VNiVERSiDAD D SALAMANCA

CAMPUS DE EXCELENCIA INTERNACIONAL

Salamanca, September of 2019
Faculty of Science
Master's degree in Physics and Mathematics

Moduli spaces of vector bundles over a smooth projective algebraic curve

Master's thesis

written by
Jorge Juan Mallo Fuentes-Lojo
under the supervision of
Carlos Tejero Prieto

Moduli spaces of vector bundles over a smooth projective algebraic curve

Master's thesis

Author: Jorge Juan Mallo Fuentes-Lojo
Supervisor: Carlos Tejero Prieto

Author: Supervisor:

Salamanca, September of 2019

Contents

Introduction

This master's thesis is a bibliographic review of geometric invariant theory and the construction of the moduli spaces of vector bundles over a smooth projective algebraic curve defined over an algebraically closed field of characteristic zero. The main objective is to provide a complete introduction to the study of moduli problems, developing all the necessary tools from geometric invariant theory and studying a concrete example in detail.

Geometric invariant theory is a collection of techniques developed by David Mumford that can be used to construct quotients of schemes by actions of algebraic groups. This theory of passage to the quotient is one of the most fundamental tools used in the study of moduli problems.

Let G be a topological group acting on a topological space X. Then, the set of G-orbits X/G is a topological space with the quotient topology. What happens when we have some geometric structure (i.e., if X is a scheme, a smooth manifold or a complex manifold and G is respectively an algebraic group, a Lie group or a complex Lie group)? In this case, the set of G-orbits X/G may present many pathological properties.

The properties that a quotient is expected to have can be isolated to give the notion of categorical quotient for a G-action, that is, the properties that an object in a given category must satisfy in order to be considered a quotient for a group action. For example, in the category of topological spaces, the categorical quotient of a topological space X with respect to an action of a topological group G always exists and is given by the set of G-orbits X/G with the quotient topology.

In the case of differential or complex geometry there is not a general theory that determines the conditions under which a categorical quotient exists. However, in algebraic geometry there is such a theory: geometric invariant theory.

Geometric invariant theory was first developed in a 1965 monograph by David Mumford, based partly on some results of Hilbert in classical invariant theory. Among other results, in this monograph Mumford proved that categorical quotients for reductive algebraic group actions on finite type schemes always existed over fields of characteristic zero (in the same monograph, Mumford conjectured that the same results held in any characteristic, and this was later confirmed).

The general idea of geometric invariant theory is that the functions of the quotient space for the G-action should be the functions that are invariant with respect to that action. Over affine schemes, this idea can be applied by considering the spectrum of the algebra of invariant functions. This construction is then generalized, after some technicalities, to arbitrary finite type schemes.

Geometric invariant theory has profound connections with symplectic quotients, equivariant cohomology or Yang-Mills theory. The standard reference for the subject is Mumford's book [MFK94]. The theory presented in this thesis assumes that we are considering actions of algebraic groups that are reductive. This is a technical condition that ensures that the quotient schemes that are obtained are of finite type. However, reductivity is not a necessary condition to obtain finite type quotients, and many important algebraic groups are not reductive (for example, the additive group is not reductive). For a survey about geometric invariant theory for non reductive algebraic groups see [DK07].

We will apply the results of geometric invariant theory to the construction of a concrete moduli space: the moduli space of vector bundles over a smooth projective algebraic curve X.

It's well known that, if $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module on X, then $V_{\mathcal{E}} := \operatorname{Spec}\operatorname{Sym}^{\bullet}\mathcal{E}^{\vee}$ is a vector bundle over X, where $\mathcal{E}^{\vee} := \underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual module. This correspondence is an equivalence of categories, so we will speak about locally free sheaves instead of vector bundles.

Intuitively, a moduli space of locally free sheaves on X is a scheme whose points parameterize locally free sheaves on X (the exact definition, in terms of schemes corepresenting a contravariant functor, is given in the thesis). The moduli problem of locally free sheaves on X consists, then, in proving the existence of those moduli spaces.

Some concrete instances of this problem are well known. For example, the moduli space of locally free sheaves of rank 1 on a curve is the Jacobian variety of the curve, and its construction is classical (see for example [CS86, Chapter VII]). The moduli problem for a genus zero curve was solved by Grothendieck in [Gro57], and the case of elliptic curves is solved in [Ati57] by Atiyah.

The techniques used by Grothendieck and Atiyah in these particular cases could not be immediately generalized to curves of higher genus. A first approach, due to André Weil, consisted of studying the problem in terms of the vector bundles that arose as associated bundles to representations of the fundamental group of the Riemann surface associated to X.

This point of view is particularly well-behaved when restricting to the case of unitary representations of the fundamental group. Narasimhan and Seshadri utilized these techniques to construct moduli spaces of sheaves of degree zero, and obtained moduli spaces for higher degree sheaves when studying unitary representations of more complicated groups.

At the same time, Mumford was working on notions of stability and semistability for locally free sheaves in the sense of geometric invariant theory (see chapter 4 in this thesis). Narasimhan and Seshadri proved that stable sheaves, in the sense of Mumford, are in a one to one correspondence with irreducible unitary representations of the groups that they considered.

Furthermore, they constructed moduli spaces of stable sheaves that were smooth, irreducible quasiprojective varieties. More generally, they proved that semistable sheaves were in a one to one correspondence with arbitrary unitary representations, and the associated moduli spaces were normal irreducible projective varieties, that were seen as natural compactifications of the moduli spaces of stable sheaves.

In this thesis, we will give a construction of the moduli spaces of semistable and stable sheaves using a method due to Simpson (see [Sim94]). This approach has the advantage of being easily generalizable to base schemes of higher dimension.

We will now give a brief description of the contents of the thesis:

- Chapter 1 contains the basic definitions and results from the theory of algebraic groups that are needed for the rest of the thesis
- In chapter 2 we introduce the notion of categorical quotient by an action of an algebraic group and study its main properties. We also define some stronger notions of quotients called good and good geometric quotients
- In chapter 3 we motivate the definition of reductive algebraic group and prove that there are always good quotients for actions of reductive algebraic groups on affine schemes
- In chapter 4 we introduce the concept of linearization of an algebraic group action in terms of equivariant sheaves. The notion of linearization allows one to define when a point is stable or semistable with respect to an algebraic group action. With these technical concepts, we generalize the results of chapter 3 and prove that, for an action of a reductive algebraic group on a finite type scheme and with respect to a fixed linearization of the action, there exists a good (resp. good geometric) quotient of the set of semistable (resp. stable) points of the action
- In chapter 5 we prove a criterion to decide if a point of a projective scheme is semistable or stable with respect to an action of a reductive algebraic group
- In chapter 6 we prove the so-called Luna's étale slice theorem, a result concerning the local structure of the quotients studied in chapter 4. This result is very useful in the study of the local properties of moduli spaces

- In chapter 7 we prove the Riemann-Roch theorem for coherent sheaves on a smooth projective algebraic curve. This allows us to introduce many properties of coherent sheaves on curves that will be useful later
- In chapter 8 we introduce the notions of semistable and stable sheaves on a smooth projective algebraic curve and study their main properties. In particular, we prove that the notion of semistability of a family of coherent sheaves on a curve is, essentially, a necessary and sufficient condition for that family to be parameterized by the points of some scheme. This property shows the fundamental importance of semistable and stable sheaves in moduli problems
- Finally, in chapter 9 we describe the moduli problem of semistable sheaves over a smooth projective algebraic curve, and give a detailed construction of the moduli space using techniques from geometric invariant theory

By convention, we will use the word scheme to mean a connected scheme of finite type over an algebraically closed field k. Sometimes, however, we will emphasize that the schemes considered are of finite type (for example, in chapter 4).

We will use the word variety to mean a separated scheme of finite type over an algebraically closed field.

If X is an integral scheme and x_0 is its generic point, we will denote $k(X) = \mathcal{O}_{X,x_0}$.

If X is a scheme, we will denote by $X^\bullet$ its functor of points. For any k-algebra A, we will denote $X^\bullet(A) = X^\bullet(\operatorname{Spec} A)$.

If $\mathcal{F}$ is a sheaf on X, we will denote $h^i(\mathcal{F}) = \dim_k H^i(X, \mathcal{F})$. We will denote with lower case letters the dimension of the space of global sections of a sheaf. For example, we will write $\hom(\mathcal{F}, \mathcal{G}) = \dim_k \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ or $\operatorname{ext}^i(\mathcal{F}, \mathcal{G}) = \dim_k \operatorname{Ext}^i(\mathcal{F}, \mathcal{G})$.

We will denote by Sch the category of schemes, Grp the category of groups and Sets the category of sets

Part I

Geometric invariant theory

Chapter 1

Actions of algebraic groups

In this chapter we will define what is an algebraic group, study some of their main properties and present some examples. We will also define the notion of an action of an algebraic group on a scheme and see how the usual notions associated to group actions (orbits, stabilizers...) behave in algebraic geometry. We will finish this chapter by studying the algebro-geometric structure of the orbits of an action. Some references for the topics covered in this chapter are [Mil17, Chapters 1, 2 and 3] and [Bor91, Chapter I]

1.1 Basic notions of algebraic groups

Definition 1.1.1. *A scheme* G *is an algebraic group if there are morphisms of schemes* $m : G \times G \to G$, $i : G \to G$ *and* $e : \operatorname{Spec} k \to G$ *such that the following diagrams are commutative*

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{(\mathrm{Id}_G, m)} & G \times G \\ \Big\downarrow{\scriptstyle (m,\mathrm{Id}_G)} & & \Big\downarrow{\scriptstyle m} \\ G \times G & \xrightarrow[m]{} & G \end{array} \qquad \begin{array}{ccccc} \operatorname{Spec} k \times G & \xrightarrow{(e,\mathrm{Id}_G)} & G \times G & \xleftarrow{(\mathrm{Id}_G, e)} & G \times \operatorname{Spec} k \\ & {\scriptstyle \simeq}\searrow & \Big\downarrow{\scriptstyle m} & \swarrow{\scriptstyle \simeq} & \\ & & G & & \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{Id}_G, i)} & G \times G & \xleftarrow{(i,\mathrm{Id}_G)} & G \\ \Big\downarrow & & \Big\downarrow{\scriptstyle m} & & \Big\downarrow \\ \operatorname{Spec} k & \xrightarrow[e]{} & G & \xleftarrow[e]{} & \operatorname{Spec} k \end{array}$$

where the vertical morphisms $G \to \operatorname{Spec} k$ *in the bottom diagram are the structure morphism of* G *as a* k*-scheme*

Observation 1.1.1. *Let* (G, m, i, e) *be an algebraic group. We will usually call* $m : G \times G \to G$ *the multiplication operation of the group,* $i : G \to G$ *the inversion operation and* $e : \operatorname{Spec} k \to G$ *the identity element of the group. When there is no danger of confusion, we will simply say that* G *is an algebraic group, without making any reference to* m, i *or* e

We will now give an alternative presentation of algebraic groups that is more natural for many purposes

Lemma 1.1.1. *Let* G *be an algebraic group and* $G^\bullet : \mathrm{Sch} \to \mathrm{Sets}$ *the functor of points of* G. *There is a unique functor* $\mathcal{F}_G : \mathrm{Sch} \to \mathrm{Grp}$ *from the category of schemes to the category of groups such that the diagram of functors*

$$\begin{array}{ccc} \mathrm{Sch} & \xrightarrow{G^\bullet} & \mathrm{Sets} \\ {\scriptstyle \mathcal{F}_G}\downarrow & \nearrow{\scriptstyle \mathrm{for}} & \\ \mathrm{Grp} & & \end{array}$$

is commutative, where $\mathrm{for} : \mathrm{Grp} \to \mathrm{Sets}$ *is the forgetful functor*

Proof. Let $m : G \times G \to G$ and $i : G \to G$ be the group operations in G. Let S be a scheme. For any $f, g \in G^\bullet(S) = \mathrm{Hom}(S, G)$, define

$$\begin{aligned} f + g &:= m \circ (f, g) \\ -f &:= f \circ i \\ 0 &:= e \circ \mathrm{str}, \text{ where } \mathrm{str} : S \to \mathrm{Spec}\, k \text{ is the structure morphism of } S \end{aligned}$$

It's easy to prove that $(G^\bullet(S), +)$ is a group with identity element 0 and such that the inverse element of any $f \in G^\bullet(S)$ is $-f$. For example, given $f, g, h \in G^\bullet(S)$ we have that

$$\begin{aligned} (f+g)+h &= m \circ (f+g, h) = \\ &= m \circ (m \circ (f, g), h) = \\ &= (m \circ (m, \mathrm{Id}_G)) \circ (f, g, h) = \\ &= (m \circ (\mathrm{Id}_G, m)) \circ (f, g, h) = \\ &= m \circ (f, m \circ (g, h))) = \\ &= f + (g+h) \end{aligned}$$

so $+$ is associative in $G^\bullet(S)$. The rest of the group axioms for $G^\bullet(S)$ follow similarly.

We define $\mathcal{F}_G(S) := (G^\bullet(S), +)$. Clearly, this defines a functor $\mathcal{F}_G : \mathrm{Sch} \to \mathrm{Grp}$ that satisfies the conditions of the lemma □

Definition 1.1.2. *Let* (G, m_G, i_G, e_G) *and* (H, m_H, i_H, e_H) *be algebraic groups. A morphism of schemes* $\varphi : G \to H$ *is a morphism of algebraic groups if the following diagrams are commutative*

$$\begin{array}{ccc} G \times G & \xrightarrow{\varphi \times \varphi} & H \times H \\ {\scriptstyle m_G}\downarrow & & \downarrow{\scriptstyle m_H} \\ G & \xrightarrow[\varphi]{} & H \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ {\scriptstyle e_G}\uparrow & \nearrow{\scriptstyle e_H} & \\ \mathrm{Spec}\, k & & \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ {\scriptstyle i_G}\downarrow & & \downarrow{\scriptstyle i_H} \\ G & \xrightarrow[\varphi]{} & H \end{array}$$

We will denote by AlgGrp the category of algebraic groups with morphisms given by definition 1.1.2. The next proposition will allow us to work with algebraic groups using the language of points with values in a scheme

Proposition 1.1.1. *There is an equivalence of categories*

$$\mathrm{AlgGrp} = \begin{matrix} \textit{Representable functors} \\ \mathrm{Sch} \to \mathrm{Grp} \end{matrix}$$
$$G \mapsto \mathcal{F}_G$$

where $\mathcal{F}_G : \mathrm{Sch} \to \mathrm{Grp}$ *is the group-valued functor associated to* G*, as defined in lemma 1.1.1*

Proof. Let $\mathcal{F} : \mathrm{Sch} \to \mathrm{Grp}$ be a representable functor. Then, there is a scheme X such that $\mathcal{F} = X^\bullet$. Let's prove that X has an algebraic group structure such that $F = \mathcal{F}_X$.

We have to give group operations $m : X \times X \to X$, $i : X \to X$ and a closed point $e : \operatorname{Spec} k \to X$ making the diagrams in definition 1.1.1 commutative. Since $(X \times X)^\bullet = X^\bullet \times X^\bullet$, by Yoneda's lemma we just have to give morphisms of functors $m^\bullet : X^\bullet \times X^\bullet \to X^\bullet$, $i^\bullet : X^\bullet \to X^\bullet$ and $e^\bullet : (\operatorname{Spec} k)^\bullet \to X^\bullet$ making the diagrams of definition 1.1.1 commutative at the level of valued points.

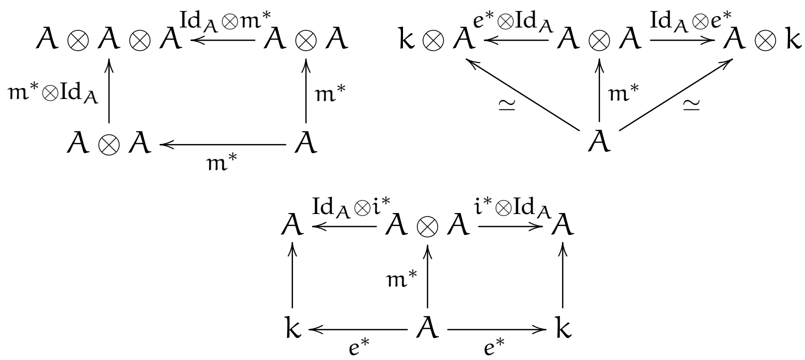
For every scheme S, by definition there is a group operation $+_S : X^\bullet(S) \times X^\bullet(S) \to X^\bullet(S)$. We can define $m^\bullet_S := +_S$. We define $i^\bullet$ and $e^\bullet$ similarly. It's now easy to prove that (X, m, i, e) is an algebraic group such that $\mathcal{F}_X = \mathcal{F}$ □

Observation 1.1.2. *If* G *is an algebraic group and* $\mathcal{F}_G$ *is the group-valued functor associated to* G*, we will denote* $\mathcal{F}_G = G^\bullet$

Definition 1.1.3. *An algebraic group* G *is affine if* G *is an affine scheme*

Since the category of affine schemes is anti-equivalent to the category of commutative rings, we can define the notion of an affine algebraic group solely in terms of the ring of regular functions of the scheme. The notion that arises is that of a Hopf algebra

Definition 1.1.4. *Let* A *be a finite type* k*-algebra.* A *is a Hopf algebra if there are* k*-algebra homomorphisms* $m^* : A \to A \otimes_k A$, $i^* : A \to A$ *and* $e^* : A \to k$ *such that the following diagrams are commutative*



where the vertical morphisms $k \to A$ *in the bottom diagram are given by the structure morphism of* A *as a* k*-algebra*

Observation 1.1.3. *Note that the diagrams of definition 1.1.4 are, essentially, dual to the diagrams of definition 1.1.1*

We can define a morphism of Hopf algebras as a morphism of k-algebras that preserves the Hopf algebra structures in the obvious way. We will denote by HopfAlg the category of Hopf algebras and by AffAlgGrp the category of affine algebraic groups

Lemma 1.1.2. *There is an anti-equivalence of categories*

$$\begin{aligned} \text{HopfAlg} &= \text{AffAlgGrp} \\ (A, m^*, i^*, e^*) &\mapsto (\operatorname{Spec} A, m, i, e) \\ (\mathcal{O}(G), m^*, i^*, e^*) &\leftarrow (G, m, i, e) \end{aligned}$$

where we denote by $f^* : B \to A$ *the* k*-algebra homomorphism induced by a morphism of affine schemes* $f : \operatorname{Spec} A \to \operatorname{Spec} B$

