Finite element approximation of fractional diffusion

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DeustoTech – Universidad de Deusto
July 12, 2019

Local jump random walk

- Consider a random walk of a particle along the real line.
- Let $h\mathbb{Z} = \{hz : z \in \mathbb{Z}\}$ be the set of possible states of the particle.
- Let $u(x,t)$ be the probability of the particle to be at $x \in h\mathbb{Z}$ at time $t \in \tau\mathbb{N}$.
- **Local jump random walk:** at each time step of size τ, the particle jumps to the left or right with probability $1/2$.

$$u(x, t+\tau) = \frac{1}{2}u(x+h, t) + \frac{1}{2}u(x-h, t)$$

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$$u(x,t+\tau) = \frac{1}{2}u(x+h,t) + \frac{1}{2}u(x-h,t)$$

If we consider $2\tau = h^2$, then we obtain

$$\frac{u(x,t+\tau) - u(x,t)}{\tau} = \frac{u(x+h,t) + u(x-h,t) - 2u(x,t)}{h^2}$$

Letting $h, \tau \downarrow 0$ yields the heat equation

$$\partial_t u - \Delta u = 0$$

Long jump random walk

- The probability that the particle jumps from the point $hk \in h\mathbb{Z}$ to the point $hm \in h\mathbb{Z}$ is $\mathcal{K}(k-m) = \mathcal{K}(m-k)$:

$$u(x, t+\tau) = \sum_{k \in \mathbb{Z}} \mathcal{K}(k) u(x+hk, t).$$

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- Since $\sum_{k\in\mathbb{Z}} \mathcal{K}(k) = 1$, this yields

$$u(x, t+\tau) - u(x,t) = \sum_{k\in\mathbb{Z}} \mathcal{K}(k) \left(u(x+hk, t) - u(x,t)\right)$$

- If $\mathcal{K}(k) \sim |k|^{-(1+2s)}$ with $s \in (0,1)$ and $\tau = h^{2s}$, then $\frac{\mathcal{K}(k)}{\tau} = h\mathcal{K}(kh)$. Letting $h, \tau \downarrow 0$ yields the fractional heat equation

$$\partial_t u(x,t) = C \int_{\mathbb{R}} \frac{u(x+y,t) - u(x,t)}{|y|^{1+2s}} dy \quad \Leftrightarrow \quad \partial_t u + (-\Delta)^s u = 0.$$

Goal & outline

Consider fractional-order problems on bounded domains. Derive **Sobolev regularity estimates** and perform a **finite element analysis.**

- (Linear) Dirichlet problem.
  - Finite element discretizations.
  - Regularity of solutions.
  - A BPX preconditioner.
- Fractional obstacle problem.
- Fractional minimal graphs.

Integral definition for $\Omega \subset \mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and let $f : \Omega \to \mathbb{R}$ be sufficiently smooth.

- **Boundary value problem:**

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

- **Integral representation:**

$$(-\Delta)^s u(x) = C(n,s) \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \, dy = f(x), \quad x \in \Omega.$$

- **Boundary conditions:** imposed in $\Omega^c = \mathbb{R}^n \setminus \overline{\Omega}$

$$u = 0 \quad \text{in } \Omega^c.$$

- **Probabilistic interpretation:** it is the same as over $\mathbb{R}^n$ except that particles are killed upon reaching Ω^c.

Setting

- **Fractional Sobolev spaces in $\mathbb{R}^n$:**

$$H^s(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n) \colon |v|_{H^s(\mathbb{R}^n)} < \infty \right\}$$

with

$$(v, w)_s = \frac{C(n,s)}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(v(x) - v(y))(w(x) - w(y))}{|x-y|^{n+2s}} dydx,$$

$$|v|_{H^s(\mathbb{R}^n)} = \sqrt{(v,v)_s}, \quad \|v\|_{H^s(\mathbb{R}^n)} = \left(\|v\|^2_{L^2(\mathbb{R}^n)} + |v|^2_{H^s(\mathbb{R}^n)} \right)^{\frac{1}{2}}.$$

- **Zero-extension Sobolev spaces in Ω:**

$$\widetilde{H}^s(\Omega) = \left\{ v \in H^s(\mathbb{R}^n) \colon \operatorname{supp}(v) \subset \overline{\Omega} \right\}.$$

We define the norm $\|v\|_{\widetilde{H}^s(\Omega)} := |v|_{H^s(\mathbb{R}^n)}$. (Poincaré inequality in $\widetilde{H}^s(\Omega)$)

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- **Weak formulation:** given $f \in H^{-s}(\Omega) = \left[\widetilde{H}^s(\Omega)\right]^*$, find $u \in \widetilde{H}^s(\Omega)$ such that

$$(u, v)_s = \langle f, v \rangle \; \forall v \in \widetilde{H}^s(\Omega).$$

Existence, uniqueness & stability follow from Lax-Milgram.

Regularity of solutions

Theorem (Vishik & Èskin, 1965; Grubb, 2015)

Let $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^\infty$ and $f \in H^r(\Omega)$ for some $r \geq -s$. Then,

$$u \in \begin{cases} H^{2s+r}(\Omega) & \text{if } s+r < 1/2, \\ H^{s+1/2-\varepsilon}(\Omega), \quad \forall \varepsilon > 0, & \text{if } s+r \geq 1/2. \end{cases}$$

- **Boundary behavior:** typically,

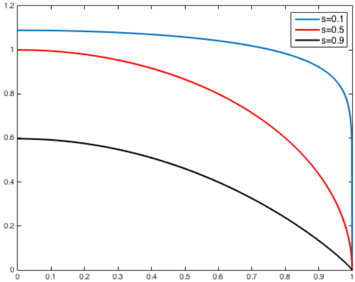
$$u(x) \approx d(x, \partial\Omega)^s + v(x),$$

  with v smooth and vanishing on $\partial\Omega$.

