.

Our proof for the null-controllability of (1.1) follows lines similar to [12], but with several technical differences. In fact, with small adjustments, the control strategy we present in this work also yields a local null-controllability result for both the solution and the free boundary of the Stefan problem (1.4), namely $\ell(T) = \ell_*$ and $v(T,\cdot) = 0$ in $(0,\ell_*)$ whenever v_0 and $\ell_0 - \ell_*$ are small enough.

1.1.1. *Comparison with fluid-structure interaction problems.* Free boundary problems which arise in fluid-structure interaction have also been addressed. Doubova & Fernández-Cara [10] as well as Liu, Takahashi & Tucsnak [17] consider the system

$$\begin{cases} v_t - v_{zz} + vv_z = 0 & \text{in } (0,T)\times(-1,\ell(t))\cup(\ell(t),1) \\ v(t,-1) = u_1(t), \quad v(t,1) = u_2(t) & \text{in } (0,T) \\ v(t,\ell(t)) = \ell'(t) & \text{in } (0,T) \\ m\ell''(t) = [v_z](t,\ell(t)) & \text{in } (0,T) \\ v(0,z) = v_0(z), \quad \ell(0) = \ell_0, \quad \ell'(0) = \ell_1 & \text{in } (-1,\ell_0)\cup(\ell_0,1), \end{cases} \tag{1.5}$$

which is first introduced by Vázquez & Zuazua [23, 24], where global in-time well-posedness, self-similar asymptotics and particle collision are addressed (see also [18] for a related study). The free boundary $\ell(t)$ represents the displacement/position of a solid particle of mass $m > 0$, which splits the domain in two parts. The null-controllability of (1.5) refers to controlling three components: the fluid velocity $v(T,\cdot) = 0$, the particle velocity $\ell'(T) = 0$, and the particle's position $\ell(T) = 0$.

In [10], controls u_1, u_2 are used on both boundaries in view of applying a Carleman based strategy. Such an approach is not feasible when there is a control at only one end (i.e. $u_2 = 0$) because of the lack of connectivity of the fluid domain. This issue was mended in [17], where the authors introduce a systematic methodology for tackling the null-controllability of parabolic systems in spite of source terms, without requiring Carleman inequalities (they thus use spectral techniques). We also refer to the work of Cindea, Micu, Roventa and Tucsnak [7], where the authors consider a control actuating only on the moving particle: $m\ell''(t) = [v_z](t,\ell(t)) + u(t)$. They prove global null-controllability (in large time) for the fluid and particle velocities, and approximate controllability for the particle's position. The lack of connectivity of the fluid domain does not appear in two and three dimensions, and the Carleman-based approach has been successfully applied for proving local null-controllability results for fluid-rigid-body systems (see [4, 15] and the references therein) where the control is generally actuating along a part of the fixed boundary.

Remark 1.1. *At this point we remark that there is a notable difference between problems of the type* (1.5) *and* (1.1). *Indeed, the former system has a stronger coupling than the latter systems due to the presence of two equations for the free boundary ℓ. This can be seen when linearizing both systems around their trivial trajectory (after fixing the domain). In the linearization of* (1.1) *(see* (2.3) *with $a \equiv 1$, $b,c,d,e \equiv 0$ and Section 2 for details),*

$$\begin{cases} y_t - y_{xx} = 0 & \textit{in } (0,T)\times(0,1) \\ y(t,0) = u(t), \quad y_x(t,1) = 0 & \textit{in } (0,T) \\ \ell'(t) = y(t,1) & \textit{in } (0,T) \\ y(0,x) = y_0(x), \quad \ell(0) = \ell_0 & \textit{in } (0,1), \end{cases}$$

the PDE and ODE components are decoupled, as the linear PDE may be solved without any knowledge of the ODE component. On the other hand, the linearization of (1.5) *around the trivial solution (see* [17]*)*

$$\begin{cases} y_t - y_{xx} = 0 & \text{in } (0,T)\times(-1,0)\cup(0,1) \\ y(t,-1) = u(t), \quad y(t,1) = 0 & \text{in } (0,T) \\ y(t,0) = \ell'(t) & \text{in } (0,T) \\ m\ell''(t) = [y_x](t,0) & \text{in } (0,T) \\ y(0,x) = y_0(x), \quad \ell(0) = \ell_0, \quad \ell'(0) = \ell_1 & \text{in } (-1,\ell_0)\cup(\ell_0,1), \end{cases}$$

preserves the coupling of the PDE component and the ODE component because of the presence of two equations for the latter.

In the above-cited works on fluid-structure problems, the controllability problem addressed is that of controlling the PDE component to zero and the ODE component(s) to some given reference points. For the case of non-trivial stationary solutions and trajectories as targets, much less is known. In [1] Badra & Takahashi prove feedback stabilization to non-trivial stationary solutions for (1.5). Therein, it can also be seen that the question of controllability to non-trivial stationary solutions is not straightforward. This is observed on the level of the system linearized around the target, which contains several trace terms (as in (2.3)). As a result, in terms of the adjoint problem one obtains non-local boundary conditions (similar to (5.3)) for which observability inequalities are lacking.

We also refer to Koga, Diagne & Krstic [16] and the references therein for feedback stabilization of the Stefan problem (1.4), see also Phan & Rodrigues [22] for stabilization to trajectories for general parabolic problems.

As discussed in what precedes, up to the best of our knowledge, the question of controllability to non-trivial trajectories (or even non-trivial stationary states) for parabolic free boundary problems such as (1.1), (1.4), (1.5) has not been addressed in the literature. We aim to present some of the difficulties which appear in solving this kind of control problem through this work.

1.2. **Scope.** In Section 2, we reformulate the control problem (1.1) on the time-independent domain $(0,1)$. We give the linearization of (1.1) around an arbitrary smooth trajectory, and provide a brief discussion on the possible strategies for the general controllability to trajectories problem, see also Section 5. In Section 3, we prove the null-controllability of the system linearized around $(\overline{v},\overline{\ell})$. The PDE component is a linear heat equation with a source term, and the ODE component is simply an integrator of the heat solution's Dirichlet trace. The controllability requirement for the second component may thus be seen as a linear constraint on the control. An improved observability inequality along with an adaptation of the standard HUM method provide the desired controllability result for both components of the linearized system. In Section 4, we come back to the nonlinear problem by means of a Banach fixed point argument.

2. Reformulation of the problem

Transformation. To take advantage of a simplified functional setting, it is more advantageous to reformulate (1.1) in a domain which is time-independent. In view of linearizing, perturbations around the target trajectory would be defined in the same domain. To this end, let us define the pull-back velocity function $w : (0,1) \to \mathbb{R}$ by

$$w(t,x) = v(t,z), \quad x = \frac{z}{\ell(t)} \quad \text{for } x \in (0,1). \tag{2.1}$$

A simple application of the chain rule gives the following system of equations for w:

$$\begin{cases} w_t - \frac{1}{\ell^2} w_{xx} - \frac{\ell'}{\ell} x w_x + \frac{1}{\ell} w w_x = 0 & \text{in } (0,T) \times (0,1) \\ w(t,0) = u(t), \quad w_x(t,1) = 0 & \text{in } (0,T) \\ \ell'(t) = w(t,1) & \text{in } (0,T) \\ w(0,x) = w_0(x), \quad \ell(0) = \ell_0 & \text{in } (0,1), \end{cases} \tag{2.2}$$

where $w_0(x) = v_0(\ell_0 x)$. As (1.1) and (2.2) are equivalent provided $\ell(t) > 0$ in $[0,T]$, we will henceforth concentrate our controllability analysis on the latter system.