Proof. Let (G, m, i, e) be an affine algebraic group. G is, as an affine scheme, completely determined by its ring of regular functions $\mathcal{O}(G)$, and the morphisms of schemes $m : G \times G \to G$, $i : G \to G$ and $e : \operatorname{Spec} k \to G$ induce k-algebra homomorphisms

$$\begin{aligned} m^* : \mathcal{O}(G \times G) \simeq \mathcal{O}(G) \otimes \mathcal{O}(G) &\to \mathcal{O}(G) \\ i^* : \mathcal{O}(G) &\to \mathcal{O}(G) \\ e^* : \mathcal{O}(G) &\to k \end{aligned}$$

that clearly make commutative the diagrams of definition 1.1.1 on regular functions. These diagrams are exactly the ones in definition 1.1.4, so $(\mathcal{O}(G), m^*, i^*, e^*)$ is a Hopf algebra. The same argument proves that every Hopf algebra defines an affine algebraic group □

We will now give some basic examples of algebraic groups

Example 1.1.1. *Denote* $G_a := \operatorname{Spec} k[t]$. G_a *is an affine algebraic group, called the additive group. By lemma 1.1.2, giving the structure of an algebraic group on* G_a *is the same as giving a Hopf algebra structure on* $k[t]$. *Consider the following* k*-algebra homomorphisms*

$$\begin{aligned} m^* : k[t] &\to k[t] \otimes_k k[t] \\ t &\mapsto t \otimes 1 + 1 \otimes t \\ i^* : k[t] &\to k[t] \\ t &\mapsto -t \\ e^* : k[t] &\to k \\ t &\mapsto 0 \end{aligned}$$

it's easy to check that $(k[t], m^*, i^*, e^*)$ *is a Hopf algebra. For example, let's prove that* $(\mathrm{Id}_A \otimes e^*) \circ m^* = \mathrm{Id}_A$. *We have that*

$$\begin{aligned} ((\mathrm{Id}_A \otimes e^*) \circ m^*)(t) &= (\mathrm{Id}_A \otimes e^*)(t \otimes 1 + 1 \otimes t) = \\ &= t \otimes 1 + 1 \otimes 0 = \\ &= t \otimes 1 = \\ &= (\textit{via } A \otimes k \simeq A) = \\ &= t \end{aligned}$$

This proves that (G_a, m, i, e) *is an affine algebraic group. For every* k*-algebra* $(A, +, \cdot)$*, we have that*

$$G_a^\bullet(A) = \mathrm{Hom}_{k-alg}(k[t], A) = (A, +)$$

In virtue of proposition 1.1.1, this could have been an alternative definition of the additive group: the algebraic group representing the group-valued functor that assigns to every k*-algebra* $(A, +, \cdot)$ *its additive group* $(A, +)$

Example 1.1.2. *Denote* $G_m := \mathrm{Spec}\, k[t, t^{-1}]$*.* G_m *is an affine algebraic group called the multiplicative group. We can give a Hopf algebra structure on* $k[t, t^{-1}]$ *by considering the following* k*-algebra homomorphisms*

$$\begin{aligned} m^* : k[t, t^{-1}] &\to k[t, t^{-1}] \otimes k[t, t^{-1}] \\ t &\mapsto t \otimes t \\ i^* : k[t, t^{-1}] &\to k[t, t^{-1}] \\ t &\mapsto t^{-1} \\ e^* : k[t, t^{-1}] &\to k \\ t &\mapsto 1 \end{aligned}$$

it's easy to check that $(k[t, t^{-1}], m^*, i^*, e^*)$ *is a Hopf algebra, so* (G_m, m, i, e) *is an affine algebraic group. For every* k*-algebra* $(A, +, \cdot)$*, we have that*

$$G_m^\bullet(A) = \mathrm{Hom}_{k-alg}(k[t, t^{-1}], A) = (A^\times, \cdot)$$

because every k*-algebra homomorphism* $k[t, t^{-1}] \to A$ *is determined by the image of* t *in* A*, and the image of* t *must be invertible in* A*, because* t *is invertible in* $k[t, t^{-1}]$*. The multiplicative group* G_m *is, then, the representative of the group-valued functor that assigns to every* k*-algebra* $(A, +, \cdot)$ *its multiplicative group* $(A^\times, \cdot)$

Example 1.1.3. *Let* E *be a* k*-vector space. Consider the group-valued functor*

$$\begin{aligned} \underline{\mathrm{GL}}(E) : \mathrm{Alg}_k &\to \mathrm{Grp} \\ A &\mapsto \underline{\mathrm{GL}}(E)(A) := \mathrm{Aut}_{A-mod}(A \otimes_k E) \end{aligned}$$

where Alg_k *denotes the category of* k*-algebras. The representative of* $\underline{\mathrm{GL}}(E)$*, if it exists, is an affine algebraic group denoted by* $\mathrm{GL}(E)$ *and called the general linear group of* E*.*

For example, $\underline{\mathrm{GL}}(E)$ *is representable if* E *is finite-dimensional over* k*. Suppose in particular that* $E = k^n$*. Then* $\mathrm{GL}(k^n)$ *will be denoted by* $\mathrm{GL}(k^n) = \mathrm{GL}(n, k)$ *and* $\mathrm{GL}(n, k)$ *is, as a scheme, the open subset of* $\mathrm{Spec}\, k[\{x_{ij}\}_{i,j=1,\dots,n}]$ *given by the complement of the zero set of the determinant* $\det \in k[\{x_{ij}\}_{i,j=1,\dots,n}]$*. In other words,*

$$\mathrm{GL}(n, k) = \mathrm{Spec}\, k[\{x_{ij}\}_{i,j=1,\dots,n}]_{\det}$$

and the group operations are the ones induced by the usual multiplication of matrices

Example 1.1.4. *The usual closed subgroups of the general linear group are algebraic groups. For example, the special linear group* $\mathrm{SL}(n, k) := \{\det - 1 = 0\}$ *is a closed algebraic subgroup of* $\mathrm{GL}(n, k)$*. It's not hard to define the classical groups* $\mathrm{PGL}(n, k)$*,* $\mathrm{PSL}(n, k)$*,* $\mathrm{O}(n, k)$*,* $\mathrm{SO}(n, k)$*... in the category of affine algebraic groups*

Example 1.1.5. *A very important family of examples of algebraic groups is the one given by abelian varieties. An abelian variety is a separated algebraic group that is proper over* k*. For example, every elliptic curve (smooth projective algebraic curve of genus 1 with a rational point) is an abelian variety. For more on abelian varieties, see [Mum70]*

For more examples of algebraic groups, see [Mil17, Chapter 2].

We are now going to study some basic properties of algebraic groups

Lemma 1.1.3. *Every algebraic group is a separated scheme*

Proof. Let G be an algebraic group and denote by $\delta_G = (\mathrm{Id}_G, \mathrm{Id}_G) : G \to G \times G$ the diagonal morphism of G. It's easy to check that there is a cartesian diagram

$$\begin{array}{ccc} G & \longrightarrow & \operatorname{Spec} k \\ {\scriptstyle \delta_G}\downarrow & & \downarrow{\scriptstyle e} \\ G \times G & \xrightarrow[m \circ (\mathrm{Id}_g \times i)]{} & G \end{array}$$

where the horizontal top arrow $G \to \operatorname{Spec} k$ is the structure morphism of G as a k-scheme.

By definition, the identity element $e : \operatorname{Spec} k \to G$ is a closed point, and thus the diagonal $\delta_G : G \to G \times G$ is the base change of a closed immersion, hence a closed immersion, so G is a separated scheme □

Algebraic groups come with many automorphisms

Definition 1.1.5. *Let* G *be an algebraic group and let* $g : \operatorname{Spec} k \to G$ *be a closed point. The left translation by* g *is the morphism of schemes* $L_g : G \to G$ *defined by* $L_g := m \circ (g, \mathrm{Id}_G)$

Observation 1.1.4. *Note that, by proposition 1.1.1, the set* $G^\bullet(k)$ *of closed points of* G *is a group*

Lemma 1.1.4. *The map*

$$\begin{aligned} G^\bullet(k) &\to \mathrm{Aut}_{k-\mathrm{sch}}(G) \\ g &\mapsto L_g \end{aligned}$$

is a group homomorphism. The induced left action of $G^\bullet(k)$ *on* G *is transitive when restricted to* $G^\bullet(k)$

Proof. For every $g, h \in G^\bullet(k)$, denote $i(g) = g^{-1}$ and $m(g, h) = gh$. Then it's easy to prove that, for every $g_1, g_2, g \in G^\bullet(k)$, we have that

$$\begin{aligned} L_{g_1 g_2} &= L_{g_1} \circ L_{g_2} \\ (L_g)^{-1} &= L_{g^{-1}} \\ L_e &= \mathrm{Id}_G \end{aligned}$$

so $G^\bullet(k) \to \mathrm{Aut}_{k-\mathrm{sch}}(G)$ is a group homomorphism. For any $g_1, g_2 \in G^\bullet(k)$, we have that

$$g_2 = (g_1 \cdot g_2^{-1}) \cdot g_2 = L_{g_1 g_2^{-1}}(g_2)$$

and thus the restriction of the induced left action to $G^\bullet(k)$ is transitive □

The fact that $G^{\bullet}(k)$ acts transitively on itself has many interesting consequences. Schemes satisfying the condition that their groups of automorphisms act transitively on their sets of closed points are called homogeneous schemes. As an example, we have the following result

Proposition 1.1.2 ([Mil17] Proposition 1.18)**.** *Let G be an algebraic group. Then*

$$G \textit{ is reduced} \Leftrightarrow G \textit{ is smooth}$$

Observation 1.1.5. *If $\mathrm{char}(k) = 0$, then it can be proven that every algebraic group is reduced ([Mil17, Corollary 10.36]) so, by lemma 1.1.3 and proposition 1.1.2, every algebraic group defined over a field of characteristic zero is a smooth algebraic variety*

1.2 Actions of algebraic groups

A good reference for this section is [Mil17, Chapter 9]

Definition 1.2.1. *Let X be a scheme and G an algebraic group. A left action of G on X is a morphism of schemes $\sigma : G \times X \to X$ making the following diagrams commutative*

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{Id}_G \times \sigma} & G \times X \\ \downarrow{\scriptstyle m \times \mathrm{Id}_X} & & \downarrow{\scriptstyle \sigma} \\ G \times X & \xrightarrow[\sigma]{} & X \end{array} \qquad \begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ \uparrow{\scriptstyle e \times \mathrm{Id}_X} & & \uparrow{\scriptstyle \mathrm{Id}_X} \\ \mathrm{Spec}\, k \times X & \xrightarrow[\simeq]{} & X \end{array}$$

Right actions of an algebraic group can be defined in a completely analogous way. In general, we will always work with left actions unless we say the contrary. A pair (X, σ) given by a scheme X and an algebraic group action $\sigma : G \times X \to X$ will be called a G-scheme

Definition 1.2.2. *Let (X, σ_X) and (Y, σ_Y) be G-schemes. A morphism of schemes $f : X \to Y$ is G-equivariant if the square*

$$\begin{array}{ccc} G \times X & \xrightarrow{\mathrm{Id}_G \times f} & G \times Y \\ \downarrow{\scriptstyle \sigma_X} & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is commutative.

If G acts trivially on Y (i.e. if $\sigma_Y : G \times Y \to Y$ is the natural projection on Y) and $f : X \to Y$ is G-equivariant, we will say that f is G-invariant

We will denote by Sch_G the category of G-schemes with G-equivariant morphisms.

For almost every purpose, it's easier to work with actions of algebraic groups on schemes defined in a functorial way

Definition 1.2.3. *Let* G *be an algebraic group and* X *a scheme. A functorial action of the group-valued functor* $G^\bullet : \mathrm{Sch} \to \mathrm{Grp}$ *(recall observation 1.1.2) on* $X^\bullet : \mathrm{Sch} \to \mathrm{Sets}$ *is a morphism of functors*

$$\sigma^\bullet : G^\bullet \times X^\bullet \to X^\bullet$$

such that, for any scheme S*,* $\sigma^\bullet_S : G^\bullet(S) \times X^\bullet(S) \to X^\bullet(S)$ *is an action of* $G^\bullet(S)$ *on* $X^\bullet(S)$

Definition 1.2.4. *Let* G *be an algebraic group and let* X *and* Y *be schemes. Let* $\sigma^\bullet_X : G^\bullet \times X^\bullet \to X^\bullet$ *and* $\sigma^\bullet_Y : G^\bullet \times Y^\bullet \to Y^\bullet$ *be functorial actions of* $G^\bullet$ *on* $X^\bullet$ *and* $Y^\bullet$ *respectively. A morphism of schemes* $f : X \to Y$ *is functorially* $G^\bullet$*-equivariant if, for every scheme* S *and every* $p \in X^\bullet(s)$*,* $g \in G^\bullet(S)$*, we have that*

$$f^\bullet_S(\sigma^\bullet_{X,S}(g, x)) = \sigma^\bullet_{Y,S}(g, f^\bullet_S(x))$$

where $f^\bullet : X^\bullet \to Y^\bullet$ *is the morphism of functors induced by* $f : X \to Y$*.*

$f : X \to Y$ *is functorially* $G^\bullet$*-invariant if* f *is functorially* $G^\bullet$*-equivariant and the functorial action of* $G^\bullet$ *on* $Y^\bullet$ *is trivial (i.e. if* $\sigma^\bullet_Y : G^\bullet \times Y^\bullet \to Y^\bullet$ *is the natural projection on* $Y^\bullet$*)*

Denote by FunSch_G the category of functors of points of schemes with a functorial $G^\bullet$-action and functorially $G^\bullet$-equivariant morphisms

Proposition 1.2.1. *There is an equivalence of categories*

$$\mathrm{Sch}_G = \mathrm{FunSch}_G$$
$$(X, \sigma) \mapsto (X^\bullet, \sigma^\bullet)$$

where $\sigma^\bullet : (G \times X)^\bullet = G^\bullet \times X^\bullet \to X^\bullet$ *is the morphism of functors induced by* $\sigma : G \times X \to X$

Proof. The proof of this proposition follows easily by Yoneda's lemma using the same ideas as in proposition 1.1.1 □

The action of an algebraic group on a scheme induces a notion of functions that are invariant with respect to the action. We will now introduce some basic definitions in regard to this concept

Definition 1.2.5. *Let* (G, m, i, e) *be an affine algebraic group and let* A *be a* k*-algebra (resp. a vector space). A dual action of* G *on* A *is a* k*-algebra homomorphism (resp. a* k*-linear homomorphism)* $\sigma^* : A \to \mathcal{O}(G) \otimes A$ *making the following diagrams commutative*

$$\begin{array}{ccc} A & \xrightarrow{\sigma^*} & \mathcal{O}(G) \otimes A \\ \downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle m^* \otimes \mathrm{Id}_A} \\ \mathcal{O}(G) \otimes A & \xrightarrow[\sigma^* \otimes \mathrm{Id}_A]{} & \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes A \end{array} \qquad \begin{array}{ccc} k \otimes A \simeq A & \xrightarrow{\mathrm{Id}_A} & A \\ \downarrow{\scriptstyle e^* \otimes \mathrm{Id}_A} & \nearrow{\scriptstyle \sigma^*} & \\ \mathcal{O}(G) \otimes A & & \end{array}$$

Definition 1.2.6. *Let* G *be an affine algebraic group and let* (A, σ^*) *be a* k*-algebra (resp. a* k*-vector space) with a dual action of* G*. An element* $a \in A$ *is* G*-invariant if* $\sigma^*(a) = 1 \otimes a$

Definition 1.2.7. *Let* $B \subseteq A$ *be a subalgebra (resp. a subspace) of* A*.* B *is* G*-invariant or* G*-stable with respect to the dual action* σ^* *on* A *if* $\sigma^*(B) \subseteq \mathcal{O}(G) \otimes B$

We state without proof the following easy result

Lemma 1.2.1. *Let* G *be an affine algebraic group and let* (A, σ^*) *be a* k*-algebra (resp. a* k*-vector space) with a dual action of* G*. Then, the set of* G*-invariant elements of* A

$$A^G := \{a \in A : \sigma^*(a) = 1 \otimes a\}$$

is a subalgebra (resp. a vector subspace) of A

There is a natural notion of a morphism between k-algebras that have a dual action of an affine algebraic group. We denote by Alg_G the category of k-algebras with a dual action of the affine algebraic group G.