- **Example:** if $\Omega = B(0,1)$, $f \equiv 1$, then

$$u(x) = C(1 - |x|^2)_+^s,$$

  which does not belong to $H^{s+1/2}(\Omega)$.



Hölder regularity

Assume Ω is Lipschitz but satisfies a uniform exterior ball condition.

Theorem (Ros-Oton & Serra, 2014)

If $f \in L^\infty(\Omega)$, then $u \in C^s(\mathbb{R}^n)$ and

$$\|u\|_{C^s(\mathbb{R}^n)} \leq C(\Omega, s)\|f\|_{L^\infty(\Omega)}.$$

Furthermore, defining $\delta(x) := dist(x, \partial\Omega)$, the function u/δ^s can be continuously extended to $\overline{\Omega}$.

- **Boundary behavior:** if $f \in C^\beta(\overline{\Omega})$ ($\beta < 2 - 2s$), then

$$\sup_{x,y\in\Omega} \delta(x,y)^{\beta+s} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{\beta+2s-1}} \leq C_1, \qquad \sup_{x\in\Omega} \delta(x)^{1-s}|\nabla u(x)| \leq C_2,$$

where $\delta(x) := \mathrm{dist}(x, \partial\Omega)$ and $\delta(x,y) = \min\{\delta(x), \delta(y)\}$.

Weighted fractional Sobolev regularity (Acosta & B., 2017)

- **Definition of space $H^{1+\theta}_\alpha(\Omega)$:** let $\alpha \geq 0$ and $\theta \in (0,1)$. Define

$$\|v\|^2_{\widetilde{H}^{1+\theta}_\alpha(\Omega)} := \|v\|^2_{H^1_\alpha(\Omega)} + \iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{|\nabla v(x) - \nabla v(y)|^2}{|x-y|^{n+2\theta}} \, \delta(x,y)^{2\alpha} dx\, dy,$$

with $\|v\|^2_{H^1_\alpha(\Omega)} = \|(v + \nabla v)\, \delta(\cdot)^\alpha\|^2_{L^2(\Omega)}$.

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Theorem (regularity)

Let $f \in C^{1-s}(\overline{\Omega})$, Ω be a bounded Lipschitz domain satisfying an exterior ball condition and $\varepsilon > 0$ be small. Then, the solution u of $(-\Delta)^s u = f$ which vanishes in Ω^c belongs to $\widetilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)$ and satisfies the estimate

$$\|u\|_{\widetilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)} \leq \frac{C(\Omega, s)}{\varepsilon} \|f\|_{C^{1-s}(\overline{\Omega})}.$$

(Based on results by Ros-Oton & Serra).

Besov regularity on Lipschitz domains (B. & Nochetto)

- **Besov estimate:** given $v \in H^s(B)$ let $v_h = v(\cdot + h)$ be a translation; then

$$|v|_{B^{s+\sigma}_{2,\infty}(B)} \lesssim \sup_{h\in\mathcal{C}} \frac{1}{|h|^\sigma} |v - v_h|_{H^s(B)},$$

  where $\mathcal{C} = \mathcal{C}(x_0)$ is a cone of directions centered at x_0.
- **Functional:** if u is a minimizer of $\mathcal{F}(v) := \mathcal{F}_2(v) - \mathcal{F}_1(v)$ with $\mathcal{F}_2(v) := \frac{1}{2}\|v\|^2_{\widetilde{H}^s(\Omega)}$ and $\mathcal{F}_1(v) := \langle f, v\rangle$, then

$$\frac{1}{2}\|u - v\|^2_{\widetilde{H}^s(\Omega)} = \mathcal{F}(v) - \mathcal{F}(u).$$

- **Translation operator (Savaré, 1998):** let $T_h v = \phi v_h + (1-\phi)v$ be a convex combination of v_h and v with ϕ a cut-off function with support of $\phi \subset B_{2\rho}(x_j)$ and $\phi = 1$ in $B_j = B_\rho(x_j)$ for $x_j \in \Omega$. Then

$$\frac{|u - T_h u|_{H^s(B)}}{|h|^\sigma} \lesssim \sup_{h\in\mathcal{C}} \frac{\mathcal{F}(T_h u) - \mathcal{F}(u)}{|h|^\sigma}.$$

- **Localization:** if $\Omega = \cup_{j=1}^m B_j$ is a covering with finite overlap, then

$$\|v\|_{B^s_{p,q}(\Omega)} \approx \sum_{j=1}^m \|v\|_{B^s_{p,q}(B_j)}.$$

Besov Regularity on Lipschitz Domains (continued)

- **Linear part:** if $\gamma \in (0,1)$, then

$$\sup_{h \in \mathcal{C}} \frac{\mathcal{F}_1(T_h v) - \mathcal{F}_1(v)}{|h|^\gamma} \leq \|f\|_{L^2(B_{2\rho})} |v|_{H^\gamma(B_{3\rho})}.$$