Linearization. To illustrate some key difficulties related to the controllability to trajectories for free boundary problems such as (1.1), we will linearize the equivalent transformed system (2.2) around an arbitrary smooth time-dependent trajectory $(\overline{w}, \overline{\ell})$ of (2.2), associated to initial and boundary data $(\overline{w}_0, \overline{\ell}_0, \overline{u})$. Such a couple may be obtained, for instance, by considering a free trajectory of (1.1) and applying the transformation (2.1).

To proceed with the linearization, we consider perturbations around the smooth solution $(\overline{w}, \overline{\ell})$ of (2.2), namely we set $w = \overline{w} + \varepsilon y$ and $\ell = \overline{\ell} + \varepsilon h$ in (2.2), differentiate with respect to $\varepsilon > 0$ and finally evaluate the resulting system at $\varepsilon = 0$. This will give the linearized system for the new unknowns (y,h). Equivalently, we may write $w = \overline{w} + y$ and $\ell = \overline{\ell} + h$, and keep all the terms which are linear with respect to (y,h). The linearized system reads

$$\begin{cases} y_t - a y_{xx} + b y_x + c y + d h' + e h = 0 & \text{in } (0,T) \times (0,1) \\ y(t,0) = u(t) - \overline{u}(t), \quad y_x(t,1) = 0 & \text{in } (0,T) \\ h'(t) = y(t,1) & \text{in } (0,T) \\ y(0,x) = y_0(x), \quad h(0) = h_0 & \text{in } (0,1) \end{cases} \tag{2.3}$$

where $\overline{u}(t) = \overline{w}(t,0)$, $y_0(\cdot) = w_0(\cdot) - \overline{w}(0,\cdot)$, $h_0 = \ell_0 - \overline{\ell}(0)$, and the smooth, bounded coefficients are given by

$$\begin{aligned} a(t) = \frac{1}{\overline{\ell}(t)^2}, \quad b(t,x) &= \frac{\overline{w}(t,x) - \overline{\ell}(t)' x}{\overline{\ell}(t)}, \quad c(t,x) = \frac{\overline{w}_x(t,x)}{\overline{\ell}(t)} \\ d(t,x) &= -\frac{x \overline{w}_x(t,x)}{\overline{\ell}(t)} \\ e(t,x) &= \frac{2\overline{w}_t(t,x)}{\overline{\ell}(t)} + \frac{\overline{w}(t,x)\overline{w}_x(t,x) - x\overline{\ell}(t)'\overline{w}_x(t,x)}{\overline{\ell}(t)^2}. \end{aligned} \tag{2.4}$$

To proceed in using a fixed-point argument, it is good to have knowledge of the nonlinear terms. We note that the problem satisfied by a perturbation of the form $w = \overline{w} + y$ and $\ell = \overline{\ell} + h$ is

$$\begin{cases} y_t - a y_{xx} + b y_x + c y + d h' + e h = \mathcal{N}(y,h) & \text{in } (0,T) \times (0,1) \\ y(t,0) = u(t) - \overline{u}(t), \quad y_x(t,1) = 0 & \text{in } (0,T) \\ h'(t) = y(t,1) & \text{in } (0,T) \\ y(0,x) = y_0(x), \quad h(0) = h_0 & \text{in } (0,1) \end{cases} \tag{2.5}$$

the coefficients being the same as in (2.3), and the nonlinear term is of the form

$$\begin{aligned}\mathcal{N}(y,h) = a\big(- h^2 y_t - h^2 \overline{w}_t - 2h\overline{\ell} y_t + h' h x y_x + h' h x \overline{w}_x + h' \overline{\ell} x y_x \\ + \overline{\ell} h x y_x - h y y_x - h y \overline{w}_x - h \overline{w} y_x - \overline{\ell} y y_x \big).\end{aligned}$$

It is important to note that the nonlinearity above only consists of (at least) quadratic terms.

Remark 2.1. *At this point we notice that the linearized problem* (2.3) *contains the terms dh' and eh, which are non-local as they may be expressed in terms of the Dirichlet trace of y at $x = 1$. As these terms act on a single point in space, at the level of the adjoint problem one could expect to obtain a non-local integral boundary condition over all points in space (see* (5.3)*). See Section 5.1 for more detail.*

In the special case of the explicit solution $\overline{w} \equiv \overline{v} \in \mathbb{R}$ and $\overline{\ell}(t) = \ell_* + \overline{v}t$ with $\ell_* > 0$ given in (1.3) we consider in this work, it may be seen that the coefficients d, e factoring the terms h and h' vanish. The coupling between the PDE and ODE components is thus done solely through the nonlinear term. To be more precise, in the case of the specific trajectory $(\overline{v}, \overline{\ell})$, the nonlinear perturbed problem (2.5) reads

$$\begin{cases} y_t - a y_{xx} + b y_x = \mathcal{N}(y,h) & \text{in } (0,T)\times(0,1) \\ y(t,0) = u(t) - \overline{v}, \quad y_x(t,1) = 0 & \text{in } (0,T) \\ h'(t) = y(t,1) & \text{in } (0,T) \\ y(0,x) = y_0(x), \quad h(0) = h_0 & \text{in } (0,1). \end{cases} \tag{2.6}$$

Distributed control problem. Taking the previous transformations into account, Theorem 1.1 would in essence be a consequence of the null-controllability of System (2.6). To prove the latter, using standard methodology for parabolic equations, we will first consider the distributed control problem

$$\begin{cases} y_t - a y_{xx} + b y_x = \mathcal{N}(y,h) + u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ y_x(t,-1) = y_x(t,1) = 0 & \text{in } (0,T) \\ h'(t) = y(t,1) & \text{in } (0,T) \\ y(0,x) = y_0(x), \quad h(0) = h_0 & \text{in } (-1,1) \end{cases} \tag{2.7}$$

where $\omega \subsetneq (-1,0)$ is an open and non-empty interval. The initial datum $y_0 \in H^1(0,1)$ is also extended to a datum $\widetilde{y}_0$ with $\|\widetilde{y}_0\|_{H^1(-1,1)} \leq \|y_0\|_{H^1(0,1)}$. By abuse of notation, we continue denoting the extended initial datum by y_0.

Once the null-controllability problem for (2.7) is solved, $u(t) := y(t,0) + \overline{v}$ would provide the desired control for Problem (2.2), which in view of the previous discussion, also provides a solution to (1.1).

To prove the null-controllability for Problem (2.7), we will first consider the associated linear problem

$$\begin{cases} y_t - a y_{xx} + b y_x = f + u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ y_x(t,-1) = y_x(t,1) = 0 & \text{in } (0,T) \\ h'(t) = y(t,1) & \text{in } (0,T) \\ y(0,x) = y_0(x), \quad h(0) = h_0 & \text{in } (-1,1), \end{cases} \tag{2.8}$$

where f is a given source term. The null-controllability at time T of the linearized system is the goal of the following section. The nonlinear term appearing in (2.7) will be seen as a small perturbation and will be dealt with by means of a fixed-point argument.