Note that the diagrams in definition 1.2.5 are dual to the diagrams that appear in definition 1.2.1. This observation yields the following

Proposition 1.2.2. *Let* G *be an affine algebraic group. There is an anti-equivalence of categories*

$$\begin{aligned} \mathrm{AffSch}_G &= \mathrm{Alg}_G \\ (X, \sigma) &\mapsto (\mathcal{O}(X), \sigma^*) \end{aligned}$$

Proof. Let (X, σ) be an affine G-scheme and let $\sigma^* : \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X)$ be the k-algebra homomorphism induced by σ. Since σ makes the diagrams of definition 1.2.1 commutative, it's easy to check that σ^* makes the diagrams of definition 1.2.5 commutative, and thus $(\mathcal{O}(X), \sigma^*)$ is a k-algebra equipped with a dual action. On the other hand, every k-algebra with a dual action of G induces, using the anti-equivalence of commutative rings and affine schemes, an action of G on its spectrum $\square$

Definition 1.2.8. *Let* G *be an algebraic group and* (X, σ) *a* G*-scheme. A subscheme* $Z \subseteq X$ *is* G*-invariant if* $Z^\bullet$ *is invariant with respect to the induced functorial action* $\sigma^\bullet : G^\bullet \times X^\bullet \to X^\bullet$*, i.e., if there is a functorial action* $\widetilde{\sigma}^\bullet : G^\bullet \times Z^\bullet \to Z^\bullet$ *such that the following diagram of functors commutes*

$$\begin{array}{ccc} G^\bullet \times X^\bullet & \xrightarrow{\sigma^\bullet} & X^\bullet \\ {\scriptstyle \mathrm{Id}_{G^\bullet} \times i^\bullet} \uparrow & & \uparrow {\scriptstyle i^\bullet} \\ G^\bullet \times Z^\bullet & \xrightarrow[\widetilde{\sigma}^\bullet]{} & Z^\bullet \end{array}$$

where $i : Z \hookrightarrow X$ *is the natural inclusion of* Z *in* X

We will now define the notion of an invariant function with respect to an action of an affine algebraic group

Lemma 1.2.2. *Let* G *be an affine algebraic group and* (X, σ) *a* G*-scheme. Let* $X^{\mathrm{aff}} = \mathrm{Spec}\, \mathcal{O}(X)$ *be the affine scheme associated to* X*. There is a unique action* $\widetilde{\sigma} : G \times X^{\mathrm{aff}} \to X^{\mathrm{aff}}$ *of* G *on* X *such that the following diagram commutes*

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ {\scriptstyle \mathrm{Id}_G \times p} \downarrow & & \downarrow {\scriptstyle p} \\ G \times X^{\mathrm{aff}} & \xrightarrow[\widetilde{\sigma}]{} & X^{\mathrm{aff}} \end{array}$$

where $p : X \to X^{\mathrm{aff}}$ *is the natural morphism of* X *to its asociated affine scheme* X^{aff}

Proof. We have a morphism $p \circ \sigma : G \times X \to X^{aff}$ of $G \times X$ on an affine scheme X^{aff}. It's easy to check that $(G \times X)^{aff} = G \times X^{aff}$, since $G^{aff} = G$ because G is an affine scheme. Using the universal property of the associated affine scheme (see [Liu02, Chapter 2, Proposition 3.25]), there is a unique morphism of schemes $\tilde{\sigma} : G \times X^{aff} \to X^{aff}$ making the diagram of the lemma commutative. It's easy to check that $\tilde{\sigma}$ is an action of G on X^{aff} □

From the definition of an invariant subscheme, lemma 1.2.2 and proposition 1.2.2, it follows that if G is an affine algebraic group acting on a scheme X and $U \subseteq X$ is a G-invariant open subset of X, then $\mathcal{O}(U)^G$ is well defined, so it makes sense to speak about functions that are invariant with respect to an algebraic group action.

We will now define some of the basic concepts associated to group actions in the context of algebraic geometry

Let G be an algebraic group

Definition 1.2.9. *Let (X, σ) be a G-scheme and $x \in X^\bullet(k)$. Let $(\mathrm{Id}_G, x) : G \to G \times X$ and $\sigma_x := \sigma \circ (\mathrm{Id}_G, x)$. The orbit of x by the G-action is*

$$G \cdot x := \mathrm{Im}(\sigma_x) \subseteq X$$

Definition 1.2.10. *Let (X, σ) be a G-scheme, $x \in X^\bullet(k)$ and $\sigma_x : G \to X$ as in definition 1.2.9. The stabilizer subgroup of G at x, denoted by G_x, is*

$$G_x := \sigma_x^{-1}(x) = G \times_X \{x\}$$

Lemma 1.2.3. *G_x is a closed algebraic subgroup of G*

Proof. By definition, the following diagram is cartesian

$$\begin{array}{ccc} G_x & \longrightarrow & G \\ \downarrow & & \downarrow \sigma_x \\ \mathrm{Spec}\, k & \xrightarrow[x]{} & X \end{array}$$

so the top horizontal arrow $G_x \to G$ is the base change of a closed embedding $x : \mathrm{Spec}\, k \to X$, and thus it's a closed embedding. On the other hand, it's easy to prove that, for every scheme S, we have that

$$G_x^\bullet(S) = \{g \in G^\bullet(S) : g \cdot x = x\} \tag{1.1}$$

where $x : \mathrm{Spec}\, k \to X$ is thought as an element of $X^\bullet(S)$ via the top horizontal arrow of the following diagram

$$\begin{array}{ccc} S & \longrightarrow & X \\ \downarrow & \nearrow_{x} & \\ \mathrm{Spec}\, k & & \end{array}$$

and the vertical arrow $S \to \mathrm{Spec}\, k$ is the structure morphism of the k-scheme S (in other words, x is thought as a S-valued point by considering it as the constant morphism $S \to X$ equal to x). Equation 1.1 clearly shows that $G_x^\bullet$ is a group-valued functor, and hence G_x is an algebraic group in virtue of proposition 1.1.1 □

The algebro-geometric structure of the orbits of an action of an algebraic group on a scheme is given by the following

Proposition 1.2.3. *Let* (X, σ) *be a* G*-scheme and* $x \in X^\bullet(k)$

(a) $G \cdot x$ *is a locally closed subset of* X*. In particular, it has a unique structure of reduced subscheme of* X

(b) The boundary $\overline{G \cdot x} - G \cdot x$ *is an union of* G*-orbits of strictly less dimension than* $G \cdot x$*. In particular,* $\overline{G \cdot x} - G \cdot x$ *contains, at least, one closed orbit of minimal dimension*

Proof. It suffices to prove that $G \cdot x$ is an open subset of $\overline{G \cdot x}$. Chevalley's theorem ensures the existence of an open subset $U \subseteq \overline{G \cdot x}$, dense in $\overline{G \cdot x}$, such that

$$U \subseteq G \cdot x \subseteq \overline{G \cdot x}$$

(see [An12, Lemma 2.1]). By definition, for every closed point $z \in G \cdot x$ there is some $g \in G^\bullet(k)$ such that $g \cdot x = z$. In particular, $z \in \sigma_g(U)$, and $\sigma_g(U)$ is an open subset of $\overline{G \cdot x}$ because $\sigma_g : X \to X$ is an automorphism of X. This proves that

$$G \cdot x = \bigcup_{g \in G^\bullet(k)} \sigma_g(U)$$

and thus $G \cdot x$ is an open subset of $\overline{G \cdot x}$. In particular, the G-orbit $G \cdot x$ is a locally closed subset of X and thus it has a unique structure of reduced subscheme.

On the other hand, since $G \cdot x$ is a G-invariant open subset of $\overline{G \cdot x}$, then the boundary $\overline{G \cdot x} - G \cdot x$ is also G-invariant, so we have that

$$\overline{G \cdot x} - G \cdot x = \bigcup_{z \in \overline{G \cdot x} - G \cdot x} G \cdot z$$

and these G-orbits are clearly of less dimension than $G \cdot x$ because they are contained in the complement of an open dense subset $G \cdot x \subseteq \overline{G \cdot x}$. In particular, $\overline{G \cdot x} - G \cdot x$ contains orbits of minimal dimension, and these orbits must be closed because if they were not closed then they would contain orbits of less dimension in their boundaries, and that's impossible because we had already assumed that their dimension is minimal □

Lemma 1.2.4. *Let* (X, σ) *be a* G*-scheme and* $x \in X^\bullet(k)$*. Then* $\sigma_x : G \to G \cdot x$ *is a flat morphism*

Proof. This morphism takes values in the reduced scheme $G \cdot x$. By generic flatness (see for example [GW10, Corollary 10.85]) there is a non empty dense open subset $U \subseteq G \cdot x$ such that $\sigma_x^{-1}(U) \to U$ is a flat morphism. The action of G on $G \cdot x$ is transitive, so we can cover $G \cdot x$ by the translates $\sigma_g(U)$, and for each g we have that $\sigma_x^{-1}(\sigma_g(U)) \to \sigma_g(U)$ is flat because it is the base change of a flat morphism $\sigma_x^{-1}(U) \to U$. This proves that $\sigma_x : G \to G \cdot x$ is locally flat, hence flat □

Proposition 1.2.4. *Let* (X, σ) *be a* G*-scheme. We have that*

(a) *For every* $x \in X^\bullet(k)$,

$$\dim G = \dim G_x + \dim (G \cdot x)$$

(b) *For every* $n \geqslant 0$, *the set* $\{x \in X^\bullet(k) : \dim G_x \geqslant n\}$ *is the set of closed points of a closed subset of* X

Proof. By lemma 1.2.4, $\sigma_x : G \to G \cdot x$ is a flat morphism of schemes. By the formulas for the dimension of the fibers of a flat morphism (see for example [GW10, Corollary 14.95]), we have that

$$\dim (G \cdot x) = \dim G - \dim \sigma_x^{-1}(x) = \dim G - \dim G_x$$

and hence (a) follows. In an analogous way, (b) is deduced from the fact that for every $n \geqslant 0$ then $\{x \in X : \dim \sigma_x^{-1}(x) \geqslant n\} = \{x \in X : \dim G_x \geqslant n\}$ is closed (see [GW10, Theorem 14.110]) □

Chapter 2

Quotient by an algebraic group action

In this chapter we will define the notion of categorical quotient. These are the type of quotients that we will seek to construct using geometric invariant theory. We will study their main properties and give some examples. Some references for this chapter are [MFK94, Chapter 0], [Hos15, Chapter 3], [New78, Chapter 3] or [Dol03, Chapter 6]

2.1 Categorical quotients

Let $\mathcal{C}$ be a locally small category with products

Definition 2.1.1. *Let* X *be an object of* $\mathcal{C}$. *An equivalence relation on* X *is a morphism* $R \to X \times X$ *such that, for every object* A *of* $\mathcal{C}$, *the induced map*

$$R^{\bullet}(A) \to X^{\bullet}(A) \times X^{\bullet}(A)$$

is an ordinary equivalence relation on the set $X^{\bullet}(A)$

Definition 2.1.2. *Let* R *be an equivalence relation on* X. *If it exists, the colimit of the diagram*

$$\begin{array}{ccc} R & \longrightarrow & X \\ \downarrow & & \\ X & & \end{array}$$

is called the categorical quotient of X *by the equivalence relation* R

Observation 2.1.1. *Clearly, if it exists, the categorical quotient by an equivalence relation is unique*

For some applications that will be developed in the following chapters, it's convenient to have an alternative characterization of categorical quotients

Definition 2.1.3. *Let* $\mathcal{F} : \mathcal{C} \to \text{Sets}$ *be a contravariant functor and let* F *be an object of* $\mathcal{C}$. *Let* $\xi : \mathcal{F} \to F^{\bullet}$ *be a morphism of functors. The pair* (F, ξ) *corepresents* $\mathcal{F}$ *if, for every object* X *of* $\mathcal{C}$

and every morphism of functors $\eta : \mathcal{F} \to X^\bullet$, *there is a unique morphism* $f : F \to X$ *such that the following diagram of functors is commutative*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\eta} & X^\bullet \\ \downarrow \xi & \nearrow f^\bullet & \\ F^\bullet & & \end{array}$$

where $f^\bullet : F^\bullet \to X^\bullet$ *is the morphism of functors induced by* $f : F \to X$

Let X be an object of $\mathcal{C}$ and $R \to X \times X$ an equivalence relation on X. Consider the following functor

$$\underline{X/R} : \mathcal{C} \to \text{Sets}$$
$$A \mapsto \underline{X/R}(A) := X^\bullet(A)/R^\bullet(A)$$

Lemma 2.1.1. *An object* Q *in* $\mathcal{C}$ *is a categorical quotient for* X *by the equivalence relation* R *if and only* Q *corepresents the functor* $\underline{X/R} : \mathcal{C} \to \text{Sets}$

Proof. Let Y be an object of $\mathcal{C}$. Then, for every object A in $\mathcal{C}$ we have that

$$\text{Hom}(X^\bullet(A)/R^\bullet(A), Y^\bullet(A)) = \left\{ \text{Commutative diagrams } \begin{array}{ccc} R^\bullet(A) & \longrightarrow & X^\bullet(A) \\ \downarrow & & \downarrow \\ X^\bullet(A) & \longrightarrow & Y^\bullet(A) \end{array} \right\}$$

from this, and from the definition of the colimit of a diagram, we have that an object Q in $\mathcal{C}$ is a categorical quotient for the equivalence relation R on X if and only if it corepresents the functor $\underline{X/R} : \mathcal{C} \to \text{Sets}$ □

Example 2.1.1. *Let* $\mathcal{C} = \text{Top}$ *be the category of topological spaces. Let* G *be a topological group and* X *a topological space. Suppose that we have a continuous action* $\sigma : G \times X \to X$ *of* G *on* X. *Then,* σ *induces an equivalence relation on* X *in the following way*

$$\psi : R = G \times X \to X \times X$$
$$(g, x) \mapsto (x, \sigma(g, x))$$

Let $X/G := \{G$*-orbits of the action* σ *in* $X\}$ *and* $\pi : X \to X/G$ *be the natural map. Then, the topological space given by* X/G *with the quotient topology induced by* $\pi : X \to X/G$ *is the categorical quotient of* X *by the equivalence relation* $\psi : G \times X \to X \times X$

The previous example shows that categorical quotients for equivalence relations defined by group actions always exist in the category of topological spaces. Does the same happen in the category of schemes?

Let G be an algebraic group and X a scheme. Let $\sigma : G \times X \to X$ be an action of G on X. This action induces the following morphism of schemes

$$\psi := (\pi_X, \sigma) : G \times X \to X \times X$$

where $\pi_X : G \times X \to X$ is the natural projection on X. It's easy to check that $\psi : G \times X \to X \times X$ is an equivalence relation on X in the sense of definition 2.1.1

Definition 2.1.4. *A categorical quotient for the* G*-action* $\sigma : G \times X \to X$ *on* X *is a categorical quotient for the equivalence relation* $\psi = (\pi_X, \sigma) : G \times X \to X \times X$ *on* X

Observation 2.1.2. *Alternatively, it's easy to check that a categorical quotient for the* G*-action on* X *is a pair* (Y, π) *where* Y *is a scheme and* $\pi : X \to Y$ *is a* G*-invariant morphism satisfying the following universal property: for every scheme* Z *and every* G*-invariant morphism* $f : X \to Z$*, there is a unique morphism of schemes* $h : Y \to Z$ *such that* $h \circ \pi = f$

Suppose that $\pi : X \to Y$ is a categorical quotient for the G-action on X. It's easy to see that there is a unique morphism $\psi_Y : G \times X \to X \times_Y X$ such that the following diagram is commutative

$$\begin{array}{ccc} G \times X & \xrightarrow{\psi} & X \times X \\ {\scriptstyle \psi_Y}\downarrow & \nearrow & \\ X \times_Y X & & \end{array}$$

where the diagonal arrow $X \times_Y X \to X \times X$ is the natural morphism induced by $\pi : X \to Y$. If $\psi_Y : G \times X \to X \times_Y X$ is surjective then, for every $x, y \in X^\bullet(k)$ such that $\pi(x) = \pi(y)$, there is some $g \in G^\bullet(k)$ such that $g \cdot x = y$. That is, the morphism $\pi : X \to Y$ parameterizes G-orbits, i.e. for every $x \in X^\bullet(k)$ we have that

$$G \cdot x = \pi^{-1}(\pi(x))$$

In general, if $\pi : X \to Y$ is a (not necessarily surjective) categorical quotient, we can only prove that $\pi(G \cdot x) = \pi(x)$, but disjoint orbits may have the same image in Y via π. We will later encounter examples of this behaviour

Definition 2.1.5. *Let* (X, σ) *be a* G*-scheme and* $\pi : X \to Y$ *a categorical quotient for the* G*-action on* X*.* $\pi : X \to Y$ *is a geometric quotient if* π *and* $\psi_Y : G \times X \to X \times_Y X$ *are surjective*

Example 2.1.2. *Consider the following action of the multiplicative group* $G_m = \operatorname{Spec} k[t, t^{-1}]$ *on the affine line* $\mathbb{A}^1 = \operatorname{Spec} k[x]$

$$\begin{aligned} \sigma^\bullet : G_m^\bullet \times (\mathbb{A}^1)^\bullet &\to (\mathbb{A}^1)^\bullet \\ (\lambda, p) &\mapsto \sigma^\bullet(\lambda, p) := \lambda \cdot p \end{aligned}$$

where λ *and* p *are understood respectively as points of* G_m *and* $\mathbb{A}^1$ *with values in some scheme. Alternatively,* $\sigma : G_m \times \mathbb{A}^1 \to \mathbb{A}^1$ *is induced by the dual action of* G_m *on* $k[x]$ *defined as (recall proposition 1.2.2)*

$$\begin{aligned} \sigma^* : k[x] &\to k[x] \otimes k[t, t^{-1}] \\ x &\mapsto x \otimes t \end{aligned}$$

We will prove later that the categorical quotient of the affine line by this action is the structure morphism $\mathbb{A}^1 \to \operatorname{Spec} k$*. The orbits of this action in* $\mathbb{A}^1$ *are the punctured lines passing through the origin in* $(\mathbb{A}^1)^\bullet(k) = k$

Example 2.1.3. *The previous example is part of a more general behaviour. Consider the following action of* $G_m = \operatorname{Spec} k[t, t^{-1}]$ *on* $\mathbb{A}^n = \operatorname{Spec} k[x_1, \ldots, x_n]$

$$\begin{aligned} \sigma^\bullet : G_m^\bullet \times (\mathbb{A}^n)^\bullet &\to (\mathbb{A}^n)^\bullet \\ (\lambda, p) &\mapsto \sigma^\bullet(\lambda, p) := \lambda \cdot p \end{aligned}$$

where λ *and* p *are points of* G_m *and* $\mathbb{A}^n$ *respectively with values in some scheme. Alternatively,* $\sigma : G_m \times \mathbb{A}^n \to \mathbb{A}^n$ *is induced by the following dual action*

$$\begin{aligned} \sigma^* : k[x_1, \ldots, x_n] &\to k[t, t^{-1}] \otimes k[x_1, \ldots, x_n] \\ x_i &\mapsto \sigma^*(x_i) = t \otimes x_i \end{aligned}$$

The categorical quotient for this action is the structure morphism $\pi : \mathbb{A}^n \to \operatorname{Spec} k$. *In particular, it is not a geometric quotient. It can be proven that* $\mathbb{A}^n - \{0\}$ *is a* G_m*-invariant open subset of* $\mathbb{A}^n$ *and that the natural morphism* $\mathbb{A}^n - \{0\} \to \mathbb{P}^{n-1}$ *is a geometric quotient for the* G_m*-action*

Example 2.1.4 ([SR16], Chapter 6, Example 4.10)**.** *Unlike in the case of topological spaces (recall example 2.1.1), the categorical quotient for an algebraic group action does not always exist.*

For example, let k *be a field of characteristic* 0 *and consider the following action of the additive group* G_a *on the space of order* 2 *square matrices* $M(2, k)$

$$\begin{aligned} \sigma^\bullet : G_a^\bullet \times M(2, k)^\bullet &\to M(2, k)^\bullet \\ (\lambda, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) &\mapsto \begin{pmatrix} a + \lambda \cdot c & b + \lambda \cdot d \\ c & d \end{pmatrix} \end{aligned}$$

(where, as usual, we are defining the action over points of G_a *and* $M(2, k)$ *valued on some scheme). This* G_a*-action on* $M(2, k)$ *does not have a categorical quotient. The main reason behind this is that the additive group* G_a *is not a reductive algebraic group. We will define what a reductive group is later*

The main objective of geometric invariant theory is to give necessary and sufficient conditions for the existence of categorical quotients for algebraic group actions on schemes.