- **Quadratic part:** Using the convexity of $\mathcal{F}_2$, we obtain

$$\sup_{h \in \mathcal{C}} \frac{\mathcal{F}_2(T_h v) - \mathcal{F}_2(v)}{|h|^\gamma} \leq \|\phi\|_{W^1_\infty(\mathbb{R}^d)} |v|_{H^\gamma(B_{3\rho})}.$$

Besov Regularity on Lipschitz Domains (continued)

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- **Basic pick-up regularity:** Let $f \in L^2(\Omega)$ and take $\gamma = s$. Then the pick-up regularity is $s + \gamma/2 = 3s/2$:

$$u \in B^{3s/2}_{2,\infty}(\Omega).$$

If $s > \frac{2}{3}$, then $B^{3s/2}_{2,\infty}(\Omega) \subset H^1(\Omega)$ and we can take $\gamma = 1$ to conclude

$$u \in B^{s+\frac{1}{2}}_{2,\infty}(\Omega).$$

If $s \leq \frac{2}{3}$, then we need to iterate this argument progressively increasing γ.

Besov regularity on Lipschitz domains: lift theorems

We deduce the following lift theorems without a uniform exterior ball condition, thus allowing for reentrant corners.

- **Lift Theorem 1:** Duality argument in the linear part can be modified to get

$$|u|_{B^{\frac{3s}{2}}_{2,\infty}(\Omega)} \lesssim \|f\|_{B^{-\frac{s}{2}}_{2,1}(\Omega)}.$$

Interpolating with $\|u\|_{\widetilde{H}^s(\Omega)} \lesssim \|f\|_{H^{-s}(\Omega)}$ yields

$$\|u\|_{\widetilde{H}^{s+t}(\Omega)} \lesssim \|f\|_{H^{-s+t}(\Omega)} \qquad 0 \le t \le \frac{s}{2}.$$

- **Lift Theorem 2:** If $f \in L^2(\Omega)$, then

$$|u|_{B^{s+\frac{1}{2}}_{2,\infty}(\Omega)} \le C\|f\|_{L^2(\Omega)} \qquad s > \frac{1}{2},$$

$$|u|_{B^{2s-\epsilon}_{2,\infty}(\Omega)} \le C_\epsilon\|f\|_{L^2(\Omega)} \qquad s \le \frac{1}{2}.$$

FEM and best approximation

- **Discrete spaces:** let $\{\mathcal{T}\}$ be a sequence of conforming, shape-regular meshes on Ω, and define

$$\mathbb{V}_h = \{v_h \in C(\overline{\Omega}) \colon v\big|_T \in \mathcal{P}_1 \, \forall T \in \mathcal{T}\}.$$

- **Discrete problem:** find $u_h \in \mathbb{V}_h$ such that $(u_h, v_h)_s = \langle f, v_h \rangle \; \forall v_h \in \mathbb{V}_h$.

- **Best approximation:** since we project over $\mathbb{V}_h$ with respect to the energy norm $|\cdot|_{\widetilde{H}^s(\Omega)}$ induced by $(\cdot,\cdot)_s$, we get

$$\|u - u_h\|_{\widetilde{H}^s(\Omega)} = \min_{v_h \in V_h} \|u - v_h\|_{\widetilde{H}^s(\Omega)}.$$

- **A priori error analysis:** must account for nonlocality and boundary behavior.

Interpolation estimates in $\widetilde{H}^s(\Omega)$

- **Hardy-type estimates**: for every $v \in \widetilde{H}^s(\Omega)$,

$$
\begin{aligned}
\|v\|_{\widetilde{H}^s(\Omega)} &\le C(\Omega, s)\|v\|_{H^s(\Omega)}, && \text{if } s \in (0, 1/2),\\
\|v\|_{\widetilde{H}^s(\Omega)} &\le C(\Omega, s)|v|_{H^s(\Omega)}, && \text{if } s \in (1/2, 1).
\end{aligned}
$$

- **Localized estimates in $H^s(\Omega)$** (Faermann, 2002):

$$
|v|^2_{H^s(\Omega)} \le \frac{C(n,s)}{2} \sum_{T\in\mathcal{T}} \left[\int_T \int_{S_T} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dydx + \frac{C(n,\sigma)}{sh_T^{2s}} \|v\|^2_{L^2(T)} \right],
$$

  where S_T is the patch associated with $T \in \mathcal{T}$ and σ is the shape regularity constant of $\mathcal{T}$.

- **Quasi-interpolation** (Ciarlet Jr., 2013): if Π_h is the Scott-Zhang operator,

$$
\int_T \int_{S_T} \frac{|(v-\Pi_h v)(x) - (v-\Pi_h v)(y)|^2}{|x-y|^{n+2s}}\, dy\, dx \lesssim h_T^{2\ell-2s} |v|^2_{H^\ell(S_{S_T})},
$$

  where the hidden constant depends on n, σ, ℓ and blows up as $s \uparrow 1$.

Quasi-uniform meshes

- **Error estimate:** if $\partial\Omega \in C^\infty$ and $f \in H^{1/2-s}(\Omega)$, then

$$\|u - u_h\|_{\widetilde{H}^s(\Omega)} \leq C(s,\sigma) h^{\frac{1}{2}} |\ln h| \, \|f\|_{H^{1/2-s}(\Omega)}.$$

- **Example:** let $\Omega = B(0,1) \subset \mathbb{R}^2$ and $f = 1$. Then, the solution is given by

$$u(x) = C(1 - |x|^2)^s_+.$$

s	0.1	0.3	0.5	0.7	0.9
Order (in h)	0.497	0.498	0.501	0.504	0.532

Rate is quasi-optimal. Is it possible to improve the order of convergence?