3. Null-controllability of the linearized system

In this Section, given $T > 0$, arbitrarily large initial data (y_0, ℓ_0), and a source term f with appropriate decay as $t \nearrow T$, we seek a trajectory (y, h) of the linearized problem (2.8) satisfying

$$y(T, \cdot) = 0 \quad \text{in } (-1,1) \quad \text{and} \quad h(T) = 0.$$

In (2.8) we are dealing with a cascade-like system, as knowing y immediately yields h, with the latter being reduced to the integrator

$$h(t) = h_0 + \int_0^t y(\tau, 1)\,\mathrm{d}\tau.$$

In other words, the null-controllability of (2.8), would follow from solving the linear control problem (recall that $a = a(t) > 0$)

$$\begin{cases} y_t - a y_{xx} + b y_x = f + u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ y_x(t,-1) = y_x(t,1) = 0 & \text{in } (0,T) \\ y(0,x) = y_0(x) & \text{in } (-1,1) \\ y(T,x) = 0 & \text{in } (-1,1) \end{cases} \tag{3.1}$$

subject to the linear constraint

$$h_0 + \int_0^T y(\tau,1)\,\mathrm{d}\tau = 0. \tag{3.2}$$

We will see this as a constrained controllability problem, namely with a linear constraint on the control u.

Carleman weights. Let us recall that $\omega = (\gamma_1, \gamma_2) \subsetneq (-1, 0)$. We take (a_0, b_0) with $\gamma_1 < a_0 < b_0 < \gamma_2$ and introduce a function $\alpha_0 \in C^2([-1,1])$ such that

$$\alpha_0(x) > 0 \quad \text{in } (-1,1), \quad \alpha_0(\pm 1) = 0, \quad |\alpha_{0,x}| > 0 \quad \text{in } (-1,1)\setminus(a_0,b_0),$$

and for $\lambda \geq 1$ consider the function α defined by

$$\alpha(t,x) = \theta(t)\left(e^{2\lambda\|\alpha_0\|_{L^\infty}} - e^{\lambda\alpha_0(x)}\right), \quad \text{in } (0,T)\times(-1,1), \tag{3.3}$$

where $\theta \in C^2([0,T))$ is given by

$$\theta(t) = \begin{cases} \dfrac{4}{T^2} & \text{on } \left[0, \dfrac{T}{2}\right] \\ \dfrac{1}{t(T-t)} & \text{on } \left[\dfrac{T}{2}, T\right). \end{cases}$$

Notice that the weight $\theta(t)$ does not blow up as $t \searrow 0$. This is because in view of the fixed-point argument, we will need to work with source-terms which do not vanish at $t = 0$.

The main goal of this section is prove the following result.

Theorem 3.1. *Let $T > 0$ be given. There exists $s \geq 1$ such that for any data $y_0 \in L^2(-1,1)$, $h_0 \in \mathbb{R}$ and $f \in L^2(0,T;L^2(-1,1))$ with*

$$\int_0^T \int_{-1}^1 \theta^{-3} e^{2s\alpha} |f|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{3.4}$$

there exists a control $u \in L^2(0,T;L^2(\omega))$ such that the associated solution

$$y \in L^2(0,T;H^1(-1,1)) \cap C^0([0,T];L^2(-1,1)) \quad \textit{and} \quad h \in H^1(0,T)$$

of Problem (2.8) *satisfies* $y(T,\cdot)=0$ *and* $h(T)=0$. *Moreover,*

$$\|u\|_{L^2(0,T;L^2(\omega))}+\|e^{s\alpha}y\|_{L^2(0,T;L^2(-1,1))} \leq C\left(\|y_0\|_{L^2(-1,1)}+|h_0|+\left\|\theta^{-\frac{3}{2}}e^{s\alpha}f\right\|_{L^2(0,T;L^2(-1,1))}\right)$$

holds for some $C=C(T,\omega,s)>0$.

It is well-known that a Carleman inequality (see Lemma 3.1) along with the HUM method yield the null-controllability of the linear heat equation (3.1) with a source term f as in (3.4).

To control the second component h to zero at time T, we will reformulate the constraint (3.2) by introducing an augmented adjoint problem for the heat equation with a non-homogeneous boundary condition at $x=1$. The requirement $h(T)=0$ may then be achieved by adding a corrector term to the HUM control for the heat equation. To guarantee the existence of this control by means of the HUM method, we will need to prove an improved observability inequality. This idea appears in the work of Nakoulima [21], and is standard in works on fluid-structure interaction problems (see [4, 10] for instance) where the structure's displacement at time T is deduces after having controlled the fluid and structure velocities.

3.1. **An improved observability inequality.** We will make use of the following Carleman inequality for solutions to the adjoint heat equation

$$\begin{cases}-\zeta_t-a\zeta_{xx}-(b\zeta)_x=g & \text{in } (0,T)\times(-1,1)\\ \zeta_x(t,-1)=\zeta_x(t,1)=0 & \text{in } (0,T)\\ \zeta(T,x)=\zeta_T(x) & \text{in } (-1,1),\end{cases} \tag{3.5}$$

and the weights defined in (3.3). The proof is standard, and follows by combining the well-known inequality shown in Fursikov & Imanuvilov [14, Lemma 1] with the parameters $s\geq s_0\geq 1$ and $\lambda\geq\lambda_0\geq 1$ appearing therein being henceforth fixed, and energy estimates as done in [11, Section 3].

Lemma 3.1. *Let* $T>0$. *There exists* $C=C(T,\omega,s,\lambda)>0$ *such that for every* $\zeta_T\in L^2(-1,1)$ *and* $g\in L^2(0,T;L^2(-1,1))$, *the unique weak solution* ζ *to* (3.5) *satisfies*

$$\int_0^T\int_{-1}^1\theta^3e^{-2s\alpha}|\zeta|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{-1}^1|\zeta(0,x)|^2\,\mathrm{d}x \leq C\left(\int_0^T\int_{-1}^1e^{-2s\alpha}|g|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\omega\theta^3e^{-2s\alpha}|\zeta|^2\,\mathrm{d}x\,\mathrm{d}t\right). \tag{3.6}$$

The Carleman inequality (3.6) guarantees the coercivity and continuity of the strictly convex HUM functional, the unique minimizer of which yields a solution to the adjoint heat equation (3.5) and subsequently a solution to the control problem (3.1) after investigating the corresponding Euler-Lagrange equation.

To take care of the constraint $h(T)=0$, let us consider the augmented adjoint problem

$$\begin{cases}-\psi_t-a\psi_{xx}-(b\psi)_x=0 & \text{in } (0,T)\times(-1,1)\\ \psi_x(t,-1)=0,\quad \psi_x(t,1)=1 & \text{in } (0,T)\\ \psi(T,x)=0 & \text{in } (-1,1).\end{cases} \tag{3.7}$$

Multiplying the heat equation appearing in System (2.8) by the unique weak solution $\psi\in L^2(0,T;H^1(-1,1))\cap C^0([0,T];L^2(-1,1))$ of (3.7) and integrating, we see that due to (3.2), a control u is such that the corresponding solution of (2.8) satisfies $h(T) = 0$ if and only if

$$\int_0^T \int_\omega u\psi \,\mathrm{d}x \,\mathrm{d}t = -\int_{-1}^1 y_0(x)\psi(0,x)\,\mathrm{d}x + h_0 - \int_0^T \int_{-1}^1 f\psi \,\mathrm{d}x\,\mathrm{d}t. \tag{3.8}$$

Let us define the projector

$$\mathbb{P}_\zeta := \frac{\displaystyle\int_{(0,T)\times\omega} \psi\zeta \,\mathrm{d}x\,\mathrm{d}t}{\displaystyle\int_{(0,T)\times\omega} |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t} \qquad \text{for all } \zeta \in L^2(0,T;L^2(-1,1)).$$

An important property of the projector $\mathbb{P}_\cdot$ is that it has finite-dimensional range (in fact, one-dimensional), and it is thus a compact operator. Our next result is the desired improved observability inequality. The proof follows a standard compactness-uniqueness argument (see [4, 10, 15, 17]). We assume the setting of Lemma 3.1.