Let's study some basic properties of categorical quotients for algebraic group actions

Lemma 2.1.2. *Let* G *be an algebraic group and* (X, σ) *a* G*-scheme. Suppose that there is a categorical quotient* $\pi : X \to Y$ *for the* G*-action. Then,* $\pi : X \to Y$ *is an epimorphism in the category of schemes*

Proof. Suppose that there are morphisms $f, g : Y \rightrightarrows Z$ such that $f \circ \pi = g \circ \pi$. Clearly, $f \circ \pi$ and $g \circ \pi$ are G-invariant morphisms from X to Z. By the definition of the categorical quotient for an algebraic group action, we have that $f = g$ $\square$

Proposition 2.1.1. *Let* G *be an algebraic group and* X *a* G*-scheme. Let* $\pi : X \to Y$ *be a* G*-invariant morphism*

(a) *If there is an open cover* $\{U_i\}_{i\in I}$ *of* Y *such that the restriction* $\pi_{|\pi^{-1}(U_i)} : \pi^{-1}(U_i) \to U_i$ *is a categorical quotient for every* $i \in I$*, then* $\pi : X \to Y$ *is a categorical quotient*

(b) *Suppose that* $\pi : X \to Y$ *is a categorical quotient for the* G*-action on* X*. If* X *is a reduced (resp. connected, irreducible, integral, normal) scheme, then* Y *is also a reduced (resp. connected, irreducible, integral, normal) scheme*

Proof. Let's see (a). Suppose that $f : X \to Z$ is a G-invariant morphism. Then, for every $i \in I$, $\pi^{-1}(U_i)$ is a G-invariant open subset of X and the restriction $f_{|\pi^{-1}(U_i)} : \pi^{-1}(U_i) \to Z$ is G-invariant. By the definition of categorical quotient, there is a unique morphism $h_i : U_i \to Z$ such that $h_i \circ \pi_{|\pi^{-1}(U_i)} = f_{|\pi^{-1}(U_i)}$. The family of morphisms $\{h_i : U_i \to Z\}_{i\in I}$ glues to a unique morphism $h : Y \to Z$ such that $h_{|U_i} = h_i$ for every $i \in I$. Besides, $h \circ \pi = f$ by construction. Since this holds for every G-invariant morphism $f : X \to Z$, we conclude that $\pi : X \to Y$ is a categorical quotient.

Let's prove (b). Suppose that X is a reduced scheme and that $\pi : X \to Y$ is a categorical quotient for the G-action on X. Let $j : Y^{red} \hookrightarrow Y$ be the canonical inclusion of the reduced model of Y in Y. Since X is a reduced scheme, there is a unique morphism $\widetilde{\pi} : X \to Y^{red}$ such that $j \circ \widetilde{\pi} = \pi$, and $\widetilde{\pi} : X \to Y^{red}$ is G-invariant because $\pi : X \to Y$ is. Besides, $\pi : X \to Y$ is a categorical quotient, so there is a unique morphism of schemes $i : Y \to Y^{red}$ such that $\widetilde{\pi} = i \circ \pi$. We then have that

$$j \circ i \circ \pi = j \circ \widetilde{\pi} = \pi$$
$$i \circ j \circ \widetilde{\pi} = i \circ \pi = \widetilde{\pi}$$

and thus

$$j \circ i = \mathrm{Id}_Y \text{ and } i \circ j = \mathrm{Id}_{Y^{red}}$$

because $\pi : X \to Y$ and $\widetilde{\pi} : X \to Y^{red}$ are epimorphisms in the category of schemes. The remaining properties are proven in a very similar way. For example, if X is a normal scheme, then $\pi : X \to Y$ factors through the normalization of Y and the same arguments as above apply □

Example 2.1.5 ([Muk03], Example 5.1)**.** *If* X *is a separated* G*-scheme and* $\pi : X \to Y$ *is a categorical quotient for the* G*-action, then* Y *is not a separated scheme in general.*

For example, consider the open subscheme of $\mathbb{A}^2$ *given by* $X = \mathbb{A}^2 - \{(0,0)\}$ *and define a* G_m*-action by*

$$\begin{aligned} \sigma : G_m^\bullet \times X^\bullet &\to X^\bullet \\ (\lambda, (x, y)) &\mapsto \sigma(t, (x, y)) := (t \cdot x, t^{-1} \cdot y) \end{aligned}$$

where λ *and* (x, y) *are valued points of* G_m *and* X*. The categorical quotient for the* G_m*-action on* X *exists and is isomorphic to the affine line with a double origin, so it's not a separated scheme. However, the quotients that we will construct using techniques from geometric invariant theory will always be separated schemes*

For the rest of this chapter, we will suppose that G is an affine algebraic group

Observation 2.1.3. *Let X be a G-scheme and let $\pi : X \to Y$ be a G-invariant morphism. For every open subset $U \subseteq Y$, $\pi^{-1}(U)$ is a G-invariant open subset of X, so the k-algebra $\mathcal{O}_X(\pi^{-1}(U))^G$ is well defined.*

Let $\pi_*\mathcal{O}_X^G$ be the sheaf of algebras on Y defined by $\pi_*\mathcal{O}_X^G(U) := \mathcal{O}_X(\pi^{-1}(U))^G$ for every open subset $U \subseteq Y$. It's easy to check that the image of the sheaf homomorphism $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ induced by π is contained in $\pi_*\mathcal{O}_X^G$

In practice, the quotients that we will construct will satisfy some additional properties that are given in the following definition

Definition 2.1.6. *Let X be a G-scheme. A morphism $\pi : X \to Y$ is a good quotient for the G-action if it satisfies the following properties*

(a) $\pi : X \to Y$ *is surjective*

(b) $\pi : X \to Y$ *is G-invariant*

(c) The sheaf homomorphism $\mathcal{O}_Y \to \pi_*\mathcal{O}_X^G$ (see observation 2.1.3) is an isomorphism

(d) For every G-invariant closed subset $C \subseteq X$, the image $\pi(C)$ is a closed subset of Y

(e) For every pair of disjoint G-invariant closed subsets $C_1, C_2 \subseteq X$, their images $\pi(C_1)$ and $\pi(C_2)$ are also disjoint

If, besides, the natural morphism $\psi_Y : G \times X \to X \times_Y X$ is surjective (see definition 2.1.5), then $\pi : X \to Y$ is called a good geometric quotient

Lemma 2.1.3. *Let X be a G-scheme and let $\pi : X \to Y$ be a good quotient for the G-action. Then, Y has the quotient topology of $\pi : X \to Y$*

Proof. Let $U \subseteq Y$ be a subset of Y such that $\pi^{-1}(U) \subseteq X$ is an open subset of X and $C := X - \pi^{-1}(U)$. C is a G-invariant closed subset of X, because $\pi^{-1}(U)$ is G-invariant. By condition (d) of definition 2.1.6, $\pi(C)$ is a closed subset of Y, and clearly $\pi(C) \subseteq Y - U$. On the other hand, if $y \in Y - U$, then there is some $x \in X$ such that $\pi(x) = y$, and necessarily $x \in X - \pi^{-1}(U) = C$. This proves that $Y - U = \pi(C)$ is closed, and hence $U = Y - \pi(C)$ is open □

Example 2.1.6. *The categorical quotient for the G_m-action of example 2.1.3 was given by the structure morphism $\mathbb{A}^n \to \operatorname{Spec} k$. A direct computation shows that $k[x_1, \ldots, x_n]^{G_m} = k$, so this categorical quotient is a good quotient. However, it is not a good geometric quotient. The action of G_m induced on the G_m-invariant open subset $\mathbb{A}^n - \{0\} \subseteq \mathbb{A}^n$ had the categorical quotient $\mathbb{A}^n - \{0\} \to \mathbb{P}^{n-1}$. This is a good geometric quotient*

Proposition 2.1.2. *Every good quotient is a categorical quotient*

Proof. Let X be a G-scheme and $\pi : X \to Y$ a good quotient for the G-action. Let $f : X \to Z$ be a G-invariant morphism. If we prove that there is a unique morphism $h : Y \to Z$ such that $h \circ \pi = f$, then $\pi : X \to Y$ is a categorical quotient.

Let $\{U_i\}_{i=1}^n$ be an affine open cover of Z. For every $i = 1, \dots, n$, since f is G-invariant, then $C_i := X - f^{-1}(U_i)$ is a G-invariant closed subset of X, and clearly $\cap_{i=1}^n C_i = \emptyset$ because $X = \cup_{i=1}^n f^{-1}(U_i) = f^{-1}(\cup_{i=1}^n U_i) = f^{-1}(Z)$. Properties (d) and (e) in definition 2.1.6 show that $\{\pi(C_i)\}_{i=1}^n$ is a family of closed subsets of Y such that $\cap_{i=1}^n \pi(C_i) = \emptyset$. Let $V_i := Z - \pi(C_i)$. Then $\{V_i\}_{i=1}^n$ is an open cover of Y, and besides $\pi^{-1}(V_i) \subseteq f^{-1}(U_i)$ by construction. For every $i = 1, \dots, n$, let $h_i^* : \mathcal{O}_Z(U_i) \to \mathcal{O}_Y(V_i)$ be defined by the following diagram

$$\begin{array}{ccc}
\mathcal{O}_Z(U_i) & \xrightarrow{h_i^*} & \mathcal{O}_Y(V_i) \\
{\scriptstyle f^*}\downarrow & & \uparrow{\scriptstyle (\pi^*)^{-1}} \\
\mathcal{O}_X(f^{-1}(U_i))^G & \xrightarrow[\text{res}]{} & \mathcal{O}_X(\pi^{-1}(V_i))
\end{array}$$

where $\text{res} : \mathcal{O}_X(f^{-1}(U_i))^G \to \mathcal{O}_X(\pi^{-1}(V_i))^G$ is the restriction homomorphism induced by the inclusion $\pi^{-1}(V_i) \subseteq f^{-1}(U_i)$ and $\pi^* : \mathcal{O}_Y(V_i) \to \mathcal{O}_X(f^{-1}(V_i))^G$ is an isomorphism by condition (c) in definition 2.1.6.

Recall that we have chosen the open subsets $U_i \subseteq X$ to be affine, and thus $h_i^* : \mathcal{O}_Z(U_i) \to \mathcal{O}_Y(V_i)$ induce morphisms $h_i : V_i \to U_i$ that are uniquely determined. The morphisms $\{h_i : V_i \to U_i\}_{i=1}^n$ glue to a unique morphism $h : Y \to Z$ such that $h_{|V_i} = h_i$ for every $i = 1, \dots, n$ and, by construction, $f = h \circ \pi$. We conclude that $\pi : X \to Y$ is a categorical quotient for the G-action on X □

Proposition 2.1.3. *Let X be a G-scheme and $\pi : X \to Y$ a G-invariant morphism*

(a) *If $\pi : X \to Y$ is a good quotient for the G-action, then (Y, π) is the unique pair satisfying the properties of definition 2.1.6*

(b) *For every $x \in X^\bullet(k)$, the quotient $\pi : X \to Y$ is constant on $\overline{G \cdot x}$, i.e.*

$$\pi(\overline{G \cdot x}) = \pi(x)$$

(c) *For every $x, y \in X^\bullet(k)$, we have that*

$$\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset \Rightarrow \pi(x) = \pi(y)$$

Furthermore, if $\pi : X \to Y$ is a good quotient for the G-action, then

$$\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset \Leftrightarrow \pi(x) = \pi(y)$$

(d) *If $\pi : X \to Y$ is a good quotient for the G-action, then, for every $y \in Y^\bullet(k)$, there is a unique closed G-orbit contained in $\pi^{-1}(y)$*

(e) Suppose that $\pi : X \to Y$ is a good quotient for the G-action. Then, $\pi : X \to Y$ is a good geometric quotient if and only if the G-action on X is closed, i.e., if and only if for every $x \in X^{\bullet}(k)$ the G-orbit $G \cdot x$ is a closed subset of X

(f) If there is an open cover $\{U_i\}_{i\in I}$ of Y such that $\pi_{|\pi^{-1}(U_i)} : \pi^{-1}(U_i) \to U_i$ is a good quotient (resp. good geometric quotient) for every $i \in I$, then $\pi : X \to Y$ is a good quotient (resp. good geometric quotient)

Proof. (a) is a direct consequence of proposition 2.1.2, because categorical quotients are uniquely determined for satisying a universal property.

Let's prove (b). $\pi : X \to Y$ is a continuous map and thus for every subset $A \subseteq X$ we have that $\pi(\overline{A}) \subseteq \overline{\pi(A)}$. In particular, for every $x \in X^{\bullet}(k)$ we have

$$\begin{aligned}\pi(\overline{G\cdot x}) \subseteq \overline{\pi(G\cdot x)} &= \\ &= (\pi \text{ is G-invariant}) = \\ &= \overline{\pi(x)} = \\ &= \pi(x)\end{aligned}$$

and thus $\pi(\overline{G\cdot x}) = \pi(x)$.

Property (c) is a direct consequence of (b). Indeed, given $x, y \in X^{\bullet}(k)$ such that $\overline{G\cdot x} \cap \overline{G\cdot y} \neq \emptyset$,

$$\pi(x) = \pi(\overline{G\cdot x}) = \pi(\overline{G\cdot x} \cap \overline{G\cdot y}) = \pi(\overline{G\cdot y}) = \pi(y)$$

Suppose now that $\pi : X \to Y$ is a good quotient for the G-action and that $\overline{G\cdot x} \cap \overline{G\cdot y} = \emptyset$. Then, by properties (d) and (e) from definition 2.1.6 we have that

$$\pi(\overline{G\cdot x}) \cap \pi(\overline{G\cdot y}) = \{\pi(x)\} \cap \{\pi(y)\} = \emptyset$$

and thus $\pi(x) \neq \pi(y)$. This proves that $\overline{G\cdot x} \cap \overline{G\cdot y} = \emptyset \Rightarrow \pi(x) \neq \pi(y)$, and thus $\pi(x) = \pi(y) \Rightarrow \overline{G\cdot x} \cap \overline{G\cdot y} \neq \emptyset$.

Let's now prove (d). If there were two closed G-orbits $G\cdot x, G\cdot z \subseteq \pi^{-1}(y)$ then by (c) we would have $G\cdot x \cap G\cdot y \neq \emptyset$ and thus $G\cdot x = G\cdot y$. Condition (e) follows easily from conditions (c) and (d), because $\pi : X \to Y$ is a good geometric quotient if and only if the fiber of any point in Y is a unique orbit, and there is always a unique closed orbit contained in the fiber of each point. Finally, (f) is proven in a very similar way to the analogous statement for categorical quotients □

Definition 2.1.7. *Let G be an algebraic group and (X, σ) a G-scheme. Suppose that a categorical (resp. good, resp. good geometric) quotient $\pi : X \to Y$ for the G-action exists. Then, π is a universal (resp. uniform) categorical (resp. good, resp. good geometric) quotient if for every scheme Z and every morphism (resp. every flat morphism) $f : Z \to Y$, then the natural morphism $\widetilde{\pi} : X \times_Y Z \to Z$ is a categorical (resp. good, resp. good geometric) quotient, where the G-action on $X \times_Y Z$ is given by*

$$\widetilde{\sigma} = (\sigma \circ (\mathrm{Id}_G, \pi_X), \pi_Z) : G \times (X \times_Y Z) \to X \times_Y Z$$

and $\pi_X : G \times (X \times_Y Z) \to X$, $\pi_Z : G \times (X \times_Y Z) \to Z$ are the natural projections

Chapter 3

Quotients on affine schemes

In this chapter we will prove that the categorical quotient for an action of a reductive algebraic group on an affine scheme always exists, and it's an affine scheme given by the spectrum of the algebra of invariant functions. We will give some examples and prove that we can obtain geometric quotients for the action if we restrict to the so-called open subset of stable points. Some references for this chapter are [MFK94, Chapter 1], [Hos15, Chapter 4] or [New78, Chapter 3]

3.1 Reductive groups

Definition 3.1.1. *Let* G *be a group,* V *a vector space and* $\sigma : G \times V \to V$ *an action of* G *on* V *by linear automophisms.* σ *is rational if, for every* $v \in V$*, there is a finite dimensional* G*-invariant subspace* $W \subseteq V$ *such that* $v \in W$

Let G be an affine algebraic group and (X, σ) an affine G-scheme. Let $\sigma^* : \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X)$ be the induced dual action of G on $\mathcal{O}(X)$. For every $g \in G^\bullet(k)$, let $\mu_g : \mathcal{O}(G) \to k = k(g)$ be the morphism to the residual field of g. We define $\sigma_g^* : \mathcal{O}(X) \to \mathcal{O}(X)$ by the composition $\sigma_g^* := (\mu_g \otimes \mathrm{Id}_{\mathcal{O}(X)}) \circ \sigma^*$

Lemma 3.1.1. *With the previous notations, the map*

$$\begin{aligned} G^\bullet(k) &\to \mathrm{Aut}_{k-alg}(\mathcal{O}(X)) \\ g &\mapsto \sigma^*_{g^{-1}} \end{aligned}$$

is a group homomorphism

Proof. For every $g \in G^\bullet(k)$, $\sigma_g^* : \mathcal{O}(X) \to \mathcal{O}(X)$ is the k-algebra homomorphism induced by $\sigma_g = \sigma \circ (g, \mathrm{Id}_X) : X \to X$, and clearly for every $g, h \in G^\bullet(k)$ we have that

$$\sigma_{gh} = \sigma_g \circ \sigma_h$$

from this, it follows that

$$\sigma^*_{(gh)^{-1}} = \sigma^*_{h^{-1}g^{-1}} = \sigma^*_{g^{-1}} \circ \sigma^*_{h^{-1}}$$

so we conclude $\square$

This shows that the action of an affine algebraic group G on an affine scheme X induces an action of $G^\bullet(k)$ on the k-algebra $\mathcal{O}(X)$. We will see now that this action is rational

Lemma 3.1.2 ([MFK94], Chapter 1, § 1)**.** *Let (G, m, i, e) be an affine algebraic group and (X, σ) an affine G-scheme. Then, $\mathcal{O}(X)$ is a union of finite dimensional G-invariant vector subspaces*

Proof. We just have to prove that, for every finite dimensional subspace $V \subseteq \mathcal{O}(V)$, there is a G-invariant finite dimensional subspace $\widetilde{V} \subseteq \mathcal{O}(X)$ such that $V \subseteq \widetilde{V}$.