Graded meshes

- **Weighted quasi-interpolation:**

$$\int_T \int_{S_T} \frac{|(v-\Pi_h v)(x) - (v-\Pi_h v)(y)|^2}{|x-y|^{n+2s}} dy dx \leq C h_T^{2(1+\theta-\alpha-s)} |v|^2_{H^{1+\theta}_\alpha(S_{S_T})}.$$

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- **Energy error estimate:** let $d=2$ and $\mathcal{T}$ be a graded mesh satisfying

$$h_T \le C(\sigma) \begin{cases} h^2, & T\cap\partial\Omega \neq \emptyset, \\ h\,\mathrm{dist}(T,\partial\Omega)^{1/2}, & T\cap\partial\Omega = \emptyset, \end{cases}$$

whence $\#\mathcal{T} \approx h^{-2}|\log h|$. Then,

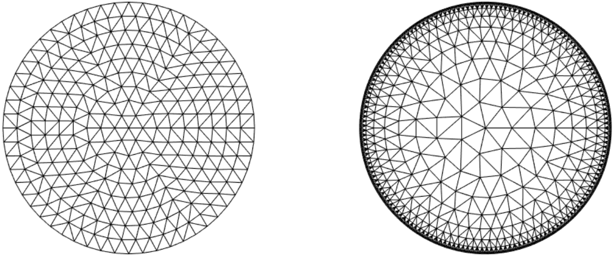
$$\|u-U\|_{\widetilde{H}^s(\Omega)} \lesssim (\#\mathcal{T})^{-\frac{1}{2}} |\log(\#\mathcal{T})| \, \|f\|_{C^{1-s}(\overline{\Omega})}.$$

- **Improvement:** this also reads $\|u-U\|_{\widetilde{H}^s(\Omega)} \lesssim h|\log h| \, \|f\|_{C^{1-s}(\overline{\Omega})}$ and is thus first order.

Numerical experiment

Exact solution: if $\Omega = B(0,1) \subset \mathbb{R}^2$ and $f = 1$, then $u(x) = C(r^2 - |x|^2)_+^s$.

Value of s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Uniform $\mathcal{T}$	0.497	0.496	0.498	0.500	0.501	0.505	0.504	0.503	0.532
Graded $\mathcal{T}$	1.066	1.040	1.019	1.002	1.066	1.051	0.990	0.985	0.977



BPX preconditioner (B., Nochetto, Wu & Xu)

- **Conditioning of the stiffness matrix** (Ainsworth, McLean & Tran, 1999):

$$\kappa(\boldsymbol{A}) \simeq \dim\left(\mathbb{V}_h\right)^{2s/n} \left(\frac{h_{\max}}{h_{\min}}\right)^{n-2s}$$

  Thus, if $n = 2$, we have
  - uniform meshes: $\kappa(\boldsymbol{A}) \simeq h^{-2s}$;
  - graded meshes ($\mu = 2$): $\kappa(\boldsymbol{A}) \simeq h^{-2} |\log h|^{s}$.

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- **Preconditioner:** assume we have a hierarchy of discrete spaces $\mathbb{V}_0 \subset \mathbb{V}_1 \subset \ldots \subset \mathbb{V}_J = \mathbb{V}$, with mesh size $h_j = \gamma^{2j}$. If
  - **Boundedness:** for every $v = \sum_{j=0}^J v_j$, $\quad |v|_s^2 \le c_1^{-1} \sum_{j=0}^J h_j^{-2s} \|v_j\|_0^2$;
  - **Stable decomposition:** for every $v \in \mathbb{V}$, there exists a decomposition $v = \sum_{j=0}^J v_j$ such that $\sum_{j=0}^J h_j^{-2s} \|v_j\|_0^2 \le c_0 |v|_s^2$;

  and $\iota_k : \mathbb{V}_k \to \mathbb{V}$ is the inclusion operator, then

$$\boldsymbol{B} = \sum_{k=0}^J h_k^{s-n} \iota_k \iota_k'$$

  leads to the condition number for $\boldsymbol{BA}$: $\quad \kappa(\boldsymbol{BA}) \le \frac{c_0}{c_1}$.

Basic Ingredients

- **Boundedness:** strengthened Cauchy-Schwarz inequality.

Let $i \leq j$. Then, for every $0 \leq \beta < \min\{s, \frac{3}{2} - s\}$,

$$(v_i, v_j)_s \lesssim \gamma^{2\beta|j-i|} h_j^{-s} |v_i|_s \|v_j\|_0 \qquad \forall v_i \in \mathbb{V}_i, v_j \in \mathbb{V}_j.$$

Proof via Fourier analysis in $\mathbb{R}^d$:

Use that $(v_i, v_j)_s \leq |v_i|_{s+\beta} |v_j|_{s-\beta}$, and inverse inequalities.

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  Use that $(v_i, v_j)_s \leq |v_i|_{s+\beta} |v_j|_{s-\beta}$, and inverse inequalities.

- **Stable decomposition:** lift theorem for Lipschitz domains.

  If Ω is Lipschitz, $\alpha < \frac{s}{2}$ and $f \in H^{-s+\alpha}(\Omega)$, then $u \in \widetilde{H}^{s+\alpha}(\Omega)$.

Numerical results

- **Example:** Dirichlet problem in the square $[-1,1]^2$ and $f = 1$. Stopping criterion: $\frac{\|b-Ax\|_2}{\|b\|_2} \leq 1 \times 10^{-6}$.