Proposition 3.1. *There exists a constant $C_{\mathrm{obs}} = C_{\mathrm{obs}}(T,\omega,s,\lambda) > 0$ such that for every $\zeta_T \in L^2(-1,1)$ and $g \in L^2(0,T;L^2(-1,1))$, the unique weak solution ζ to (3.5) satisfies*

$$\begin{aligned}\int_0^T \int_{-1}^1 \theta^3 e^{-2s\alpha}|\zeta|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{-1}^1 |\zeta(0,x)|^2\,\mathrm{d}x + |\mathbb{P}_\zeta|^2 \\ \leq C_{\mathrm{obs}} \left(\int_0^T \int_{-1}^1 e^{-2s\alpha}|g|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\omega |\zeta - \mathbb{P}_\zeta \psi|^2 \,\mathrm{d}x\,\mathrm{d}t \right).\end{aligned} \tag{3.9}$$

Proof. We will begin by showing that

$$\begin{aligned}&\int_0^T \int_{-1}^1 \theta^3 e^{-2s\alpha}|\zeta|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{-1}^1 |\zeta(0,x)|^2\,\mathrm{d}x \\ &\leq C_2 \left(\int_0^T \int_{-1}^1 e^{-2s\alpha}|g|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\omega |\zeta - \mathbb{P}_\zeta \psi|^2 \,\mathrm{d}x\,\mathrm{d}t \right)\end{aligned} \tag{3.10}$$

for some $C_2 = C_2(T,\omega,s,\lambda) > 0$ and any (ζ_T, g) as in the statement, which would cover the two leftmost terms of the desired inequality (3.9). To do so, let us assume by contradiction that (3.10) is false, thus there exist two sequences $\{\zeta_T^k\}_{k=1}^\infty$ and $\{g^k\}_{k=1}^\infty$ such that

$$\begin{aligned}1 &= \int_0^T \int_{-1}^1 \theta^3 e^{-2s\alpha}|\zeta^k|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{-1}^1 |\zeta^k(0,\cdot)|^2\,\mathrm{d}x \\ &\geq k \left(\int_0^T \int_{-1}^1 e^{-2s\alpha}|g^k|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\omega |\zeta^k - \mathbb{P}_{\zeta^k} \psi|^2 \,\mathrm{d}x\,\mathrm{d}t \right),\end{aligned} \tag{3.11}$$

for any $k \in \mathbb{N}$, with ζ^k being the corresponding solution to the adjoint problem (3.5). Elementary inequalities give

$$\begin{aligned}&\frac{1}{2}\int_0^T \int_\omega \theta^3 e^{-2s\alpha}|\mathbb{P}_{\zeta^k}\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \int_0^T \int_\omega \theta^3 e^{-2s\alpha}|\zeta^k|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\omega \theta^3 e^{-2s\alpha}|\zeta^k - \mathbb{P}_{\zeta^k}\psi|^2 \,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

thus the left-most integral is uniformly bounded for any $k \in \mathbb{N}$ in view of (3.11) (recall also the definition of the weights in (3.3)). Hence, $\mathbb{P}_{\zeta^k}$ is uniformly bounded with respect to $k \in \mathbb{N}$, and since it is a compact operator, it follows that

$$\mathbb{P}_{\zeta^k} \longrightarrow \mathbb{P}_* \quad \text{as } k \longrightarrow +\infty \tag{3.12}$$

for some $\mathbb{P}_* \in \mathbb{R}$, possibly along a subsequence. From (3.11), the functions ζ^k and $\zeta^k(0,\cdot)$ are uniformly bounded in $L^2(0,T-\varepsilon,L^2(-1,1))$ and $L^2(-1,1)$ respectively, for all $\varepsilon > 0$, as well as

$$\int_0^{T-\varepsilon}\int_{-1}^1 |g^k|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \frac{1}{k}.$$

Whence, using the well-known energy estimates for the heat equation, one also has that

$$\begin{aligned} \zeta^k(0,\cdot) &\rightharpoonup \zeta(0,\cdot) && \text{weakly in } L^2(-1,1)\\ \zeta^k &\rightharpoonup \zeta && \text{weakly in } L^2(0,T-\varepsilon,H^1(-1,1))\\ \zeta^k_t &\rightharpoonup \zeta_t && \text{weakly in } L^2(0,T-\varepsilon,H^{-1}(-1,1)) \end{aligned}$$

along subsequences as $k \longrightarrow +\infty$. It can thus be seen that ζ satisfies

$$\begin{cases} -\zeta_t - a\zeta_{xx} - (b\zeta)_x = 0 & \text{in } (0,T)\times(-1,1)\\ \zeta_x(t,-1) = 0, \quad \zeta_x(t,1) = 0 & \text{in } (0,T). \end{cases}$$

In $(0,T)\times\omega$, we have $\zeta^k = (\zeta^k - \mathbb{P}_{\zeta^k}\psi) + \mathbb{P}_{\zeta^k}\psi$, so in view of (3.11) and (3.12) we have

$$\zeta^k \longrightarrow \mathbb{P}_*\psi \quad \text{strongly in } L^2(0,T;L^2(\omega)) \tag{3.13}$$

as $k \longrightarrow +\infty$. The above convergence implies that $\zeta = \mathbb{P}_*\psi$ in $(0,T)\times\omega$. As ψ is also in the kernel of the heat operator (thus, so is $\mathbb{P}_*\psi$), by unique continuation we deduce that $\zeta = \mathbb{P}_*\psi$ in $(0,T)\times(-1,1)$. But this can only hold if $\zeta \equiv 0$ and $\mathbb{P}_* = 0$, since $\psi_x(t,1) = 1$.

From (3.13), we may deduce

$$\zeta^k \longrightarrow 0 \quad \text{strongly in } L^2(0,T;L^2(\omega))$$

as $k \longrightarrow +\infty$, and thus using (3.6) (noting that (3.11) is used for g^k) we deduce

$$\int_0^T\int_{-1}^1 \theta^3 e^{-2s\alpha}|\zeta^k|^2\,\mathrm{d}x\,\mathrm{d}t + \int_{-1}^1 |\zeta^k(0,x)|^2\,\mathrm{d}x \longrightarrow 0$$

as $k \to +\infty$, which contradicts (3.11). Consequently, (3.10) holds. Arguing as for (3.10), we can show

$$\left|\int_0^T\int_\omega \theta^3 e^{-2s\alpha}\zeta\psi\,\mathrm{d}x\,\mathrm{d}t\right|^2 \leq C_5\left(\int_0^T\int_{-1}^1 e^{-2s\alpha}|g|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\omega |\zeta - \mathbb{P}_\zeta\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right) \tag{3.14}$$

for some $C_5 = C_5(T,\omega,s) > 0$. Indeed, setting up an assumption for (3.14) as in (3.11) and applying Cauchy-Schwarz, after following the lines of the previous step, it may be seen that this would provide the necessary contradiction. □

Remark 3.1. *While Proposition 3.1 yields the desired improved observability inequality for what follows, due to the indirect argument used for the proof an explicit dependence of the newly obtained constant on the parameters (T,ω) is not guaranteed.*

3.2. **Proof of Theorem 3.1.** We are now in a position to complete the proof of Theorem 3.1, which follows by adapting the relatively standard HUM arguments.