Denote by $<,>: \mathcal{O}(G)^* \otimes \mathcal{O}(G) \to k$ the canonical duality and define

$$\alpha := (<,> \otimes \mathrm{Id}_{\mathcal{O}(X)}) \circ (\mathrm{Id}_{\mathcal{O}(G)^*} \otimes \sigma^*) : \mathcal{O}(G)^* \otimes \mathcal{O}(X) \to \mathcal{O}(X)$$

let $\widetilde{V} := \alpha(\mathcal{O}(G)^* \otimes V)$. Let's see that $\widetilde{V}$ is a finite dimensional G-invariant vector subspace of $\mathcal{O}(X)$ containing V

(a) V is a finite dimensional vector space, so there are $f_1, \dots, f_n \in V$ such that $V =< f_1, \dots, f_n >$. For every $j \in \{1, \dots, n\}$, we have

$$\sigma^*(f_j) = \sum_i h_{ij} \otimes f_{ij} \in \mathcal{O}(G) \otimes \mathcal{O}(X)$$

and thus $\widetilde{V} =< \{f_{ij}\}_{i,j} >$, so it is a finite dimensional vector space

(b) Let $(e^*)^t : k \to \mathcal{O}(G)^*$ be the transpose of $e^* : \mathcal{O}(G) \to k$. Then, the composition

$$\mathcal{O}(X) \longrightarrow \mathcal{O}(G)^* \otimes \mathcal{O}(X) \xrightarrow{\alpha} \mathcal{O}(X)$$

where the first arrow is $(e^*)^t \otimes \mathrm{Id}_{\mathcal{O}(X)}$, is equal to $\mathrm{Id}_{\mathcal{O}(X)}$, because $(e^* \otimes \mathrm{Id}_{\mathcal{O}(X)}) \circ \sigma^* = \mathrm{Id}_{\mathcal{O}(X)}$ since σ^* is a dual action. From this, we have that $V = \alpha(((e^*)^t \otimes \mathrm{Id}_{\mathcal{O}(X)})(V)) \subseteq \widetilde{V}$

(c) Finally, let's prove that $\widetilde{V}$ is invariant with respect to the dual action. From the definition of $\widetilde{V}$, we just have to prove that, for every $\omega \in \mathcal{O}(G)^*$ and every $\nu \in V$, we have that $(\sigma^* \circ \alpha)(\omega \otimes \nu) \in \mathcal{O}(G) \otimes \overline{V}$. It suffices to prove that, for every $\omega' \in \mathcal{O}(G)^*$, we have

$$< \omega', (\sigma^* \circ \alpha)(\omega \otimes \nu) > \in \widetilde{V}$$

But σ^* is a dual action and hence $(\sigma^* \otimes \mathrm{Id}_{\mathcal{O}_X}) \circ \sigma^* = (\mathrm{Id}_{\mathcal{O}(G)} \otimes \sigma^*) \circ \sigma^*$ so, if $(\sigma^*)^t$ is the transpose of σ^*, we obtain

$$< \omega', (\sigma^* \circ \alpha)(\omega \otimes \nu) > = \alpha((\sigma^*)^t(\omega' \otimes \omega) \otimes \nu) \in \widetilde{V}$$

□

Lemma 3.1.3. *Let G be an affine algebraic group and (X, σ) an affine G-scheme. The action of $G^\bullet(k)$ induced in $\mathcal{O}(X)$ by lemma 3.1.1 is rational*

Proof. For every $f \in \mathcal{O}(X)$, the subspace generated by f is clearly finite dimensional and thus, by lemma 3.1.2, there is some finite dimensional G-invariant subspace $V \subseteq \mathcal{O}(X)$ such that $f \in V$ □

We will now introduce the notion of reductive algebraic group

Definition 3.1.2. *Let G be an affine algebraic group and let E be a finite dimensional k-vector space. A linear representation of G in E is an algebraic group homomorphism (recall example 1.1.3)*

$$\rho : G \to \mathrm{GL}(E)$$

Observation 3.1.1. *Let G be an affine algebraic group and $\rho : G \to \mathrm{GL}(E)$ a linear representation of G. Denote by $\mathbb{E} := \operatorname{Spec}\operatorname{Sym}^\bullet(E^*)$ the scheme induced by E. For every k-algebra A, we have that*

$$\mathbb{E}^\bullet(A) = \mathrm{Hom}_{k-alg}(\mathrm{Sym}^\bullet(E^*), A) = \mathrm{Hom}_{k-lin}(E^*, A) = E \otimes_k A$$

and thus, the natural group action

$$\begin{aligned} \mathrm{Aut}_A(E \otimes_k A) \times (E \otimes_k A) &\to E \otimes_k A \\ (f, (e \otimes a)) &\mapsto f(e \otimes a) \end{aligned}$$

induces a functorial action of $\underline{\mathrm{GL}}(E)$ on $\mathbb{E}^\bullet$

$$\underline{\mathrm{GL}}(E) \times \mathbb{E}^\bullet \to \mathbb{E}^\bullet$$

the linear representation $\rho^\bullet : G^\bullet \to \underline{\mathrm{GL}}(E)$ induces then a functorial group action of $G^\bullet$ on $\mathbb{E}^\bullet$ via the diagram

$$\begin{array}{ccc} G^\bullet \times \mathbb{E}^\bullet & \longrightarrow & \mathbb{E}^\bullet \\ {\scriptstyle \rho^\bullet \times \mathrm{id}} \downarrow & \nearrow & \\ \underline{\mathrm{GL}}(E) \times \mathbb{E}^\bullet & & \end{array}$$

In particular, $G^\bullet(k)$ acts on $\mathbb{E}^\bullet(k) = E \otimes k = E$

Since every G-invariant morphism $\pi : X \to Y$ induces a sheaf homomorphism $\mathcal{O}_Y \to \pi_* \mathcal{O}_X^G$, a natural candidate for a categorical quotient of an affine scheme X by the action of G is $\operatorname{Spec} \mathcal{O}(X)^G$.

The problem with this construction is that, in general, $\mathcal{O}(X)^G$ is not a k-algebra of finite type. In order for this to be true, we have to impose additional conditions on G

Definition 3.1.3 ([MFK94], Chapter 0, Definition 1.4)**.** *Let G be an affine algebraic group. G is linearly reductive if every linear representation of G splits as a direct sum of irreducible subrepresentations*

Example 3.1.1. *Suppose that k is a field of characteristic 0*

- *Every classical group* $\mathrm{GL}(n,k)$, $\mathrm{SL}(n,k)$, $\mathrm{O}(n,k)$, $\mathrm{SO}(n,k)$, $\mathrm{PGL}(n,k)$, $\mathrm{PSL}(n,k)$, ... *is linearly reductive*

- *Every finite group is linearly reductive*

The following result shows the importance of linearly reductive groups

Theorem 3.1.1 ([MFK94], Theorem 1.1)**.** *Let* G *be a linearly reductive group and* X *an affine* G*-scheme. Then,* $\mathcal{O}(X)^G$ *is a* k*-algebra of finite type*

The problem is that, in positive characteristic, many important algebraic groups are not linearly reductive. In fact, in [Nag61], Nagata gave a classification of linearly reductive groups in $\mathrm{char}(k) = p > 0$: the only examples are extensions of algebraic torii $G_m^r = G_m \times \overset{r)}{\dots} \times G_m$ (where $G_m^0 = \operatorname{Spec} k$ by convention) by a finite group of order coprime with p.

This shows that the notion of linearly reductive algebraic group is not sufficient to develop geometric invariant theory over fields of arbitrary characteristic. To overcome this problem, we introduce the following concepts

Definition 3.1.4. *Let* G *be an affine algebraic group*

- G *is geometrically reductive if for every linear representation* $\rho : G \to GL(E)$ *the following property is true: for every* G*-invariant vector* $e \in E^G$ *there is a non-constant* G*-invariant homogeneous polynomial* $f \in \mathcal{O}(\mathbb{E})^G = k[\omega_1, \dots, \omega_n]^G$ *such that* $f(e) \neq 0$*, where* $\mathbb{E} = \operatorname{Spec} \operatorname{Sym}^\bullet(E^*)$ *and* $\{\omega_1, \dots, \omega_n\}$ *is a basis of* E^*
- G *is reductive if it's smooth (note that, by 1.1.5, every algebraic group over a field of characteristic* 0 *is a smooth algebraic variety) and every smooth, normal and unipotent subgroup of* G *is trivial*

Example 3.1.2. *We give some examples*

- *Every linearly reductive group is reductive and geometrically reductive*
- *Every algebraic torus* $G_m^r = G_m \times \overset{r)}{\dots} \times G_m$ *is linearly reductive (see for example [Hos15, Proposition 3.12])*
- *Every classical group is geometrically reductive*
- *Suppose that* k *is a field of characteristic* 0 *and let* G *be an affine algebraic group over* k*. Then (see [Nag61])*

$$G \text{ linearly reductive} \Leftrightarrow G \text{ geometrically reductive} \Leftrightarrow G \text{ reductive}$$

Theorem 3.1.2 (Nagata, Mumford, Haboush, Popov)**.** *Let* G *be a smooth affine algebraic group. The following properties are equivalent*

(a) G *is geometrically reductive*

(b) G *is reductive*

(c) *For every finitely generated* k*-algebra* A *and every rational action of* G *on* A*, then* A^G *is a finitely generated* k*-algebra*

Proof. Nagata proved in [Nag63] that every geometrically reductive algebraic group is reductive. The converse statement was conjectured by Mumford and proven by Haboush in [Hab75]. For a proof of the equivalence of these results with the finiteness of the algebra of invariants, see for example [New78, Theorem 3.4] □

Observation 3.1.2. *The additive group* G_a *is not a reductive algebraic group. The problem of determining if the algebra of invariants for an algebraic group action is of finite type was one Hilbert's original 23 problems. The first counterexample was given by Nagata in [Nag59] for an action of the additive group*

Corollary 3.1.1. *Let* G *be a reductive algebraic group and* X *an affine* G*-scheme. The algebra of invariants* $\mathcal{O}(X)^G$ *is of finite type over* k

Proof. This is a direct consequence of theorem 3.1.2 and lemma 3.1.3 □

3.2 Geometric invariant theory over affine schemes

We start by proving a fundamental technical result. In general, the functions of an affine scheme separate closed subsets of the scheme. The same property holds in the context of reductive algebraic group actions: the invariant functions of an affine scheme on which a reductive algebraic group acts separate invariant closed subsets

Lemma 3.2.1. *Let* G *be a reductive algebraic group and* X *an affine* G*-scheme. For every pair of closed, disjoint,* G*-invariant subsets* $C_1, C_2 \subseteq X$*, there is some* $f \in \mathcal{O}(X)^G$ *such that* $f_{|C_1} = 0$ *and* $f_{|C_2} = 1$

Proof. Since C_1 and C_2 are closed subsets of an affine scheme X, they are defined respectively by some ideals $I(C_1)$ and $I(C_2)$ of $\mathcal{O}(X)$. The condition $C_1 \cap C_2 = \emptyset$ is expressed in terms of the associated ideals as $I(C_1) + I(C_2) = \mathcal{O}(X)$, and thus there are $f_1 \in I(C_1)$ and $f_2 \in I(C_2)$ such that $f_1 + f_2 = 1$. From this, it follows that $f_{1|C_1} = 0$ and $f_{1|C_2} = 1$. We will now construct from f_1 a G-invariant function on X satisfying the same properties.

From lemma 3.1.2, $V = < G^{\bullet}(k) \cdot f_1 >$ is a G-invariant finite dimensional vector subspace of $\mathcal{O}(X)$. Let $\{h_1, \dots, h_n\}$ be a basis of V. Then there are $\{\lambda_{ij}\}_{i,j} \subseteq k$ and $\{g_{ij}\}_{i,j} \subseteq G^{\bullet}(k)$ such that

$$h_j = \sum_i \lambda_{ij} \cdot (g_{ij} \cdot f_1) \text{ for every } j = 1, \dots, n$$

so, by the definition of each h_j and the properties of f_1 we have that

$$h_{j|C_1} = \sum_i \lambda_{ij} \cdot (g_{ij} \cdot f_{1|C_1}) = 0 \text{ for every } j = 1, \dots, n$$

$$h_{j|C_2} = \sum_i \lambda_{ij} \cdot (g_{ij} \cdot f_{1|C_2}) = \nu_j \neq 0 \text{ for every } j = 1, \dots, n$$

We can define a morphism $h : X \to \mathbb{A}^n$ given by $h = (h_1, \dots, h_n)$, i.e. induced by the homomorphism of k-algebras

$$\begin{aligned} h^* : k[x_1, \dots, x_n] &\to \mathcal{O}(X) \\ x_j &\mapsto h^*(x_j) := h_j \end{aligned}$$

by definition, h satisfies

$$\begin{aligned} &h_{|C_1} = 0 \\ &h_{|C_2} = v = (v_1, \dots, v_n) \in \mathbb{A}^n \text{ with } v \neq 0 \end{aligned}$$

On the other hand, V is a G-invariant subspace of $\mathcal{O}(X)$ and thus we have that $\sigma^*(V) \subseteq \mathcal{O}(G) \otimes V$. Since $\{h_1, \dots, h_n\}$ is a basis of V, then

$$\sigma^*(h_j) = \sum_{i=1}^{n} a_{ji} \otimes h_i \text{ with } a_{ji} \in \mathcal{O}(G)$$

consider the following k-algebra homomorphism

$$\begin{aligned} \rho^* : k[\{x_{ij}\}_{i,j=1,\dots,n}] &\to \mathcal{O}(G) \\ x_{ij} &\mapsto a_{ij} \end{aligned}$$

We have that $\rho^*(\det) = \det(a_{ij}) \in \mathcal{O}(G)$. Since $\{h_i\}_{i=1,\dots,n}$ is a basis of V, for every $g \in G^\bullet(k)$ we have that $\det(a_{ij}(g)) \neq 0$, and thus $\det(a_{ij}) \in \mathcal{O}(G)^\times$. This means that ρ^* factors through $k[\{x_{ij}\}_{i,j=1,\dots,n}]_{\det} = \mathcal{O}(GL(n,k))$ and thus induces a morphism

$$\rho : G \to GL(n, k)$$

It's not hard to prove that ρ is a linear representation of G. Following the same ideas as in observation 3.1.1, $\rho : G \to GL(n,k)$ induces an action of G on $\mathbb{A}^n = \operatorname{Spec} \operatorname{Sym}^\bullet(k^n)$, and by construction $h : X \to \mathbb{A}^n$ is G-equivariant with respect to this action. In particular, we have that

$$\begin{aligned} g \cdot v = g \cdot h(C_2) &= \\ = h(g \cdot C_2) &= \\ = h(C_2) &= \\ = v & \end{aligned}$$

and thus $v \in \mathbb{A}^n$ is a non-zero G-invariant element of $\mathbb{A}^n$. G is a geometrically reductive group, so by definition there is some $\phi \in \mathcal{O}(\mathbb{A}^n)^G$ such that $\phi(v) \neq 0$. Clearly $f := \frac{h^*(\phi)}{\phi(v)}$ satisfies the conditions of the lemma □

We will now introduce the concept of Reynolds operator.

Let G be an affine algebraic group and $\rho : G \to GL(E)$ a linear representation of G. Denote by E^G the subspace of G-invariant elements of E. If G is a linearly reductive group, there is a G-equivariant decomposition of E as a direct sum of G-invariant subspaces

$$E \simeq E^G \oplus W$$

and this decomposition induces a canonical projection homomorphism $E \to E^G$. If we try to extend this to k-algebras with a dual action of G, we obtain the following

Definition 3.2.1. *Let* G *be an affine algebraic group and* (A, σ^*) *a* k*-algebra with a dual action of* G*. A Reynolds operator is a surjective* k*-algebra homomorphism* $R_A : A \to A^G$ *such that* $R_A(ab) = aR_A(b)$ *for every* $a \in A^G$ *and every* $b \in A$

Proposition 3.2.1 ([Hos15], Lemma 4.22)**.** *Let* G *be a linearly reductive algebraic group and* (A, σ^*) *a* k*-algebra with a dual action of* G*. Then,* (A, σ^*) *has a unique Reynolds operator*

Lemma 3.2.2. *Let* G *be an affine algebraic group and* (A, σ^*) *be a* k*-algebra with a dual action of* G*. Suppose that a Reynolds operator* $R_A : A \to A^G$ *exists for the dual action. Then, for every family of ideals* $\{I_j\}_{j \in J}$ *of* A^G*, we have that*

$$(\sum_{j \in J} I_j A) \cap A^G = \sum_{j \in J} I_j$$

Proof. We will prove this result for the case of a single ideal. The general case follows from the fact that $\sum_{j \in J} I_j \cdot A = (\sum_{j \in J} I_j) \cdot A$.

Let I be an ideal of A^G. Then, clearly we have the inclusion $I \subseteq (I \cdot A) \cap A^G$. Let now be $x \in (I \cdot A) \cap A^G$. We can write x as $x = \sum_i a_i b_i$ for some $a_i \in I$ and $b_i \in A$. From the definition of Reynolds operators, we have

$$x = R_A(x) = R_A(\sum_i a_i b_i) = \sum_i a_i R_A(b_i) \in I$$

so $(I \cdot A) \cap A^G \subseteq I$ and thus $I = (I \cdot A) \cap A^G$ □

The following theorem is the central result of this chapter

Theorem 3.2.1 ([MFK94], Theorem 1.1)**.** *Let* G *be a reductive algebraic group and* (X, σ) *an affine* G*-scheme. Let* $X//G := \operatorname{Spec} \mathcal{O}(X)^G$ *and let* $\pi : X \to X//G$ *be the morphism induced by the natural inclusion* $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$*. Then,* $\pi : X \to X//G$ *is a uniform good quotient for the* G*-action on* X*. Besides, if* k *is a field of characteristic 0,* $\pi : X \to X//G$ *is a universal good quotient*

Proof. Let's start by proving that $\pi : X \to X//G$ is a good quotient. We will prove this result in the case that $\operatorname{char}(k) = 0$.