J	h_J	DOFs	$s = 0.9$		$s = 0.5$	
			CG	PCG	CG	PCG
1	2^{-1}	154	4	4	4	4
2	2^{-2}	383	12	13	8	9
3	2^{-3}	1166	25	16	11	12
4	2^{-4}	3953	46	19	17	14
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- **Other contributions:** H. Gimperlein, J. Stocek, C. Urzúa-Torres (2019); M. Fautsmann, J. Melenk, M. Parvisi, D. Praetorius (2019).

- **Extensions:** theory extends to graded bisection grids.

Obstacle problem (B., Nochetto & Salgado)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f, \chi\colon \Omega \to \mathbb{R}$ be smooth enough data.

Find $u\colon \mathbb{R}^n \to \mathbb{R}$, supported in Ω, such that

$$\begin{aligned} u \geq \chi & \quad \text{in } \Omega, \\ (-\Delta)^s u \geq f & \quad \text{in } \Omega, \\ (-\Delta)^s u = f & \quad \text{whenever } u > \chi. \end{aligned}$$

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Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f, \chi \colon \Omega \to \mathbb{R}$ be smooth enough data.

Find $u \colon \mathbb{R}^n \to \mathbb{R}$, supported in Ω, such that

$$\begin{aligned} u &\geq \chi \quad && \text{in } \Omega, \\ (-\Delta)^s u &\geq f \quad && \text{in } \Omega, \\ (-\Delta)^s u &= f \quad && \text{whenever } u > \chi. \end{aligned}$$

Can be equivalently written as a **variational inequality.**

Find $u \in \mathcal{K}$ such that

$$(u, u - v)_s \leq \langle f, u - v \rangle \quad \forall v \in \mathcal{K},$$

where $\mathcal{K}$ denotes the convex set

$$\mathcal{K} = \{ v \in \widetilde{H}^s(\Omega) \colon v \geq \chi \text{ a.e. in } \Omega \}.$$

Assumptions

- **Domain:** $\partial\Omega$ is Lipschitz, and satisfies an exterior ball condition.

- **Data:** from now on,

$$\chi \in C^{2,1}(\Omega), \qquad 0 \leq f \in \mathcal{F}_s(\overline{\Omega}) = \begin{cases} C^{2,1-2s}(\overline{\Omega}), & s \in \left(0, \frac{1}{2}\right) \\ C^{1,2-2s}(\overline{\Omega}), & s \in \left[\frac{1}{2}, 1\right) \end{cases}.$$

- We assume that $\chi < 0$ on $\partial\Omega$, so that
  - the behavior of solutions near $\partial\Omega$ is dictated by a linear problem;
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  - the behavior of solutions near $\partial\Omega$ is dictated by a linear problem;
  - the nonlinearity is constrained to the interior of the domain.

- **Non-locality:** gluing interior and boundary estimates is not straightforward! If $\eta \equiv 1$ in a neighborhood of x_0, then it does not follow that

$$(-\Delta)^s(\eta u)(x_0) = (-\Delta)^s u(x_0).$$

Regularity for the obstacle problem in $\mathbb{R}^n$

Theorem (Caffarelli, Salsa & Silvestre, 2008)

For the obstacle problem in $\mathbb{R}^n$, if $f \in \mathcal{F}_s(\mathbb{R}^n)$ and $\chi \in C^{2,1}(\mathbb{R}^n)$, then the solution u belongs to $C^{1,s}(\mathbb{R}^n)$.

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Limiting cases: for the classical obstacle problem $(s = 1)$, solutions are $C^{1,1}$.

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Recall also that for the linear problem, solutions are C^s near $\partial\Omega$.

Moral: free boundary regularity is not any worse than boundary regularity for the linear problem.

Regularity for the obstacle problem on Ω

- **Interior regularity:** Caffarelli-Salsa-Silvestre's theorem + localization argument.

- **Boundary regularity:** adapt the result for the linear Dirichlet problem.

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Theorem

Let $u \in \widetilde{H}^s(\Omega)$ be the solution to the fractional obstacle problem. Then, for every $\varepsilon > 0$ we have that $u \in \widetilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)$ with the estimate

$$\|u\|_{\widetilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)} \leq \frac{C}{\varepsilon},$$

with $C > 0$ depending on $\chi, s, n, \Omega, \|f\|_{\mathcal{F}_s(\overline{\Omega})}$.

Finite element approximation ($n = 2$)

- **Discrete problem:** find $u_h \in \mathcal{K}_h = \{v_h \in \mathbb{V}_h \colon v_h \geq I_h \chi\}$ such that

$$(u_h, u_h - v_h)_s \leq \langle f, u_h - v_h \rangle \quad \forall v_h \in \mathcal{K}_h.$$

  Here, I_h is a positivity-preserving quasi-interpolation operator (Chen & Nochetto, 2000).

- Weighted Sobolev regularity $\Rightarrow$ **graded meshes** (keep $\#\mathcal{T} \approx h^{-2} |\log h|$).

- **Fractional interpolation estimates:** for quasi-interpolation operator I_h.

Error bound

Since

$$\|u - u_h\|^2_{\widetilde{H}^s(\Omega)} = (u - u_h, u - I_h u)_s + (u - u_h, I_h u - u_h)_s,$$

we have

$$\frac{1}{2}\|u - u_h\|^2_{\widetilde{H}^s(\Omega)} \leq \frac{1}{2}\|u - I_h u\|^2_{\widetilde{H}^s(\Omega)} + (u - u_h, I_h u - u_h)_s.$$

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- **Interpolation error:** $\|u - I_h u\|_{\widetilde{H}^s(\Omega)} \leq C h^{1-2\varepsilon} \|u\|_{\widetilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)}$.