Proof of Theorem 3.1. For a solution ψ of (3.7), let us henceforth denote

$$M_0 := -\int_{-1}^1 y_0(x)\psi(0,\cdot)\,\mathrm{d}x + h_0 - \int_0^T\int_{-1}^1 f\psi\,\mathrm{d}x\,\mathrm{d}t. \tag{3.15}$$

We split the proof in three steps.

Step 1: Minimization problem. Consider the functional

$$\begin{aligned}J_{\text{obs}}(\zeta_T, g) :=&\frac{1}{2}\int_0^T\int_\omega |\zeta - \mathbb{P}_\zeta\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_{-1}^1 e^{-2s\alpha}|g|^2\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_{-1}^1 f\zeta\,\mathrm{d}x\,\mathrm{d}t - \int_{-1}^1 y_0(x)\zeta(0,x)\,\mathrm{d}x - \mathbb{P}_\zeta M_0,\end{aligned}$$

initially defined for $(\zeta_T, g) \in L^2(-1,1)\times L^2(0,T;L^2(-1,1))$ with corresponding solution $\zeta \in L^2(0,T;H^1(-1,1))\cap C^0([0,T];L^2(-1,1))$ to the adjoint heat equation (3.5), and ψ being the solution to the augmented adjoint problem (3.7). We will show the existence of a minimizer to J_{obs}, which will consequently be used to build the desired control – state pair for Problem (2.8).

We remark that the quantity

$$\|(\zeta_T, g)\|_{\text{obs}}^2 = \int_0^T\int_\omega |\zeta - \mathbb{P}_\zeta\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{-1}^1 e^{-2s\alpha}|g|^2\,\mathrm{d}x\,\mathrm{d}t$$

defines a norm on $L^2(-1,1)\times L^2(0,T;L^2(-1,1))$. In order to have completeness, we thus introduce the space

$$X_{\text{obs}} := \overline{L^2(-1,1)\times L^2(0,T;L^2(-1,1))}^{\|\cdot\|_{\text{obs}}}.$$

The set X_{obs} is then endowed with the Hilbert structure given by the above norm.

On X_{obs}, the functional J_{obs} may be extended by continuity in a unique way. Indeed, the improved weighted observability inequality (3.9) implies (recall that f is assumed to satisfy (3.4))

$$\begin{aligned}\left|\int_0^T\int_{-1}^1 f\zeta\,\mathrm{d}x\,\mathrm{d}t\right| &\leq \left(\int_0^T\int_{-1}^1 \theta^{-3}e^{2s\alpha}|f|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T\int_{-1}^1 \theta^3 e^{-2s\alpha}|\zeta|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\\ &\leq C\left\|\theta^{-\frac{3}{2}}e^{s\alpha}f\right\|_{L^2(0,T;L^2(-1,1))}\|(\zeta_T,g)\|_{\text{obs}},\end{aligned}\tag{3.16}$$

as well as

$$\begin{aligned}\left|\int_{-1}^1 y_0(x)\zeta(0,x)\,\mathrm{d}x\right| &\leq \left(\int_{-1}^1 |y_0|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{-1}^1 |\zeta(0,x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\ &\leq C\|y_0\|_{L^2(-1,1)}\|(\zeta_T,g)\|_{\text{obs}}\end{aligned}\tag{3.17}$$

and

$$|\mathbb{P}_\zeta| \leq C\|(\zeta_T,g)\|_{\text{obs}}.\tag{3.18}$$

Due to (3.16) – (3.17) – (3.18), it can be seen that the functional J_{obs} is also coercive. As J_{obs} is also strictly convex on X_{obs} (since $\|\cdot\|_{\text{obs}}$ is a Hilbert norm), it admits a unique minimizer $(\widehat{\zeta},\widehat{g})\in X_{\text{obs}}$ by the direct method.

Step 2: Null-controllability requirements. Now the unique minimizer $(\widehat{\zeta_T},\widehat{g})\in X_{\text{obs}}$ of J_{obs} satisfies the Euler-Lagrange equation

$$\begin{aligned}0 =& \int_0^T\int_\omega (\widehat{\zeta} - \mathbb{P}_{\widehat{\zeta}}\psi)\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{-1}^1 e^{-2s\alpha}\widehat{g}F\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_{-1}^1 f\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_{-1}^1 y_0(x)\varphi(0,x)\,\mathrm{d}x - \mathbb{P}_\varphi M_0\end{aligned}\tag{3.19}$$

for all $(\varphi_T, F) \in X_{\text{obs}}$, where $\widehat{\zeta}$ and φ denote the solutions to (3.5) corresponding to $(\widehat{\zeta_T}, \widehat{g})$ and (φ_T, F) respectively. Comparing (3.19) with (3.21), we are led to consider the control function

$$u := -(\widehat{\zeta} - \mathbb{P}_{\widehat{\zeta}}\psi) + M_0 \left(\int_0^T \int_\omega \psi^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{-1} \psi$$

restricted to ω, where ψ is the unique solution to the augmented adjoint problem (3.7). Let $y \in L^2(0,T;H^1(-1,1)) \cap C^0([0,T];L^2(-1,1))$ be the solution to the heat equation in (2.8) with control u. Let us justify this choice. Noting that

$$\int_0^T \int_\omega u\varphi \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_\omega (\widehat{\zeta} - \mathbb{P}_{\widehat{\zeta}}\psi)\varphi \,\mathrm{d}x\,\mathrm{d}t + \mathbb{P}_\varphi M_0,$$

we come back to (3.19) and deduce that

$$\begin{aligned} 0 = &-\int_0^T \int_{-1}^1 e^{2s\alpha}\widehat{g}F \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\omega u\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \int_{-1}^1 f\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{-1}^1 y_0\varphi(0,\cdot)\,\mathrm{d}x. \end{aligned} \tag{3.20}$$

On the other hand, multiplying the heat component in (2.8) by φ solution of (3.5) with initial data φ_T and source term F, we see that

$$\int_{-1}^1 y(T,\cdot)\varphi_T = -\int_0^T \int_{-1}^1 yF + \int_0^T \int_{-1}^1 f\varphi + \int_{-1}^1 y_0\varphi(0,\cdot) + \int_0^T \int_\omega u\varphi. \tag{3.21}$$

Comparing with (3.20), for all $(\varphi_T, F) \in L^2(-1,1) \times L^2(0,T;L^2(-1,1))$

$$\int_{-1}^1 y(T,\cdot)\varphi_T \,\mathrm{d}x = \int_0^T \int_{-1}^1 (e^{2s\alpha}\widehat{g} - y)F \,\mathrm{d}x\,\mathrm{d}t.$$

Choosing $F = 0$, we get the desired control requirement $y(T,\cdot) = 0$. On the other hand, choosing $\varphi_T \equiv 0$, we see also that

$$y = \widehat{g}e^{-2s\alpha}.$$

We now define $h \in H^1(0,T)$ by

$$h(t) := h_0 + \int_0^t y(\tau,1)\,\mathrm{d}\tau.$$

It remains to be seen that the above-defined control u is such that $h(T) = 0$. Recalling the definition of M_0 in (3.15), a straightforward computation shows that

$$\int_0^T \int_\omega u\psi \,\mathrm{d}x\,\mathrm{d}t = M_0,$$

which in view of (3.8) yields the conclusion $h(T) = 0$, as desired.