Clearly, $\pi : X \to Y$ is G-invariant. Since the image of $\pi : X \to X//G$ is a constructible set by Chevalley's theorem, to prove that $\pi : X \to X//G$ is surjective it suffices to prove that $\pi^\bullet_k : X^\bullet(k) \to (X//G)^\bullet(k)$ is surjective. Let $y \in (X//G)^\bullet(k)$ and denote by $\mathfrak{m}_y \subseteq \mathcal{O}(X)^G$ the associated maximal ideal. Let $\{f_1, \ldots, f_m\} \subseteq \mathcal{O}(X)^G$ be a set of generators for $\mathfrak{m}_y$. Suppose that G is linearly reductive so that a Reynolds operator for the dual action on $\mathcal{O}(X)$ exists (for the general case, see [New78, Lemma 3.4.2]). From lemma 3.2.2, we deduce that

$$\sum_{i=1}^{m} f_i \mathcal{O}(X) \neq \mathcal{O}(X)$$

and thus there is a maximal ideal $\mathfrak{m} \subseteq \mathcal{O}(X)$ such that $\sum_{i=1}^{m} f_i \mathcal{O}(X) \subseteq \mathfrak{m}$. Let $x \in X^{\bullet}(k)$ be the point associated with $\mathfrak{m}$. By construction, $\pi(x) = y$, so $\pi : X \to Y$ is surjective.

Let's prove that $\mathcal{O}_{X//G} \simeq \pi_* \mathcal{O}_X^G$. It suffices to prove that $\mathcal{O}((X//G)_f) = (\pi_* \mathcal{O}_X^G)((X//G)_f)$ for every $f \in \mathcal{O}(X//G) = \mathcal{O}(X)^G$. Indeed, we have that

$$\begin{aligned}
\mathcal{O}((X//G)_f) &= (\mathcal{O}(X)^G)_f = \\
&= (f \text{ is G-invariant}) = \\
&= (\mathcal{O}(X)_f)^G = \\
&= \mathcal{O}(X_f)^G = \\
&= \mathcal{O}(\pi^{-1}((X//G)_f))^G = \\
&= (\pi_* \mathcal{O}_X^G)((X//G)_f)
\end{aligned}$$

so we conclude. The rest of properties of definition 2.1.6 follow directly from lemma 3.2.1 .

Let's prove that $\pi : X \to X//G$ is a uniform good quotient for the G-action. We will denote $Y = X//G$. Let $f : T \to Y$ be a flat morphism. We have to prove that the natural morphism $\widetilde{\pi} : X \times_Y T \to T$ is a good quotient for the G-action induced by σ on $X \times_Y T$. By proposition 2.1.3, this is a local condition on T, so we can suppose that $T = \operatorname{Spec} \mathcal{O}(T)$. We have the cartesian square

$$\begin{array}{ccc}
X = \operatorname{Spec} \mathcal{O}(X) & \longleftarrow & X \times_Y T = \operatorname{Spec}(\mathcal{O}(X) \otimes_{\mathcal{O}(X)^G} \mathcal{O}(T)) \\
\downarrow \pi & & \downarrow \widetilde{\pi} \\
Y = \operatorname{Spec} \mathcal{O}(X)^G & \xleftarrow[f]{} & T = \operatorname{Spec} \mathcal{O}(T)
\end{array}$$

if we prove that $\mathcal{O}(T) = [\mathcal{O}(X) \otimes_{\mathcal{O}(X)^G} \mathcal{O}(T)]^G$, then we would conclude by the previous constructions.

Consider the k-algebra homomorphism $1 \otimes \mathrm{Id}_{\mathcal{O}(X)} : \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X)$. Let $\phi := \sigma^* - 1 \otimes \mathrm{Id}_{\mathcal{O}(X)}$. Clearly, there is an exact sequence of $\mathcal{O}(X)^G$-modules

$$0 \longrightarrow \mathcal{O}(X)^G \longrightarrow \mathcal{O}(X) \xrightarrow{\phi} \mathcal{O}(G) \otimes \mathcal{O}(X)$$

Let $\widetilde{\sigma}$ be the action of G on $X \times_Y T$ induced by σ and $f : T \to Y$. Then, it's easy to check that

$$\begin{aligned}
\widetilde{\sigma}^* &= \sigma^* \otimes \mathrm{Id}_{\mathcal{O}(T)} \\
\widetilde{\sigma}^* - 1 \otimes \mathrm{Id}_{\mathcal{O}(X) \otimes_{\mathcal{O}(X)^G} \mathcal{O}(T)} &= \phi \otimes \mathrm{Id}_{\mathcal{O}(T)}
\end{aligned}$$

so, since $\mathcal{O}(T)$ is a flat $\mathcal{O}(X)^G$-module, we have the exact sequence of $\mathcal{O}(T)$-modules

$$0 \longrightarrow \mathcal{O}(T) \longrightarrow \mathcal{O}(X) \otimes_{\mathcal{O}(X)^G} \mathcal{O}(T) \longrightarrow \mathcal{O}(G) \otimes (\mathcal{O}(X) \otimes_{\mathcal{O}(X)^G} \mathcal{O}(T))$$

where the last arrow is equal to $\phi \otimes \mathrm{Id}_{\mathcal{O}(T)}$, and thus

$$[\mathcal{O}(X) \otimes_{\mathcal{O}(X)^G} \mathcal{O}(T)]^G = \mathrm{Ker}(\phi \otimes \mathrm{Id}_{\mathcal{O}(T)}) = \mathcal{O}(T)$$

so we conclude □

Observation 3.2.1. *The morphism $\pi : X \to X//G$ constructed in theorem 3.2.1 is usually called the GIT quotient*

Example 3.2.1. *Consider the action of G_m on $\mathbb{A}^n = \operatorname{Spec} k[x_1, \dots, x_n]$ given in example 2.1.3. Since G_m is a reductive algebraic group, by theorem 3.2.1 the morphism $\pi : \mathbb{A}^n \to \operatorname{Spec} k[x_1, \dots, x_n]^{G_m}$ is the categorical quotient for this action.*

By definition, for every i we have that $\sigma^*(x_i) = t \otimes x_i$, and thus $k[x_1, \dots, x_n]^{G_m} = k$. This proves that the structure morphism $\mathbb{A}^n \to \operatorname{Spec} k$ is the categorical quotient for the G_m-action. Besides, it is a good quotient by theorem 3.2.1, although it's not a good geometric quotient

Example 3.2.2 ([Hos15], Example 4.39)**.** *Consider the following action of $GL(2, k)$ on $M(2, k)$ defined by*

$$\begin{aligned} GL(2,k)^\bullet \times M(2,k)^\bullet &\to M(2,k)^\bullet \\ (A, X) &\mapsto c_A(X) := AXA^{-1} \end{aligned}$$

where A and X are points of $GL(2, k)$ and $M(2, k)$ with values in some scheme.

Clearly, for any $X, Y \in M(2, k)^\bullet(k)$ we have that

$GL(2, k) \cdot X = GL(2, k) \cdot Y \Leftrightarrow$ *The endomorphisms of k^2 defined by X and Y are equivalent*

Using the Jordan canonical decomposition and the fact that k is algebraically closed we can distinguish 3 different types of orbits

- *Case $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ with $\alpha \neq \beta$. It corresponds to the case of characteristic and minimal polynomials of the form $c(t) = p(t) = (t - \alpha)(t - \beta)$. These are closed orbits*
- *Case $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$. It corresponds to the case of $c(t) = p(t) = (t - \alpha)^2$*
- *Case $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ it corresponds to the case $c(t) = (t - \alpha)^2$ and $p(t) = (t - \alpha)$. These orbits are in the closure of orbits of the form $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$, because*

$$\lim_{t \to 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

Given some $M \in M(2,k)$*, then its characteristic polynomial can be written as*

$$c(t) = t^2 - \mathrm{tr}(M)t + \det(M)$$

so $\mathrm{tr}, \det$ *are* $GL(2,k)$*-invariant regular functions on* $M(2,k)$*. It can be shown that* $\mathcal{O}(M(2,k))^{GL(2,k)} = k[\mathrm{tr}, \det]$ *and thus the GIT quotient is, in terms of points with values,*

$$\begin{aligned} \pi^\bullet : M(2,k)^\bullet &\to (M(2,k)//GL(2,k))^\bullet = (\mathrm{Spec}\, k[\mathrm{tr}, \det])^\bullet \\ M &\mapsto (\det(M), \mathrm{tr}(M)) \end{aligned}$$

The trace and determinant of a matrix do not, however, determine its Jordan canonical form. This means that π *is not a good geometric quotient*

Example 3.2.3. *Consider the action of* $G_m = \mathrm{Spec}\, k[t, t^{-1}]$ *on* $\mathbb{A}^2 = \mathrm{Spec}\, k[x,y]$ *defined by*

$$\begin{aligned} \sigma^\bullet : G_m^\bullet \times (\mathbb{A}^2)^\bullet &\to (\mathbb{A}^2)^\bullet \\ (t,(x,y)) &\mapsto \sigma^\bullet(t,(x,y)) = (tx, t^{-1}y) \end{aligned}$$

where t *and* (x,y) *are, respectively, points of* G_m *and* $\mathbb{A}^2$ *with values in some scheme. Alternatively,* σ *is induced by the following dual action of* G_m *on* $k[x,y]$

$$\begin{aligned} \sigma^* : k[x,y] &\to k[t,t^{-1}] \otimes k[x,y] \\ x &\mapsto t \otimes x \\ y &\mapsto t^{-1} \otimes y \end{aligned}$$

Clearly, xy *is a* G_m *invariant function for this action because* $\sigma^*(xy) = (t \otimes x)(t^{-1} \otimes y) = 1 \otimes (xy)$*. It's easy to see that* $k[x,y]^{G_m} = k[xy]$*, and thus, by theorem 3.2.1, the induced morphism* $\pi : \mathbb{A}^2 \to \mathrm{Spec}\, k[xy] = \mathbb{A}^1$ *is a good quotient for the* G_m*-action* σ *on* $\mathbb{A}^2$*.*

The orbits for this action of G_m *on* $\mathbb{A}^2$ *fall into the four following types*

(a) Hyperbolas $\{xy = a\}$ *with* $a \in k - \{0\}$

(b) The X-axis $\{y = 0\}$ *minus the origin* $\{(0,0)\}$

(c) The Y-axis $\{x = 0\}$ *minus the origin* $\{(0,0)\}$

(d) The origin $\{(0,0)\}$

The orbits given by hyperbolas are closed and pairwise disjoint, and their images via the quotient morphism $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ *are given by* $\pi(\{xy = a\}) = a$*. The remaining orbits have closures that contain the origin* $(0,0) \in \mathbb{A}^2$*, so by proposition 2.1.3 they have the same image via* π

$$\pi(\{y=0\} - \{(0,0)\}) = \pi(\{x=0\} - \{(0,0)\}) = \pi((0,0)) = 0$$

This proves that $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ *is not a good geometric quotient*

In the previous example, we did not obtain a good geometric quotient because some orbits in $\mathbb{A}^2$ did not behave well: the orbits given by the axes were not closed and their boundaries overlapped, so they were identified in the quotient space.

Intuitively, if we had these orbits removed, then we would not identify disjoint orbits in the quotient because the rest of the orbits are closed. We will follow this idea to obtain good geometric quotients from the GIT quotient construction

Definition 3.2.2. *Let G be a reductive algebraic group and X an affine G-scheme. Let $x \in X^\bullet(k)$. x is stable for the G-action if*

- $G \cdot x$ *is closed*
- $\dim G_x = 0$

Observation 3.2.2. *Note that, for every closed point $y \in G \cdot x$, we have $\dim G_y = \dim G_x$ and $G \cdot y = G \cdot x$, so we can speak of stable orbits instead of stable points*

Example 3.2.4. *In example 3.2.3, the orbits given by the X-axis minus the origin and the Y-axis minus the origin are not stable because they are not closed. In the same example, the orbit given by the origin is not stable, because its stabilizer subgroup is equal to G_m, which has dimension 1.*

We ask a stable point to satisfy the condition $\dim G_x = 0$ so we can have nice topological properties for the set of stable points. The role of this condition will be made explicit in lemma 3.2.4

The following result is elementary, although very important

Lemma 3.2.3. *Let G be an algebraic group. For every $x \in X^\bullet(k)$, we have that*

$$G \cdot x \text{ is closed} \Leftrightarrow \text{ for every closed point } y \in \overline{G \cdot x} \text{ we have } \dim G_y = \dim G_x$$

Proof. If $G \cdot x$ is closed, then clearly the dimension of the stabilizer subgroups are the same for every closed point of $\overline{G \cdot x} = G \cdot x$, so the result follows.

Suppose now that for every closed point $y \in \overline{G \cdot x}$ we have that $\dim G_y = \dim G_x$. If $G \cdot x$ is not closed, then there is some closed point $z \in \overline{G \cdot x} - G \cdot x$, but by proposition 1.2.3 the dimension of $G \cdot z$ has strictly less dimension than $G \cdot x$. On the other hand, by proposition 1.2.4 and $\dim G_x = \dim G_z$ we have that

$$\dim G \cdot z = \dim G - \dim G_z = \dim G - \dim G_x = \dim G \cdot x$$

which is a contradiction □

Lemma 3.2.4. *Let G be a reductive algebraic group and X an affine G-scheme. Then, the set of stable points for the G-action are the set of closed points of a unique G-invariant open subset $X^s \subseteq X$*

Proof. Since X is a finite type scheme, it suffices to prove that the set of stable points is an open subset of $X^\bullet(k)$ with the induced topology. Let $x \in X^\bullet(k)$ be a stable point. Consider

$$X_+ := \{y \in X : \dim G_y \geqslant 1\}$$

this is a G-invariant closed subset of X by proposition 1.2.4. On the other hand, $G \cdot x$ is also a G-invariant closed subset of X by definition, and we have that $X_+ \cap G \cdot x = \emptyset$. By lemma 3.2.1, there is some $f \in \mathcal{O}(X)^G$ such that $f_{|G\cdot x} \neq 0$ and $f_{|X_+} = 0$. The complement of the set of zeroes of f, X_f, is an affine G-invariant open subset of X and clearly $x \in X_f$. If we prove that every closed point of X_f is stable, we conclude.

Let $y \in X_f$ be a closed point. Then, $f(y) \neq 0$, and thus $y \notin X_+$, so $\dim G_y = 0$. In particular, this means that the dimension of the stabilizer subgroup of any closed point of X_f is zero, so by lemma 3.2.3 every closed point of X_f is stable □

Observation 3.2.3. *We will usually call X^s the set of stable points for the G-action on X*

We will finish this chapter proving that the quotient for the induced action on the set of stable points is a good geometric quotient. In order to prove this, we will introduce some useful concepts

Definition 3.2.3. *Let G be an affine algebraic group and X a G-scheme. A G-invariant open subset $U \subseteq X$ is saturated for the G-action if, for every $x \in U^\bullet(k)$ and every $y \in X^\bullet(k)$, we have that*

$$\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset \Rightarrow y \in U$$

Lemma 3.2.5. *Let G be an affine algebraic group and X a G-scheme. Suppose that there is a good quotient for the G-action $\pi : X \to Y$ and let $U \subseteq X$ be a saturated open subset of X for the G-action. Then, $\pi(U)$ is an open subset of Y, and $\pi^{-1}(\pi(U)) = U$*

Proof. Since U is a G-invariant open subset of X, we have that $X - \pi(U)$ is a G-invariant closed subset of X and thus $\pi(X - \pi(U))$ is a closed subset of Y, because π is a good quotient. π is a surjective map, so $\pi(U) = Y - (\pi(X - \pi(U))$ and thus $\pi(U)$ is an open subset of Y.

Let's prove that $\pi^{-1}(\pi(U)) = U$. Clearly, we have that $U \subseteq \pi^{-1}(\pi(U))$. If we prove that every closed point of $\pi^{-1}(\pi(U))$ is an element of U, then we conclude because X is a finite type scheme. Let $x \in \pi^{-1}(\pi(U))$ be a closed point. By definition, there is some $y \in U^\bullet(k)$ such that $\pi(x) = \pi(y)$. By proposition 2.1.3, this means that $\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$ and thus $y \in U$ because U is saturated □

Observation 3.2.4. *In particular, the restriction $\pi_{|U} : U \to \pi(U)$ is a good quotient for the induced G-action on U by theorem 3.2.1*

Lemma 3.2.6. *Let G be a reductive algebraic group and X an affine G-scheme. Then, X^s is a saturated open subset of X for the G-action*

Proof. Let x be a closed point of X^s and $y \in X^\bullet(k)$. Suppose that $\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$. Since x is a stable point for the G-action, $\overline{G \cdot x} = G \cdot x$ and thus $G \cdot x \subseteq (\overline{G \cdot y} - G \cdot y)$.

Suppose that $y \notin X^s$. Then, $\overline{G \cdot y} \neq G \cdot y$ and thus, by propositions 1.2.3 and 1.2.4 we have

$$\begin{aligned} \dim G \cdot x < \dim G \cdot y &\leqslant \\ &\leqslant \dim G = \\ &= \dim G \cdot x - \dim G_x = \\ &= \dim G \cdot x \end{aligned}$$

and this is a contradiction, so y must be an element of X^s □

Proposition 3.2.2. *Let G be a reductive algebraic group and (X, σ) an affine X-scheme. Let $\pi : X \to X//G$ be the GIT quotient. Let X^s the open subset of stable points for the G-action and denote $X^s//G := \pi(X^s)$. Then*

- $X^s//G$ *is an open subset of* $X//G$
- *The restriction $\pi_{|X^s} : X^s \to X^s//G$ is a uniform good geometric quotient. Furthermore, if k is a field of characteristic 0, then it is a universal good geometric quotient*

Proof. By lemmas 3.2.6 and 3.2.5, $X^s//G$ is an open subset of $X//G$ and the restriction $\pi_{|X^s} : X^s \to X^s//G$ is a uniform good quotient.