Error bound

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- **Interpolation error:** $\|u-I_hu\|_{\widetilde{H}^s(\Omega)} \le Ch^{1-2\varepsilon}\|u\|_{\widetilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)}$.
- **Second term in RHS:** integrate by parts and use discrete variational inequality,

$$\begin{aligned}
(u-u_h, I_hu-u_h)_s &\le \int_\Omega (I_hu-u_h)((-\Delta)^s u - f) \\
&= \int_\Omega \Big[(u-\chi) + (I_h\chi - u_h) + (I_h(u-\chi) - (u-\chi))\Big]((-\Delta)^s u - f) \\
&\le \int_\Omega (I_h(u-\chi) - (u-\chi))\,((-\Delta)^s u - f).
\end{aligned}$$

Therefore,

$$(u - u_h, I_h u - u_h)_s \le \sum_{T \in \mathcal{T}} \int_T \left(I_h(u-\chi) - (u-\chi)\right)\left((-\Delta)^s u - f\right).$$

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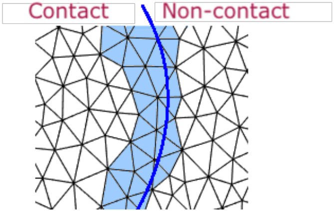

Using the interior regularity $u \in C^{1,s}(\Omega)$ we deduce:

- $u - \chi \in C^{1,s}(\Omega)$,
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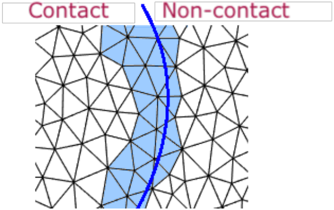

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In the light blue elements we have

$$\left|(I_h(u-\chi)-(u-\chi))\left((-\Delta)^s u - f\right)\right| \leq C h_T^{1+s} h_T^{1-s} = C h_T^2.$$

Convergence rate

Theorem

Let $0 \leq f \in \mathcal{F}_s(\overline{\Omega})$ *and assume that* $\chi \in C^{2,1}(\Omega)$ *is such that* $\chi < 0$ *on* $\partial\Omega$. *Considering shape-regular graded meshes as before, if* h *is sufficiently small, then it holds that*

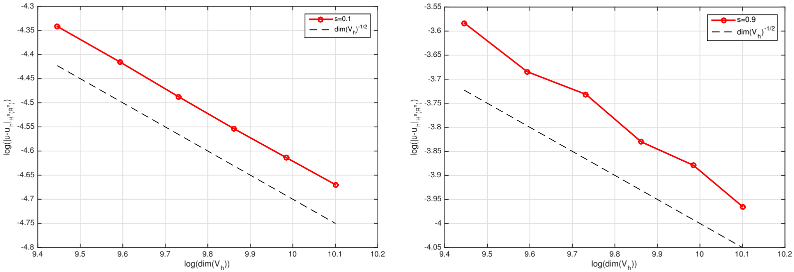
$$\|u - u_h\|_{\widetilde{H}^s(\Omega)} \leq C(\#\mathcal{T})^{1/2} |\log(\#\mathcal{T})|^{3/2}.$$

Numerical experiments

Problem: let $\Omega = B(0,1) \subset \mathbb{R}^2$, and consider f, χ so that the exact solution is

$$u(x) = (1 - |x|^2)_+^s \, p_2^{(s)}(x),$$

where $p_2^{(s)}$ is a certain Jacobi polynomial of degree two.

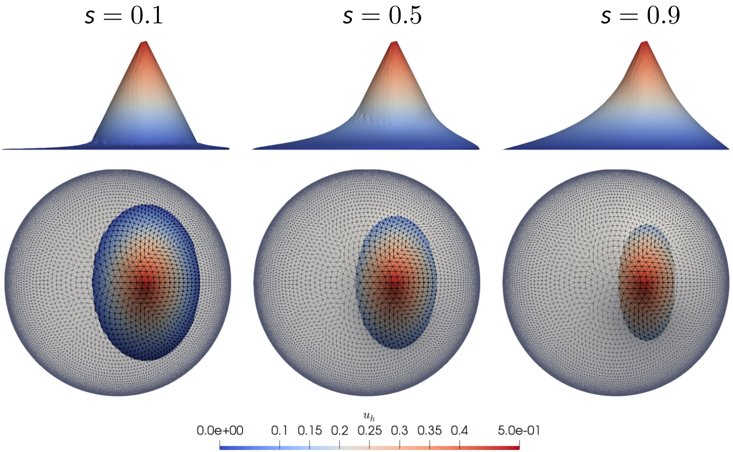


Left: $s = 0.1$; right: $s = 0.9$. The rate observed in both cases is $\approx (\#\mathcal{T})^{1/2}$.

Qualitative behavior

Problem: let $\Omega = B(0,1) \subset \mathbb{R}^2$, $f = 0$ and

$$\chi(x_1, x_2) = \frac{1}{2} - \sqrt{\left(x_1 - \frac{1}{4}\right)^2 + \frac{1}{2}\, x_2^2}.$$



Fractional minimal surfaces (B., Li & Nochetto)

- **A phase transition problem:** consider the Ginzburg-Landau energy

$$J_\varepsilon(u;\Omega) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 \, dx \, + \, \frac{1}{\varepsilon}\int_\Omega W(u)\, dx,$$

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- If $J_\varepsilon(\,\cdot\,;\Omega)$ is uniformly bounded, then there exists a subsequence $\varepsilon_k \to 0$ such that

$$u_{\varepsilon_k} \to \chi_E - \chi_{E^c} \text{ in } L^1_{loc}(\Omega),$$

  where E is a set with minimal perimeter in Ω (Modica, 1978).