Step 3: Estimates. As $J_{\text{obs}}(\widehat{\zeta_T}, \widehat{g}) \le J_{\text{obs}}(0,0) = 0$, straightforward estimates along with (3.16) – (3.18) give

$$\begin{aligned} &\|\widehat{\zeta} - \mathbb{P}_{\widehat{\zeta}}\psi\|_{L^2(0,T;L^2(\omega))} + \|e^{-s\alpha}\widehat{g}\|_{L^2(0,T;L^2(-1,1))} \\ &\quad\le C_1 \left(\|y_0\|_{L^2(-1,1)} + |h_0| + \left\|\theta^{-\frac{3}{2}}e^{s\alpha}f\right\|_{L^2(0,T;L^2(-1,1))} \right) \end{aligned} \tag{3.22}$$

for some $C_1 > 0$. On another hand, it may easily be checked that

$$\int_0^T \int_\omega u^2 \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\omega (\widehat{\zeta} - \mathbb{P}_{\widehat{\zeta}}\psi)^2 \,\mathrm{d}x\,\mathrm{d}t + M_0^2 \left(\int_0^T \int_\omega \psi^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{-1} \tag{3.23}$$

Thus, in view of the definitions of the control u and the state y and (3.22) and (3.23) lead us to conclude that

$$\begin{aligned}\|u\|_{L^2(0,T;L^2(\omega))} &+ \|e^{s\alpha}y\|_{L^2(0,T;L^2(-1,1))} \\ &\leq C_2 \left(\|y_0\|_{L^2(-1,1)} + |h_0| + \left\|\theta^{-\frac{3}{2}}e^{s\alpha}f\right\|_{L^2(0,T;L^2(-1,1))}\right)\end{aligned}$$

for some $C_2 > 0$. This concludes the proof. □

The following Lemma gives additional estimates of the controlled trajectory in the weighted spaces provided more regular initial data.

Lemma 3.2. *Let (v, y, h) denote the control-state pair given by Theorem 3.1. Assume moreover that $y_0 \in H^1(-1,1)$. Then*

$$\begin{aligned}\|\theta^{-1}e^{s\alpha}y_x\|_{L^2(0,T;L^2(-1,1))} &+ \|\theta^{-2}e^{s\alpha}y_t\|_{L^2(0,T;L^2(-1,1))} \\ &+ \|\theta^{-2}e^{s\alpha}y_{xx}\|_{L^2(0,T;L^2(-1,1))} + \|\theta^{-2}e^{s\alpha}y\|_{L^\infty(0,T;H^1(-1,1))} \\ &\leq C \left(\|y_0\|_{H^1(-1,1)} + |h_0| + \left\|\theta^{-\frac{3}{2}}e^{s\alpha}f\right\|_{L^2(0,T;L^2(-1,1))}\right)\end{aligned}$$

holds for some $C = C(T, \omega, s) > 0$.

Proof. The proof for estimating the first three norms follows standard energy estimate arguments, and we refer to [12, Lemma 3.4] for details. To obtain the weighted $L^\infty(H^1)$-estimate, we note that by interpolation

$$\|\theta^{-2}e^{s\alpha}y\|_{L^\infty(0,T;H^1(-1,1))} \lesssim \|\theta^{-2}e^{s\alpha}y\|^{\frac{1}{2}}_{L^2(0,T;H^2(-1,1))}\|\theta^{-2}e^{s\alpha}y\|^{\frac{1}{2}}_{H^1(0,T;L^2(-1,1))},$$

and the right-hand side is bounded by the properties of the Carleman weights and the three previous estimates. □

4. The nonlinear problem

We now look to conclude the proof of Theorem 1.1 by virtue of a fixed-point argument for nonlinear system

$$\begin{cases} y_t - ay_{xx} + by_x = \mathcal{N}(y,h) + u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ y_x(t,-1) = y_x(t,1) = 0 & \text{in } (0,T) \\ h'(t) = y(t,1) & \text{in } (0,T) \\ y(0,x) = y_0(x), \quad h(0) = h_0 & \text{in } (-1,1), \end{cases} \tag{4.1}$$

a restriction argument and reverting the transformations performed in Section 2. We recall that the nonlinear term in (4.1) is of the form

$$\mathcal{N}(y,h) = a\big(-hy_t(h+2\overline{\ell}) + h'y_x(hx+\overline{\ell}x) + hy_x(\overline{\ell}x+\overline{v}) - yy_x(h+\overline{\ell})\big), \tag{4.2}$$

only consisting of (at least) quadratic terms.

Let us consider the norm

$$\begin{aligned}\|y\|_{\mathcal{Y}} := \|e^{s\alpha}y\|_{L^2(0,T;L^2(-1,1))} &+ \|\theta^{-1}e^{s\alpha}y_x\|_{L^2(0,T;L^2(-1,1))} \\ &+ \|\theta^{-2}e^{s\alpha}y_t\|_{L^2(0,T;L^2(-1,1))} + \|\theta^{-2}e^{s\alpha}y_{xx}\|_{L^2(0,T;L^2(-1,1))} \\ &+ \|\theta^{-2}e^{s\alpha}y\|_{L^\infty(0,T;H^1(-1,1))}.\end{aligned}$$

We begin by the following lemma, which provides the appropriate estimates of each nonlinear term with respect to the $\|\cdot\|_{\mathcal{Y}}$ – norm.

Lemma 4.1 (Nonlinear estimates)**.** *For $y_0 \in H^1(-1,1)$, let (y,h) denote the controlled trajectory of the linearized problem (2.8) given by Theorem 3.1. Then*

$$\left\|\theta^{-\frac{3}{2}} e^{s\alpha} \mathcal{N}(y,h)\right\|_{L^2(0,T;L^2(-1,1))} \leq C\|y\|_{\mathcal{Y}}^2$$

holds for some $C = C(T,\omega,s) > 0$.

Proof. We begin by noting that $a \in L^\infty(0,T)$. Using standard interpolation estimates,

$$\|y\|_{L^\infty(L^\infty)} \leq \|y\|_{L^\infty(H^1)} \leq C\|y\|_{H^1(L^2)}^{\frac{1}{2}} \|y\|_{L^2(H^2)}^{\frac{1}{2}} \leq C\|y\|_{\mathcal{Y}}. \tag{4.3}$$

Let us begin by estimating the right-most term of (4.2). Since $h + \bar{\ell} \in L^\infty(0,T)$ as well as $\theta^{-1} \in L^\infty(0,T)$, using (4.3) one deduces

$$\begin{aligned}\left\|\theta^{-\frac{3}{2}} e^{s\alpha}(h+\bar{\ell}) y y_x\right\|_{L^2(0,T;L^2(-1,1))} &\leq C\|y\|_{L^\infty(L^\infty)} \left\|\theta^{-\frac{3}{2}} e^{s\alpha} y_x\right\|_{L^2(0,T;L^2(-1,1))} \\ &\leq C\|y\|_{\mathcal{Y}}^2. \end{aligned}\tag{4.4}$$

To estimate the two middle terms in (4.2), we first observe that since $h(T) = 0$, for any $t \in [0,T]$ we may write

$$h(t) = h(t) - h(T) \leq C(T) \sup_{t\in[0,T]} |h'(t)|. \tag{4.5}$$

Moreover, as $h'(t) = y(t,1)$ for $t \in (0,T)$, $(h+\bar{\ell})\cdot \in L^\infty((0,T)\times(-1,1))$ and $\bar{\ell}\cdot + \bar{v} \in L^\infty((0,T)\times(-1,1))$ and $\theta^{-1} \in L^\infty(0,T)$, we may estimate the middle terms using (4.5) and (4.3) as follows:

$$\begin{aligned}\left\|\theta^{-\frac{3}{2}} e^{s\alpha} h' y_x (h+\bar{\ell})\right\|_{L^2(0,T;L^2(-1,1))} &+ \left\|\theta^{-\frac{3}{2}} e^{s\alpha} h y_x (\bar{\ell}+\bar{v})\right\|_{L^2(0,T;L^2(-1,1))} \\ &\leq C\|y\|_{L^\infty(L^\infty)} \left\|e^{-\frac{3}{2}} e^{s\alpha} y_x\right\|_{L^2(0,T;L^2(-1,1))} \\ &\leq C\|y\|_{\mathcal{Y}}^2. \end{aligned}\tag{4.6}$$