By definition, the action of G on X^s is closed, so $\pi_{|X^s} : X^s \to X^s//G$ is a good geometric quotient by proposition 2.1.3.

Finally, let $f : T \to X^s//G$ be a flat morphism. By theorem 3.2.1, the natural morphism $\widetilde{\pi} : X^s \times_{X//G} T \to T$ is a good quotient for the induced G-action $\widetilde{\sigma}$ on $X^s \times_{X//G} T$. On the other hand, for every $(x, t) \in (X \times_{X//G} T)^\bullet(k)$ we have that $\widetilde{\sigma}_{(x,t)} = (\sigma_x, t)$ and thus $G \cdot (x, t) = (G \cdot x) \times \{t\}$, so $\widetilde{\sigma}$ is a closed action on $X^s \times_{X//G} T \to T$ and thus $\widetilde{\pi} : X^s \times_{X//G} T \to T$ is a good geometric quotient by proposition 2.1.3. The same argument is valid in the case that k is a field of characteristic 0 and $f : T \to X^s//G$ is a (non necessarily flat) morphism □

Observation 3.2.5. *In general, $X^s//G$ is not an affine scheme*

Example 3.2.5. *Consider the action of G_m on $\mathbb{A}^2$ given in example 3.2.3. In example 3.2.4, we saw that*

$$(\mathbb{A}^2)^s = \mathbb{A}^2 - \{xy = 0\}$$

so, by proposition 3.2.2, the restriction

$$\begin{aligned} \pi : \mathbb{A}^2 - \{xy = 0\} &\to \mathbb{A}^1 - \{0\} \\ (x, y) &\mapsto xy \end{aligned}$$

is a good geometric quotient

Example 3.2.6. *The set of stable points for an action might be an empty set. Consider the action of G_m on $\mathbb{A}^n$ given in example 2.1.3. Let $(x_1, \ldots, x_n)$ be a closed point in $\mathbb{A}^n$. Then*

- *If* $(x_1, \dots, x_n) = (0, \dots, 0)$ *then* $G_m \cdot (0, \dots, 0) = \{(0, \dots, 0)\}$ *is closed, but* $(G_m)_{(0,\dots,0)} = G_m$*, so* $(0, \dots, 0)$ *is not a stable point*
- *If* $(x_1, \dots, x_n) \neq (0, \dots, 0)$ *then* $(G_m)_{(x_1,\dots,x_n)} = \{e\}$*, but* $G_m \cdot (x_1, \dots, x_n)$ *is not closed, because* $(0, \dots, 0) \in (\overline{G_m \cdot (x_1, \dots, x_n)} - G_m \cdot (x_1, \dots, x_n))$*, so it's not a stable point*

This proves that $(\mathbb{A}^n)^s = \emptyset$

Chapter 4

Quotients on finite type schemes

In this chapter we will extend the construction of quotients we gave for affine schemes to the general case of a finite type scheme over k.

Let G be a reductive algebraic group and X a G-scheme. We would like to construct a categorical quotient for the G-action on X. In chapter 3, we constructed categorical quotients in the case when X were an affine scheme. In general, we could consider an open cover $\{U_i\}_{i \in I}$ of X by affine G-invariant open subsets, take the GIT quotient $\pi_i : U_i \to U_i//G$ for each i (recall theorem 3.2.1) and glue these morphisms to a global morphism $\pi : X \to Y$ such that $\pi_{|U_i} = \pi_i$ for every $i \in I$. $\pi : X \to Y$ would then be a categorical quotient for the G-action by proposition 2.1.1.

This approach presents two main problems

- In general, a scheme X does not have an affine G-invariant open cover, so we will construct rational categorical quotients $\pi : X \dashrightarrow Y$, i.e., categorical quotients for a G-invariant open subset $U \subseteq X$
- There is not a canonical way of choosing G-invariant open subsets of X, so we may have many different rational categorical quotients for a given action on X, defined on different G-invariant open subsets of X. We will solve this problem by fixing additional data: a linearization of the G-action. Different linearizations will yield different rational categorical quotients for the G-action. The study of how these rational quotients vary is a subject called variation of the GIT quotient (see for example [DH98])

Let's see an example of how we can solve the previous problems in the particular case that our scheme is projective

Example 4.0.1. *Suppose that X is a projective scheme. We will fix the following data*

- *A closed immersion $i : X \hookrightarrow \mathbb{P}^n = \operatorname{Proj} k[x_0, \dots, x_n]$. With respect to this immersion, we can write $X = \operatorname{Proj} k[x_0, \dots, x_n]/I_x$ for some homogeneous ideal $I_X \subseteq k[x_0, \dots, x_n]$*
- *A linear representation $\rho : G \to GL(n+1, k)$ inducing a G-action on $\mathbb{P}^n = \operatorname{Proj} k[x_0, \dots, x_n]$*

- *We ask for the previous data to satisfy a compatibility condition, in particular, we ask that* $i : X \hookrightarrow \mathbb{P}^n$ *is* G*-equivariant*

Later, we will prove that the natural rational morphism $\pi : X \dashrightarrow \operatorname{Proj}(k[x_0, \ldots, x_n]/I_X)^G$ *is a good categorical quotient for the* G*-action on an open subset of* X*. The* G*-invariant open subset on which* π *is defined is called the subset of semistable points for the* G*-action on* X

In the previous example, the data that we fixed was essentially a presentation of X as a subscheme of a projective space in which G acted by a linear representation. This could have been expressed in terms of the invertible sheaf on X associated to the closed immersion $i : X \hookrightarrow \mathbb{P}^n$. This is the approach that we will follow for a general finite type scheme.

Some references for this chapter include [MFK94, Chapter 1] and [Dol03, Chapters 7-8]

4.1 G-equivariant sheaves

Let G be an affine algebraic group and let (X, σ) be a G-scheme. Let $p_2 : G \times X \to X$ denote the natural projection on X and $p_{23} : G \times G \times X \to G \times X$ the natural projection on the second and third components

Definition 4.1.1. *A* G*-equivariant sheaf on* (X, σ) *is a pair* $(\mathcal{M}, \Phi)$*, where* $\mathcal{M}$ *is a coherent sheaf on* X *and* $\Phi : \sigma^*\mathcal{M} \to p_2^*\mathcal{M}$ *is a* $\mathcal{O}_{G\times X}$*-module isomorphism such that*

1. $(m \times \mathrm{id}_X)^*\Phi = p_{23}^*\Phi \circ (\mathrm{Id}_G \times \sigma)^*\Phi$
2. $(e \times \mathrm{Id}_X)^*\sigma^* : \mathcal{M} \to \mathcal{M}$ *is the identity morphism*

Definition 4.1.2. *Let* (X, σ) *be a* G*-scheme and let* $(\mathcal{M}, \Phi)$, $(\mathcal{N}, \Psi)$ *be* G*-equivariant sheaves on* X*. A morphism of* G*-equivariant sheaves from* $(\mathcal{M}, \Phi)$ *to* $(\mathcal{N}, \Psi)$ *is a morphism of* $\mathcal{O}_X$*-modules* $\alpha : \mathcal{M} \to \mathcal{N}$ *such that the following diagram commutes*

$$\begin{array}{ccc} \sigma^*\mathcal{M} & \xrightarrow{\Phi} & p_2^*\mathcal{M} \\ {\scriptstyle \sigma^*(\alpha)}\downarrow & & \downarrow{\scriptstyle p_2^*(\alpha)} \\ \sigma^*\mathcal{N} & \xrightarrow[\Psi]{} & p_2^*\mathcal{N} \end{array}$$

Every G-equivariant sheaf comes with a natural notion of sections that are invariant with respect to the action of G

Lemma 4.1.1. *Let* (X, σ) *be a* G*-scheme and* $(\mathcal{M}, \Phi)$ *a* G*-equivariant sheaf on* X*. There is a dual action of* G *on* $H^0(X, \mathcal{M})$ *induced by* $\Phi : \sigma^*\mathcal{M} \to p_2^*\mathcal{M}$

Proof. Denote by $p_1 : G \times X \to G$ the natural projection on G. The Künneth formula (see for example [Kem80]) yields the following isomorphism

$$H^0(G \times X, p_2^*\mathcal{M}) = H^0(G \times X, p_1^*\mathcal{O}_G \otimes p_2^*\mathcal{M}) \simeq \mathcal{O}(G) \otimes H^0(X, \mathcal{M})$$

Now it's easy to prove, using the conditions from definition 4.1.1, that the composition

$$H^0(X,\mathcal{M}) \xrightarrow{\sigma^*} H^0(G\times X,\sigma^*\mathcal{M}) \xrightarrow[\simeq]{\Phi^*} H^0(G\times X,p_2^*\mathcal{M}) \xrightarrow{\simeq} \mathcal{O}(G)\otimes H^0(X,\mathcal{M})$$

is a dual action of G on $H^0(X,\mathcal{M})$ □

Observation 4.1.1. *In particular, for every* G*-equivariant sheaf* $\mathcal{M}$ *on* X*, the space* $H^0(X,\mathcal{M})^G$ *is well defined*

Definition 4.1.3. *Let* X *be a* G*-scheme. A linearization of the* G*-action is a* G*-equivariant invertible sheaf on* X

Observation 4.1.2. *A linearization of the* G*-action will also be called a* G*-linearized invertible sheaf*

Example 4.1.1. *Let* (X,σ) *be a* G*-scheme and let* $\mathcal{O}_X$ *be the structure sheaf. Clearly, we have that*

$$\sigma^*\mathcal{O}_X = p_2^*\mathcal{O}_X = \mathcal{O}_{X\times G}$$

and also

$$\left\{ \textit{Isomorphisms } \mathcal{O}_{G\times X}\to\mathcal{O}_{G\times X} \right\} = \mathcal{O}(G\times X)^\times$$

$$(\Phi:\mathcal{O}_{G\times X}\to\mathcal{O}_{G\times X}) \mapsto \Phi_{G\times X}(1)$$

In particular, if we take $\Phi = \mathrm{Id}_{\mathcal{O}_{G\times X}}$*, then it's easy to prove that* $(\mathcal{O}_X,\mathrm{Id}_{\mathcal{O}_{G\times X}})$ *is a linearization of the* G*-action on* X*. This is called the trivial linearization*

The set of isomorphism classes of G-linearized invertible sheaves on a G-scheme has a group structure induced from the group structure of the Picard group of the scheme

Definition 4.1.4. *Let* X *be a* G*-scheme. The* G*-equivariant Picard group of* X *is*

$$\mathrm{Pic}^G(X) := \{(\mathcal{L},\Phi) \textit{ G-linearized invertible sheaves on } X\}/\simeq$$

Lemma 4.1.2. $\mathrm{Pic}^G(X)$ *is an abelian group*

Proof. We define the product of isomorphism classes of G-linearized invertible sheaves as

$$(\mathcal{L},\Phi)\cdot(\mathcal{L}',\Phi') := (\mathcal{L}\otimes\mathcal{L}',\Phi\otimes\Phi')$$

using the compatibility of the tensor product and inverse image operations, we have that

$$\begin{aligned}(m\times \mathrm{Id}_X)^*(\Phi\otimes\Phi') &= (m\times\mathrm{Id}_X)^*\Phi\otimes(m\times\mathrm{Id}_X)^*\Phi' = \\ &= (p_{23}^*\Phi\circ(\mathrm{Id}_G\times\sigma)^*\Phi)\otimes(p_{23}^*\Phi'\circ(\mathrm{Id}_G\times\sigma)^*\Phi') = \\ &= p_{23}^*(\Phi\otimes\Phi')\circ(\mathrm{Id}_G\times\sigma)^*(\Phi\otimes\Phi')\end{aligned}$$

and thus $(\mathcal{L}\otimes\mathcal{L}',\Phi\otimes\Phi')$ is a G-linearized invertible sheaf on X. This product is commutative and its identity element is the trivial G-linearization $(\mathcal{O}_X,\mathrm{Id}_{\mathcal{O}_{G\times X}})$. It's easy to check that, for every $(\mathcal{L},\Phi)\in\mathrm{Pic}^G(X)$, we have that

$$(\mathcal{L},\Phi)\cdot(\mathcal{L}^{-1},(\Phi^t)^{-1}) \simeq (\mathcal{O}_X,\mathrm{Id}_{\mathcal{O}_{G\times X}})$$

and thus $(\mathcal{L},\Phi)^{-1} = (\mathcal{L}^{-1},(\Phi^t)^{-1})$ □

Example 4.1.2. *As we have seen in example 4.1.1, a linearization of the* G*-action on the structure sheaf* $\mathcal{O}_X$ *is an invertible element of* $\mathcal{O}(G \times X)$ *plus some additional data. Explicitly, there is a group isomorphism*

$$\{G\text{-linearizations of } \mathcal{O}_X\} \xrightarrow{\simeq} Z^1 := \left\{ \begin{array}{l} \phi \in \mathcal{O}(G \times X)^\times \text{ such that} \\ (m \times \mathrm{Id}_X)^*\phi = (p_{23}^*\phi) \cdot (\mathrm{Id}_G \times \sigma)^*\phi \end{array} \right\}$$
$$(\mathcal{O}_X, \Phi) \mapsto \phi = \Phi_{G\times X}(1)$$

Definition 4.1.5. *A character of* G *is an algebraic group homomorphism* $\chi : G \to G_m$

Lemma 4.1.3. *The set of characters of* G *is a group denoted by*

$$X(G) = \mathrm{Hom}_{alg-grp}(G, G_m) = \{\phi \in \mathcal{O}(G)^\times : m^*\phi = \phi \otimes \phi\}$$

Proof. We have that

$$G_m^\bullet(G) = \mathrm{Hom}(G, G_m) = \mathrm{Hom}_{k-alg}(k[t, t^{-1}], \mathcal{O}(G)) = \mathcal{O}(G)^\times$$

The characters of G can be identified with the elements of $\mathcal{O}(G)^\times$ that induce algebraic group homomorphisms. Using definition 1.1.2, the conditions of the lemma follow □

Example 4.1.3. *Via the projection on the first component* $p_1 : G \times X \to G$*, the elements of* $\mathcal{O}(G)^\times$ *can be thought as elements of* $\mathcal{O}(G \times X)^\times$*. If* Z^1 *is the group defined in example 4.1.2,* p_1 *induces an injective group homomorphism* $X(G) \hookrightarrow Z^1 \subseteq \mathrm{Pic}^G(X)$*. In other words, every character of* G *induces a linearization of the* G*-action on* $\mathcal{O}_X$

In general, $\mathrm{Pic}^G(X)$ is not a subgroup of $\mathrm{Pic}(X)$, because a fixed invertible sheaf on X may have many different linearizations. For example, as we have just shown, every character of G induces a linearization of the action on $\mathcal{O}_X$. In general, there is a group homomorphism

$$\alpha : \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$$
$$(\mathcal{L}, \Phi) \mapsto \mathcal{L}$$

and clearly

$$\mathrm{Ker}\,\alpha = \{G\text{-linearizations on } \mathcal{O}_X\}/\simeq\, = Z^1/\simeq$$

We have a bijection

$$\{\Psi : \sigma^*\mathcal{O}_X \xrightarrow{\simeq} p_2^*\mathcal{O}_X\} = \{\psi \in \mathcal{O}(X)^\times : \phi' \cdot \sigma^*\psi = p_2^*\psi \cdot \phi\}$$

where $\psi = \Psi_X(1)$, $\phi = \Phi_{G\times X}(1)$ and $\phi' = \Phi'_{G\times X}(1)$. There is a group isomorphism

$$\{\psi \in \mathcal{O}(X)^\times : \phi' \cdot \sigma^*\psi = p_2^*\psi \cdot \phi\} \simeq B^1 := \{\sigma^*\psi \cdot (p_2^*\psi)^{-1} : \psi\}$$

and thus

$$(\mathcal{O}_X, \Phi) \simeq (\mathcal{O}_X, \Phi') \Leftrightarrow \phi' \cdot \phi^{-1} \in B^1$$

we have proven the following

Lemma 4.1.4. *Let* X *be a* G*-scheme and* $\alpha : \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$*. There is a group isomorphism*

$$\mathrm{Pic}^G(X) \simeq Z^1/B^1$$

Observation 4.1.3. *The quotient group* Z^1/B^1 *is isomorphic to the first cohomology group of* G *with values in the* G*-module* $\mathcal{O}(X)^\times$*. See [Dol03, Theorem 7.1]. For more on group cohomology, see [dS01, Chapter 4], or [Mil17, Chapter 16]*

There is an exact sequence

$$0 \longrightarrow Z^1/B^1 \longrightarrow \mathrm{Pic}^G(X) \xrightarrow{\alpha} \mathrm{Pic}(X)$$

Example 4.1.4. *Let* $X = \mathbb{A}^n$*. It's well known that* $\mathrm{Pic}(\mathbb{A}^n) = 0$ *and thus* $\mathrm{Pic}^G(\mathbb{A}^n) \simeq Z^1/B^1$*. A direct computation shows that*

$$Z_1/B_1 = X(G)/ < 1 >= X(G)$$

So the group of G*-linearized invertible sheaves for a* G*-action on* $\mathbb{A}^n$ *is isomorphic to* $X(G)$

Since we will construct categorical quotients for actions of algebraic groups by fixing a linearization of the action on an invertible sheaf, it's fundamental to know under which conditions we can ensure the existence of linearizations.

Suppose that G is a reductive algebraic group and that (X, σ) is a normal G-scheme.