A unified framework for minimal surfaces

Consider now

$$J_\varepsilon(u;\Omega) = \frac{\varepsilon^{2s}}{2} \iint_{Q_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy + \int_\Omega W(u)\, dx,$$

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Up to an appropriate scaling in ε, we have:

Theorem (Savin & Valdinoci, 2012)

Assume the energies $J_\varepsilon(u_\varepsilon;\Omega)$ are uniformly bounded. Then, there exists a subsequence $\varepsilon_k \to 0$ such that $u_{\varepsilon_k} \to \chi_E - \chi_{E^c}$ in $L^1(\Omega)$.
Moreover, let u_ε be a sequence of minimizers.

- *If $s \in (0,1/2)$ and $u_\varepsilon \rightharpoonup \chi_{E_0}$* in Ω^c, *then E minimizes a fractional perimeter among all the sets $\{F \subset \mathbb{R}^n : F \cap \Omega^c = E_0\}$.*
- *If $s \in [1/2,1)$, then E is a set with minimal* classical *perimeter.*

Minimal sets

- **Classical perimeter:**

$$P(E;\mathbb{R}^n) = |\chi_E|_{BV(\mathbb{R}^n)} \text{ “}=\text{” } |\chi_E|_{W^{1,1}(\mathbb{R}^n)}.$$

- **Fractional perimeter:** (Imbert, 2009; Caffarelli, Roquejoffre & Savin, 2010)

$$P_s(E;\mathbb{R}^n) = |\chi_E|_{W^{2s,1}(\mathbb{R}^n)} = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+2s}}dxdy, \quad s\in(0,1/2).$$

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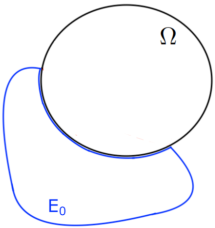

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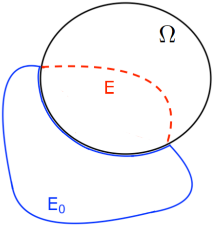
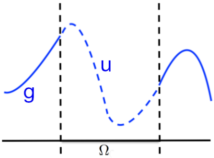


Fractional minimal graphs



- **Problem:** find a locally s-minimal set E in $\widetilde{\Omega} = \Omega \times \mathbb{R} \in \mathbb{R}^{n+1}$ s.t. $E \cap \widetilde{\Omega}^c = \widetilde{E}_0$, where

$$\widetilde{E}_0 = \{\widetilde{x} = (x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} < g(x)\},$$

  for a given bounded continuous function $g : \Omega^c \to \mathbb{R}$.

- In this setting, there exists a unique minimal set E, and is actually a subgraph (Dipierro, Savin & Valdinoci, 2016, Lombardini, 2016).

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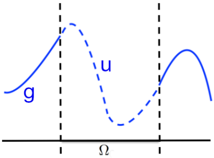

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- Fractional perimeter $P_s(E;\Omega)$ in terms of u:

$$P_s(E;\Omega) = \iint_{Q_\Omega} F_s\left(\frac{u(x)-u(y)}{|x-y|}\right) \frac{1}{|x-y|^{n+2s-1}}\, dxdy \;=: I_s[u],$$

  where

$$F_s(a) = \int_0^a \frac{a-r}{(1+r^2)^{(n+1+2s)/2}}\, dr.$$

- **Classical perimeter:** this extends surface area $I[u] = \int_\Omega \sqrt{1+|\nabla u(x)|^2} dx$.

Variational formulation

Consider the spaces

$$\mathbb{V}^g = \{v\colon \mathbb{R}^n \to \mathbb{R} \,:\, v\big|_\Omega \in W^{2s,1}(\Omega),\ v = g \text{ in } \Omega^c\}, \quad \mathbb{V}^0 = \mathbb{V}^g \text{ for } g \equiv 0.$$

Taking first variation of $I_s[u]$ gives: find $u \in \mathbb{V}^g$ such that

$$\iint_{Q_\Omega} \widetilde{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+1+2s}} dxdy = 0$$

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Finding a s-minimal graph in $\mathbb{R}^{n+1}$ becomes a nonhomogeneous problem in $\mathbb{R}^n$ for a nonlinear, degenerate diffusion operator of order $s + \frac{1}{2}$.

Discretization

- **Finite element space:** let

$$\mathbb{V}_h^g = \{v \in C(\mathbb{R}^n) \colon \; v|_T \in \mathcal{P}_1 \; \forall T \in \mathcal{T}, \; v|_{\Omega^c} = \Pi_h g\}, \quad \mathbb{V}_h^0 = \mathbb{V}_h^g \text{ for } g \equiv 0,$$

  where Π_h is a quasi-interpolation operator.

- **Discrete problem:** find $u_h \in \mathbb{V}_h^g$ such that for all $v_h \in \mathbb{V}_h^0$,

$$\iint_{Q_\Omega} \widetilde{G}_s \left(\frac{u_h(x) - u_h(y)}{|x-y|} \right) \frac{(u_h(x) - u_h(y))(v_h(x) - v_h(y))}{|x-y|^{n+1+2s}} \, dxdy = 0.$$

- Solve discrete problems using either a **semi-implicit L^2-gradient flow** or a **damped Newton method**.