To estimate the leftmost term, we need further arguments. Indeed, arguing as above we deduce

$$\left\|\theta^{-\frac{3}{2}} e^{s\alpha} h y_t (h+2\bar{\ell})\right\|_{L^2(0,T;L^2(-1,1))} \leq C\left\|\theta^{\frac{1}{2}} h\right\|_{L^\infty(0,T)} \left\|\theta^{-2} e^{s\alpha} y_t\right\|_{L^2(0,T;L^2(-1,1))}.$$

The desired estimate would thus follow provided

$$\left\|\theta^{\frac{1}{2}} h\right\|_{L^\infty(0,T)} \lesssim \|y\|_{\mathcal{Y}} \tag{4.7}$$

holds. To prove (4.7), let $0 < \bar{\alpha} < \min_{x\in(-1,1)}(e^{2\lambda\|\alpha_0\|_{L^\infty}} - e^{\lambda\alpha_0})$ and we first notice that since $h(T) = 0$ and $e^{-\frac{s\bar{\alpha}\theta(T)}{2}} = 0$, by the Cauchy mean-value theorem

$$\left|\frac{h(t)}{e^{-\frac{s\bar{\alpha}\theta}{2}}}\right| = \left|\frac{h(t)-h(T)}{e^{-\frac{s\bar{\alpha}\theta(t)}{2}} - e^{-\frac{s\bar{\alpha}\theta(T)}{2}}}\right| \lesssim \left\|\frac{h'}{\left(e^{-\frac{s\bar{\alpha}\theta}{2}}\right)'}\right\|_{L^\infty(0,T)} \lesssim_T \left\|\frac{h'}{e^{-\frac{s\bar{\alpha}\theta}{2}}}\right\|_{L^\infty(0,T)} \tag{4.8}$$

for $t \in [0,T]$. We proceed in estimating the right-most term in (4.8). For $t \in [0,T]$, using trace estimates and the decay properties of the Carleman weights,

$$\begin{aligned} e^{s\overline{\alpha}\theta(t)}|h'(t)|^2 &= e^{s\overline{\alpha}\theta(t)}|y(t,1)|^2 \\ &\lesssim \sup_{t\in[0,T]}\int_{-1}^1 e^{s\overline{\alpha}\theta(t)}|y(t,x)|^2\,\mathrm{d}x + \sup_{t\in[0,T]}\int_{-1}^1 e^{s\overline{\alpha}\theta(t)}|y_x(t,x)|^2\,\mathrm{d}x \\ &\lesssim_T \sup_{t\in[0,T]}\int_{-1}^1 \theta^{-4}e^{2s\alpha}|y|^2\,\mathrm{d}x + \sup_{t\in[0,T]}\int_{-1}^1 \theta^{-4}e^{2s\alpha}|y_x|^2\,\mathrm{d}x, \end{aligned} \tag{4.9}$$

and the right-most terms are bounded by Lemma 3.2. By (4.9), (4.8) holds, and the latter rewrites as

$$|h(t)| \lesssim_T e^{-\frac{s\overline{\alpha}\theta(t)}{2}} \left\| \frac{h'}{e^{-\frac{s\overline{\alpha}\theta}{2}}} \right\|_{L^\infty(0,T)}. \tag{4.10}$$

Consequently, (4.10) along with the decay properties of the Carleman weights yield (4.7), which concludes the proof. □

We are now in a position to state and prove the null-controllability result for Problem (4.1).

Theorem 4.1. *Let $T > 0$ and $\omega = (\gamma_1,\gamma_2) \subsetneq (-1,0)$ be non-empty. There exists $r > 0$ such that for all $(y_0,h_0) \in H^1(-1,1)\times\mathbb{R}$ satisfying $\|y_0\|_{H^1(-1,1)} + |h_0| \leq r$, there exists a control $u \in L^2(0,T;L^2(\omega))$ such that the corresponding strong solution*

$$y \in L^2(0,T;H^2(-1,1)) \cap C^0([0,T];H^1(-1,1)) \quad h \in H^1(0,T)$$

of (4.1) satisfies $y(T,\cdot) = 0$ in $(-1,1)$ and $h(T) = 0$.

The proof follows a Banach fixed point argument. For $r > 0$, we consider the associated ball of $H^1(-1,1)$:

$$B_r := \{y_0 \in H^1(-1,1)\colon \|y_0\|_{H^1(-1,1)} \leq r\},$$

and we also set

$$\mathcal{F}_r = \left\{ f \in L^2(0,T;L^2(-1,1))\colon \left\|\theta^{-\frac{3}{2}}e^{s\alpha}f\right\|_{L^2(0,T;L^2(-1,1))} \leq r \right\}.$$

We construct a map $\mathcal{N} : B_r \times (-r,r) \times \mathcal{F}_r \to \mathcal{F}_r$ by setting, for $y_0 \in B_r$, $h_0 \in (-r,r)$ and $f \in \mathcal{F}_r$,

$$\mathcal{N}(y_0,h_0,f) = \mathcal{N}(y,h),$$

where (y,h) is the controlled trajectory provided by Theorem (3.1).

Proof of Theorem 4.1. We split the proof in 3 steps.

Step 1. For each $y_0 \in B_r$ and $h_0 \in (-r,r)$, the application $\mathcal{N}(y_0,h_0,\cdot)$ maps $\mathcal{F}_r$ to itself whenever $r > 0$ is small enough. Indeed, by Lemma 4.1 and Lemma 3.2

$$\begin{aligned} \left\|\theta^{-\frac{3}{2}}e^{s\alpha}\mathcal{N}(y_0,h_0,f)\right\|_{L^2(0,T;L^2(-1,1))} &\leq C_1\|y\|_{\mathcal{Y}}^2 \\ &\leq C_1C_2^2\left(\|y_0\|_{H^1(-1,1)} + |h_0| + \left\|\theta^{-\frac{3}{2}}e^{s\alpha}f\right\|_{L^2(0,T;L^2(-1,1))}\right)^2 \leq \frac{r}{2} \end{aligned}$$

whenever $r \leq \frac{1}{18C_1C_2^2}$ (where $C_1 > 0$ is the constant from Lemma 4.1 and $C_2 > 0$ the constant from Lemma 3.2).

Step 2. For each $y_0 \in B_r$ and $h_0 \in (-r,r)$ with $r > 0$ small enough, the application $\mathcal{N}(y_0,h_0,\cdot)$ is a contraction on $\mathcal{F}_r$ with a uniform constant < 1. This follows by estimating similarly as in Lemma 4.1 and Step 1, and closely follows the estimates in [17].

Step 3. Thanks to the Banach fixed point theorem, given $r > 0$ small enough, for any $y_0 \in B_r$ and $h_0 \in (-r, r)$, the application $\mathcal{N}(y_0, h_0, \cdot)$ admits a unique fixed point $f \in \mathcal{F}_r$, and consequently a unique solution to the control problem for (4.1). ☐

We may thus conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. The result follows by virtue of the transformations performed in Section 2 and Theorem 4.1. Indeed, given initial data $(v_0, \ell_0) \in H^1(0, \ell_0) \times \mathbb{R}^*_+$, we consider $y_0(\cdot) := v_0(\ell_0 \cdot) - \overline{v}$ and $h_0 = \ell_0 - \ell_*$. As $y_0 \in H^1(0,1)$, we may extend it to a function $\widetilde{y}_0 \in H^1(-1,1)$, which coincides with y_0 on $(0,1)$. Let $\omega = (\gamma_1, \gamma_2) \subset (-1,0)$ be a non-empty set. By Theorem 4.1, there exists $r > 0$ such that whenever $\|\widetilde{y}_0\|_{H^1(-1,1)} + |h_0| \leq r$, there exists a control $\widetilde{u} \in L^2(0,T; L^2(\omega))$ such that the solution (y,h) to (4.1) satisfies $y(T,\cdot) = 0$ in $(-1,1)$ and $h(T) = 0$. This in turn implies that the control $u(t) := y(t,0) + \overline{v}$ guarantees the null-controllability of the boundary control system (2.6) on $(0,1)$, with initial data (y_0, h_0). We now set $w(t,x) := y(t,x) + \overline{v}$ in $[0,T] \times [0,1]$ and $\ell(t) = h(t) + \overline{\ell}(t)$ in $[0,T]$. It is readily seen that (w, ℓ) satisfy (2.2) for initial data $(v(\ell_0 \cdot), \ell_0)$, as well as $w(T,\cdot) = \overline{v}$ in $(0,1)$ and $\ell(T) = \overline{\ell}(T)$. As the result is local, one also has $\ell(t) > 0$ in $[0,T]$ by continuity, and thus reversing the transformation (2.1) gives the desired result. ☐

5. Conclusion and perspectives

In this work, we addressed the local controllability of a one-dimensional free boundary problem governed by the viscous Burgers equation. By means of a control actuating along the fixed boundary, we showed that we may steer the fluid to constant velocity and also control the position of its free surface, whenever the difference between the initial velocities and the interface positions respectively is small enough. While the existence of this non-trivial trajectory is a particularity of the system under consideration, our result also implies its null-controllability.

We present hereinafter a non-exhaustive list of perspectives related to our work.

5.1. **Controllability to arbitrary trajectories.**

A challenging problem to which we *have not* given a solution in this work is the controllability to arbitrary smooth trajectories for parabolic free boundary problems. Up to the best of our knowledge, this problem has not been addressed in the literature, even in the one-dimensional case. Let us give a brief overview of the issues that may arise in doing so for System (1.1).

Given an arbitrary smooth trajectory $(\overline{v}, \overline{\ell})$ of (1.1), we recall that as per Section 2, after fixing the domain the linearization of (1.1) around this trajectory is of the form

$$\begin{cases} y_t - a y_{xx} + b y_x + c y + d h' + e h = f & \text{in } (0,T) \times (0,1) \\ y(t,0) = u(t) - \overline{v}(t,0), \quad y_x(t,1) = 0 & \text{in } (0,T) \\ h'(t) = y(t,1) & \text{in } (0,T) \\ y(0,x) = y_0(x), \quad h(0) = h_0 & \text{in } (0,1) \end{cases} \tag{5.1}$$

where $y_0(\cdot) = v_0(\ell_0 \cdot) - \overline{v}(0, \overline{\ell}(0) \cdot)$, $h_0 = \ell_0 - \overline{\ell}(0)$, and the regular coefficients are given in (2.4). Contrary to the case we treated in this paper, there is no reason why the factors d, e would vanish for an arbitrary trajectory $(\overline{v}, \overline{\ell})$. As done in Section 2, let us first consider a distributed control system in the extended domain $(-1,1)$:

$$\begin{cases} y_t - ay_{xx} + by_x + cy + dh' + eh = f + u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ y_x(t,-1) = y_x(t,1) = 0 & \text{in } (0,T) \\ h'(t) = y(t,1) & \text{in } (0,T) \\ y(0,x) = y_0(x), \quad h(0) = h_0 & \text{in } (-1,1), \end{cases} \tag{5.2}$$

where the coefficients and initial data are extended accordingly. The localized control $u = u(t,x)$ actuates inside some open, non-empty set $\omega \subsetneq (-1,0)$. Since we consider the case $d, e \not\equiv 0$ (so unlike the particular trajectory (1.3) we considered), the adjoint problem one obtains is more difficult to handle.

We remark that by applying a Banach fixed-point argument to the source term $dh' + eh$, it can be shown that both linearized problems (5.1), (5.2) are well-posed in $X_T = L^2(0,T;H^1(-1,1)) \cap C^0([0,T];L^2(-1,1))$.

For simplicity, let us assume $f \equiv 0$; multiplying (5.2) by a pair of smooth functions (ζ, s) and integrating leads us to the adjoint problem

$$\begin{cases} -\zeta_t - a\zeta_{xx} - (b\zeta)_x + c\zeta = 0 & \text{in } (0,T)\times(-1,1) \\ \zeta_x(t,-1) = 0, \quad \zeta_x(t,1) = -\displaystyle\int_{-1}^1 d\,\zeta\,\mathrm{d}x + s(t) & \text{in } (0,T) \\ s'(t) = \displaystyle\int_{-1}^1 d\,\zeta\,\mathrm{d}x & \text{in } (0,T) \\ \zeta(T,x) = \zeta_T(x), \quad s(T) = s_T & \text{in } (-1,1). \end{cases} \tag{5.3}$$

The adjoint problem (5.3) is much like the forward problem appearing in certain works on population dynamics, see [19] for instance. The authors prove an observability inequality for (5.2), which in our case is the forward problem. Up to the best of our knowledge, an observability inequality for (5.3) has not been shown in the literature.

Another possible strategy for tackling the null-controllability of (5.2) is to "absorb" the nonlocal terms dh' and eh in the source term f. These terms being linear would raise an issue in proving the invariance of the fixed-point map (Step 1 in Proof of Theorem 4.1). An idea which is used in several papers on the controllability to trajectories for the non-homogeneous Navier-Stokes equations (see [8] and the references therein) is to keep the Carleman constants $s, \lambda \geq 1$ arbitrary throughout the proofs. Thus, when proving the fixed-point, one may appeal to these constants as an additional degree of freedom which could render the linear terms small. The main issue in applying this strategy is the compactness-uniqueness method used to prove the improved observability inequality in Proposition 3.1. Indeed, the indirect nature of this proof means that the explicit dependence of the new observability constant on the parameters s, λ is be a priori unknown. Hence, taking s, λ arbitrarily large a posteriori may not be feasible.

5.2. **Global results.** It would most certainly be interesting to know whether one may prove a global null-controllability result in large time, namely without assuming a smallness condition on the initial data as in Theorem 1.1 provided a large enough time horizon. This question is in fact also open in the simpler case of the one-phase Stefan problem (1.4), and also in the fluid-structure problem (1.5). A problem which arises in these cases is that while the PDE may possess an inherent dissipative mechanism, the asymptotic position of the free boundary is not known. These issues will be addressed in a future work.

5.3. **Multi-dimensional problem.** One may also consider an appropriate controllability problem for the incompressible Navier-Stokes equations with a free surface, as encountered in the works of Beale [2, 3]. This would represent a natural extension of our work to the multi-dimensional setting. Up to the best of our knowledge, this question has not been addressed in the literature.

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