Convergence

- **Interior regularity:** minimizer satisfies $u \in C^\infty(\Omega)$.
  (Barrios, Figalli, & Valdinoci, 2014; Cabré & Cozzi, 2017; Figalli & Valdinoci, 2017)

- **Stickiness phenomenon:** boundary datum may not be attained continuously!

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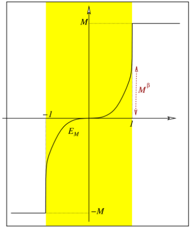

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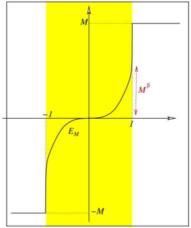


Theorem

Let u be the solution of the continuous problem and u_h be the discrete solutions, then

$$\lim_{h \to 0} I_s[u_h] = I_s[u],$$

and

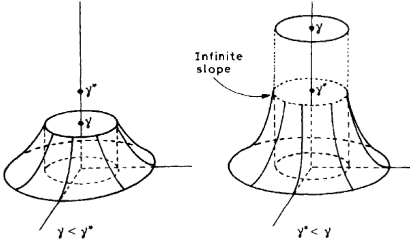
$$\lim_{h \to 0} \|u - u_h\|_{W^{2\sigma,1}(\Omega)} = 0, \quad \forall \sigma \in [0, s).$$

Experiments

- **Stickiness in 1d:** $\Omega = (-1,1)$, $g(x) = \mathrm{sign}(x)$ and $s = 0.01\colon 0.01\colon 0.49$.
  - For the classical minimal surface problem, the minimizer is $u(x) = x$, $x \in (-1,1)$.

- **Effect of $s \in (0,1/2)$:** $\Omega = B(0,1)$, $g = \chi_{B(0,3/2)}$ and $s = 0.01\colon 0.01\colon 0.49$.
  - For the classical minimal surface problem, the minimizer is flat.

- **Effect of $\gamma \in [0,1]$:** $\Omega = B(0,1) \setminus B(0,1/2)$, $g = \gamma\chi_{B(0,1/2)}$, where $\gamma = 0.02\colon 0.02\colon 1$, and $s = 0.25$.
  - For the classical minimal surface problem, when $\gamma > \gamma^* \approx 0.66$, there is no classical solution.



Graphs with prescribed nonlocal mean curvature

- Prescribed classical mean curvature:

$$\int_\Omega \frac{1}{\sqrt{1+|\nabla u|^2}} \nabla u \cdot \nabla v \, dx = \int_\Omega f v \quad \forall v \in H_0^1(\Omega).$$

- Prescribed nonlocal mean curvature:

$$\frac{1-2s}{\alpha_d} \iint_{Q_\Omega} \widetilde{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+1+2s}} dxdy = \int_\Omega f v,$$

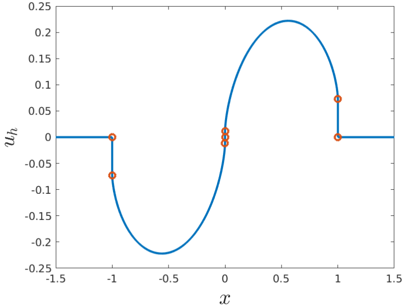
for all $v \in \mathbb{V}^0$.

The scaling factor $\frac{1-2s}{d\alpha_d}$ makes the nonlocal mean curvature converge to the classic one when $s \to 1/2$.

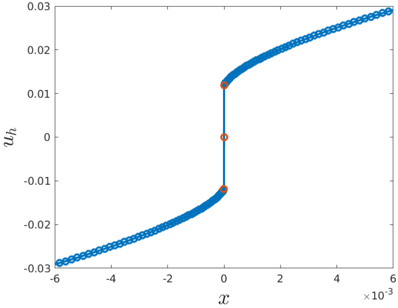
Experiment: prescribed nonlocal mean curvature

Stickiness inside the domain: let $\Omega = (-1,1) \subset \mathbb{R}$, $s = 0.01$, $g = 0$ and

$$f(x) = 1.5 * \mathrm{sign}(x).$$



Plot of u_h in $(-1,1)$.



Zoom-in near the origin.

Concluding remarks

- **Fractional Laplacian $(-\Delta)^s$:** nonlocal operator of order $0 < 2s < 2$.
  - Computational challenges: non-integrable singularities, unbounded domains;
  - Analytical challenges: nonlocality, boundary singularity;
  - Energy error estimate: weighted Sobolev spaces, localization of fractional norms, graded meshes.

- **BPX preconditioning:** mesh-independent condition number.
  - Separation of scales (SCS inequality): via Fourier analysis in $\mathbb{R}^d$;
  - Duality argument: lift property for Lipschitz domains.

- **Fractional obstacle problem:**
  - Hölder regularity: $C^{1,s}$ near the free boundary and C^s near domain boundary;
  - Energy error estimate: $C^{1,s}$ regularity near the free boundary, weighted Sobolev spaces, graded meshes.

- **Fractional minimal graphs:** nonlinear, degenerate problem of order $s + \frac{1}{2}$.
  - Solutions may be discontinuous across $\partial\Omega$ (stickiness phenomenon);
  - Convergence (without rates) in $W^{2\sigma,1}(\Omega)$ for $\sigma < s$.

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Thank you!