Let $x_0 \in X^\bullet(k)$ and define

$$\delta : \mathrm{Pic}(X) \to \mathrm{Pic}(G)$$
$$\mathcal{L} \mapsto \delta(\mathcal{L}) := (p_2^*\mathcal{L} \otimes \sigma^*\mathcal{L}^{-1})_{|G\times\{x_0\}}$$

If $\mathcal{L} \in \mathrm{Ker}\,\delta$ then

$$(p_2^*\mathcal{L} \otimes \sigma^*\mathcal{L}^{-1})_{|G\times\{x_0\}} \simeq \mathcal{O}_G$$

so we have an isomorphism

$$p_2^*\mathcal{L}_{|G\times\{x_0\}} \simeq \sigma^*\mathcal{L}_{|G\times\{x_0\}}$$

It can be proven that the fact that X is normal and G is smooth implies that this isomorphism extends to an isomorphism of $\mathcal{O}_{G\times X}$-modules

$$p_2^*\mathcal{L} \to \sigma^*\mathcal{L}$$

this extension can be modified to give a linearization of the G-action on $\mathcal{L}$ (see [Dol03, Lemma 7.2]), so $\mathcal{L} \in \mathrm{Im}\,\alpha$. This proves that $\mathrm{Ker}\,\delta = \mathrm{Im}\,\alpha$ so we have the exact sequence

$$0 \longrightarrow Z^1/B^1 \longrightarrow \mathrm{Pic}^G(X) \xrightarrow{\alpha} \mathrm{Pic}(X) \xrightarrow{\delta} \mathrm{Pic}(G)$$

Theorem 4.1.1 ([Dol03], Corollary 7.2)**.** *Let* G *be a reductive algebraic group and* (X, σ) *a normal* G*-scheme. Let* $\mathcal{L} \in \mathrm{Pic}(X)$*. There is a positive integer* N *such that* $\mathcal{L}^{\otimes N}$ *admits a linearization of the* G*-action*

Proof. Under the hypotheses of the theorem, it can be proven that $\mathrm{Pic}(G)$ is a finite group, and thus $\mathrm{Pic}(X)/\mathrm{Im}\,\alpha \simeq \mathrm{Ker}\,\delta \hookrightarrow \mathrm{Pic}(G)$ is finite. In particular, every element of $\mathrm{Pic}(X)/\mathrm{Im}\,\alpha$ has finite order, and that condition is exactly what we wanted to prove □

4.2 Geometric invariant theory over finite type schemes

Let G be a reductive algebraic group and (X, σ) a G-scheme

Definition 4.2.1. *Let $\mathcal{L}$ be a G-linearized invertible sheaf on X and $x \in X^\bullet(k)$*

- *x is semistable with respect to $\mathcal{L}$ if there are $n > 0$ and $s \in H^0(X, \mathcal{L}^{\otimes n})^G$ such that $X_s := \{y \in X : s(y) \neq 0\}$ is a G-invariant affine open subset of X and $x \in X_s$*
- *x is stable with respect to $\mathcal{L}$ if x is semistable, the action of G on X_s is closed, and $\dim G_x = 0$*

Observation 4.2.1. *For every $s \in H^0(X, \mathcal{L}^{\otimes n})^G$, X_s is a G-invariant open subset of X, so the important condition in definition 4.2.1 is that X_s is affine. The fact that X_s is affine will allow us to form its GIT quotient, using theorem 3.2.1*

The proof of the following result is analogous to the proof of lemma 3.2.4, so we omit it

Lemma 4.2.1. *Let $\mathcal{L}$ be a G-linearized invertible sheaf on X. The set of semistable and stable points with respect to $\mathcal{L}$ are, respectively, the sets of closed points of G-invariant open subsets $X^{ss}(\mathcal{L})$ and $X^s(\mathcal{L})$ of X. Besides, $X^s(\mathcal{L})$ is a saturated open subset of X for the G-action*

Observation 4.2.2. *Let $\mathcal{L}$ be a G-linearized invertible sheaf on X and $s \in H^0(X, \mathcal{L})^G$. For every $n > 0$, let $s^{\otimes n} := s \otimes \overset{n)}{\ldots} \otimes s \in H^0(X, \mathcal{L}) \otimes \overset{n)}{\ldots} \otimes H^0(X, \mathcal{L}) \subseteq H^0(X, \mathcal{L}^{\otimes n})$. It's easy to check that $X_{s^{\otimes n}} = X_s$, so we have that*

$$X^{ss}(\mathcal{L}^{\otimes n}) = X^{ss}(\mathcal{L}) \quad \textit{and} \quad X^s(\mathcal{L}^{\otimes n}) = X^s(\mathcal{L}) \quad \textit{for every } n > 0$$

Observation 4.2.3. *Recall that an invertible sheaf $\mathcal{L}$ on X is ample if and only if there is some $n > 0$ and sections $s_1, \ldots, s_r \in H^0(X, \mathcal{L}^{\otimes n})$ such that X_{s_i} are affine open subsets of X and $X = X_{s_1} \cup \ldots \cup X_{s_r}$ (see [Sta19, Tag 01PR]). As a consequence, if $\mathcal{L}$ is a G-linearized invertible sheaf on X, then the restriction $\mathcal{L}_{|X^{ss}(\mathcal{L})}$ is ample*

Example 4.2.1. *Suppose that X is an affine scheme. Consider the trivial linearization $(\mathcal{O}_X, \mathrm{Id}_{\mathcal{O}_{G \times X}})$ (recall example 4.1.1). For every $n > 0$, we have that*

$$H^0(X, \mathcal{O}_X^{\otimes n})^G = H^0(X, \mathcal{O}_X)^G = \mathcal{O}(X)^G$$

and thus $1 \in H^0(X, \mathcal{O}_X^{\otimes n})^G$ for every $n > 0$. This implies that $X^{ss}(\mathcal{O}_X) = X$. On the other hand, it can be proven easily that

$$X^s(\mathcal{O}_X)^\bullet(k) = \left\{ \begin{array}{c} x \in X^\bullet(k) : G \cdot x \textit{ is closed} \\ \textit{and } \dim G_x = 0 \end{array} \right\}$$

When we constructed the affine GIT quotient in section 3.2, we were working implicitly with respect to the trivial linearization. This is the reason why we did not have to define the notion of semistable point

The following theorem is the most fundamental result of geometric invariant theory

Theorem 4.2.1 ([MFK94], Theorem 1.10)**.** *Let* G *be a reductive algebraic group and* (X, σ) *a* G*-scheme. Let* $\mathcal{L}$ *be a* G*-linearized invertible sheaf on* X

(a) There is a uniform (resp. universal if $\mathrm{char}(k) = 0$*) good quotient for the* G*-action*

$$\pi : X^{ss}(\mathcal{L}) \to X^{ss}(\mathcal{L})//G$$

Furthermore, π *is an affine morphism and there is an ample invertible sheaf* $\mathcal{M}$ *on* $X^{ss}(\mathcal{L})//G$ *such that* $\pi^*\mathcal{M} = \mathcal{L}^{\otimes N}_{|X^{ss}(\mathcal{L})}$ *for some* $N > 0$*. In particular,* $X^{ss}(\mathcal{L})//G$ *is a quasiprojective scheme*

(b) There is an open subset $X^s(\mathcal{L})//G \subseteq X^{ss}(\mathcal{L})//G$ *such that* $\pi^{-1}(X^s(\mathcal{L})//G) = X^s(\mathcal{L})$ *and the restriction* $\pi_{|X^s(\mathcal{L})} : X^s(\mathcal{L}) \to X^s(\mathcal{L})//G$ *is a uniform (resp. universal if* $\mathrm{char}(k) = 0$*) good geometric quotient for the* G*-action*

Proof. From definition 4.2.1, we deduce that there is a sufficiently large integer $N > 0$ and G-invariant sections $s_1, \dots, s_r \in H^0(X, \mathcal{L}^{\otimes N})^G$ such that, denoting $U_i = X_{s_i}$, we have

$$X^{ss}(\mathcal{L}) = U_1 \cup \ldots \cup U_r$$

where U_i is an affine G-invariant open subset of X for every $i = 1, \dots, r$.

By theorem 3.2.1, for every $i = 1, \dots, r$ there is a good quotient $\pi_i : U_i \to Y_i = U_i//G$ for the induced G-action on U_i. The idea will be to glue the morphisms $\{\pi_i : U_i \to Y_i\}_{i=1}^r$ to a good quotient for the G-action defined on $X^{ss}(\mathcal{L})$. This will be possible because the categorical quotient for an algebraic group action is unique.

For every $j = 1, \dots, r$, we have that $\{s_j\}$ is a basis of the free $\mathcal{O}(U_j)$-module $\mathcal{L}^{\otimes N}(U_j)$, and thus there is a regular function $\frac{s_i}{s_j} \in \mathcal{O}(U_j)$ such that $s_i = \frac{s_i}{s_j} \cdot s_j$ as elements of $\mathcal{L}^{\otimes N}(U_j)$, for every $i = 1, \dots, r$. Besides, since s_i and s_j are G-invariant sections, we have that $\frac{s_i}{s_j} \in \mathcal{O}(U_j)^G = \mathcal{O}(Y_j)$.

Denote by $\phi_{ij} \in \mathcal{O}(Y_j)$ the element of $\mathcal{O}(Y_j)$ induced by $\frac{s_i}{s_j}$, and consider $D(\phi_{ij}) = \{\phi_{ij} \neq 0\} \subseteq Y_j$. Clearly, we have that $\pi_j^{-1}(D(\phi_{ij})) = U_i \cap U_j$ for every $i, j = 1, \dots, r$, and the restriction $\pi_j : U_i \cap U_j \to D(\phi_{ij})$ is a good quotient for the induced G-action. We have the commuting diagram

$$\begin{array}{ccccccc}
U_i & \longleftarrow & U_i \cap U_j & \xrightarrow{\simeq} & U_i \cap U_j & \longrightarrow & U_j \\
\downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_j} & & \downarrow{\scriptstyle \pi_j} \\
Y_i & \longleftarrow & D(\phi_{ji}) & \xrightarrow[\alpha_{ij}]{\simeq} & D(\phi_{ij}) & \longrightarrow & Y_j
\end{array}$$

where $\alpha_{ij} : D(\phi_{ji}) \to D(\phi_{ij})$ is an isomorphism that is uniquely determined, because $D(\phi_{ij})$ and $D(\phi_{ji})$ are both categorical quotients for the same G-action on $U_i \cap U_j$, and the categorical quotient is unique. The triple $(\{Y_i\}_{i=1}^r, \{D(\phi_{ji})\}_{i,j=1}^r, \{\alpha_{ij}\}_{i,j=1}^r)$ determines a glueing data for the family of morphisms $\{\pi_i : U_i \to Y_i\}_{i=1}^r$, so there is a scheme Y and a morphism $\pi : X^{ss}(\mathcal{L}) \to Y$ such that $\pi_{|U_i} = \pi_i$ for every $i = 1, \dots, r$.

By proposition 2.1.3, $\pi : X^{ss}(\mathcal{L}) \to Y$ is a good quotient for the G-action. By theorem 3.2.1, π is also a uniform (resp. universal if $\mathrm{char}(k) = 0$) quotient, since this property is local on the target scheme.

Consider $h_{ij} := \phi_{ij|Y_i \cap Y_j} \in \mathcal{O}(Y_i \cap Y_j)^\times$. It's easy to check that $\{h_{ij}\}_{i,j=1}^r$ determines a Cech 1-cocycle for the sheaf $\mathcal{O}_Y^\times$, i.e.

$$[\{h_{ij}\}] \in H^1(X, \mathcal{O}_Y^\times) \simeq \mathrm{Pic}(Y)$$

and thus $[\{h_{ij}\}]$ defines an invertible sheaf $\mathcal{M}$ on Y. Besides, $\pi^*\mathcal{M} \simeq \mathcal{L}^{\otimes N}_{|X^{ss}(\mathcal{L})}$, because $[\{\frac{s_i}{s_j}\}] \in H^1(X^{ss}(\mathcal{L}), \mathcal{O}^{\otimes}_{X^{ss}(\mathcal{L})})$ is a Cech 1-cocycle associated to the invertible sheaf $\mathcal{L}^{\otimes N}_{|X^{ss}(\mathcal{L})}$ on $X^{ss}(\mathcal{L})$.

Let $t_1, \ldots, t_r \in H^0(Y, \mathcal{M})$ be such that $t_{j|Y_i} = \phi_{ij}$ for every $i = 1, \ldots, r$. Clearly, $Y_{t_j} = Y_j$ for every $j = 1, \ldots, r$ and $Y_1 \cup \ldots \cup Y_r = Y$, so $\mathcal{M}$ is an ample invertible sheaf on Y.

Finally, the action of G on $X^s(\mathcal{L})$ is closed by definition and $X^s(\mathcal{L})$ is a saturated open subset of X for the G-action. By proposition 2.1.3 and lemma 3.2.5, the restriction $\pi : X^s(\mathcal{L}) \to X^s(\mathcal{L})//G$ is a good geometric quotient for the G-action □

The following proposition formalizes example 4.0.1

Proposition 4.2.1. *Let G be a reductive algebraic group and X a projective G-scheme. Let $\mathcal{L}$ be an ample G-linearized invertible sheaf on X. Then, $X^{ss}(\mathcal{L})//G$ is a projective scheme*

Proof. Let $R = \oplus_{n \geqslant 0} H^0(X, \mathcal{L}^{\otimes n})$. R is a k-algebra of finite type, because $\mathcal{L}$ is ample. Let $i : X = \mathrm{Proj}\, R \hookrightarrow \mathbb{P}^N$ be the closed immersion given by $\mathcal{L}$ and let $R^G = \oplus_{n \geqslant 0} H^0(X, \mathcal{L}^{\otimes n})^G$. Clearly, R^G is a finitely generated k-algebra by theorem 3.1.2. Besides, we can suppose that R^G is generated by degree 1 elements $\{s_i\}_i$, so $\mathrm{Proj}\, R^G$ is a projective scheme.

The inclusion $R^G \subseteq R$ induces a rational morphism $\pi : \mathrm{Proj}\, R \dashrightarrow \mathrm{Proj}\, R^G$ defined on the open subset $\mathrm{Proj}\, R - (R^G_+)_0$, where R^G_+ is the irrelevant ideal of R^G. Clearly, $X^{ss}(\mathcal{L}) = \mathrm{Proj}\, R - (R^G_+)_0$. Let $Y_i = \mathrm{Spec}\, [R^G_{s_i}]_0$. We have that $[R^G_{s_i}]_0 = ([R_{s_i}]_0)^G$ and thus

$$\mathcal{O}(Y_i) = [R^G_{s_i}]_0 = ([R_{s_i}]_0)^G = \mathcal{O}(X_{s_i})^G$$

so, for every i, $\pi_i : X_{s_i} \to Y_i$ is a good quotient for the G-action by theorem 3.2.1, and by proposition 2.1.3 we have that $\pi : X^{ss}(\mathcal{L}) \to \mathrm{Proj}\, R^G$ is a good quotient. Since the categorical quotient for a G-action is unique, there is a natural isomorphism $X^{ss}(\mathcal{L})//G \simeq \mathrm{Proj}\, R^G$, and thus $X^{ss}(\mathcal{L})//G$ is a projective scheme □

Observation 4.2.4. *Let G be a reductive algebraic group and X a projective G-scheme. Let $\mathcal{L}$ be an ample G-linearized invertible sheaf on X. By proposition 4.2.1, the geometric quotient $X^s(\mathcal{L})//G$ is an open subset of a projective scheme $X^{ss}(\mathcal{L})//G$. In this way, $X^{ss}(\mathcal{L})//G$ can be seen as a compactification of the orbit space $X^s(\mathcal{L})//G$, obtained by adding semistable orbits*

4.3 Example: the Proj construction

Consider the following categories

- The category of $\mathbb{Z}$-graded k-algebras: the objects of this category are finite type k-algebras A with a decomposition $A = \oplus_{n\in\mathbb{Z}} A_n$ as a direct sum of abelian groups such that $A_n \cdot A_m \subseteq A_{n+m}$. The morphisms of this category are k-algebra homomorphisms $f : A \to B$ such that $f(A_n) \subseteq B_n$ for every $n \in \mathbb{Z}$

- The category of affine G_m-schemes (see section 1.2)

Lemma 4.3.1. *The category of $\mathbb{Z}$-graded k-algebras is anti-equivalent to the category of affine G_m-schemes*

Proof. Let $(X = \operatorname{Spec} A, \sigma)$ be an affine G_m-scheme. The action $\sigma : G_m \times X \to X$ induces a k-algebra homomorphism

$$\sigma^* : A \to k[t, t^{-1}] \otimes A$$
$$a \mapsto \sum_{n\in\mathbb{Z}} t^n \otimes a_n$$

Define $\rho_n : A \to A$ as $\rho_n(a) = a_n$ where $\sigma^*(a) = \sum_{n\in\mathbb{Z}} t^n \otimes a_n$ for each $a \in A$. Clearly, ρ_n is a group homomorphism. Denote $A_n = \operatorname{Im} \rho_n$ for every $n \in \mathbb{Z}$. It's easy to check that

$$A_n = \{a \in A : \sigma^*(a) = t^n \otimes a\}$$

and there is a direct sum decomposition $A = \oplus_{n\in\mathbb{Z}} A_n$, because $a = \sum_{n\in\mathbb{Z}} a_n$ for every $a \in A$. This proves that we can associate a $\mathbb{Z}$-graded k-algebra to every affine G_m-scheme.

Consider now a $\mathbb{Z}$-graded k-algebra $A = \oplus_{n\in\mathbb{Z}} A_n$ and let $X = \operatorname{Spec} A$. Define $\sigma : G_m \times X \to X$ as the morphism of affine schemes induced by the k-algebra homomorphism

$$\sigma^* : A \to k[t, t^{-1}] \otimes A$$
$$a \mapsto \sum_{n\in\mathbb{Z}} t^n \otimes a_n$$

where $a = \sum_{n\in\mathbb{Z}} a_n$ is the decomposition of a as a $\mathbb{Z}$-graded sum of homogeneous elements of $A = \oplus_{n\in\mathbb{Z}} A_n$. It's easy to prove that $\sigma : G_m \times X \to X$ is a G_m-action on X and that these constructions give an anti-equivalence of categories □

Observation 4.3.1. *Let $X = \operatorname{Spec} A$ be an affine G_m-scheme and let $A = \oplus_{n\in\mathbb{Z}} A_n$ be the associated $\mathbb{Z}$-graded decomposition. There is a bijection*

$$\left\{ \begin{array}{c} G_m\text{-invariant} \\ \text{closed subsets of } X \end{array} \right\} = \left\{ \begin{array}{c} \text{Homogeneous ideals of} \\ A = \oplus_{n\in\mathbb{Z}} A_n \end{array} \right\}$$

Let $X = \operatorname{Spec} A$ be an affine G_m-scheme and let $A = \oplus_{n\in\mathbb{Z}} A_n$ be the associated $\mathbb{Z}$-graded decomposition. Then, we have that

$$A^{G_m} = \{a \in A : \sigma^*(a) = 1 \otimes a\} = A_0$$

In particular, the quotient for the G_m-action on X with respect to the trivial G_m-linearization is the morphism $X \to \operatorname{Spec} A_0$ induced by the natural inclusion $A_0 \subseteq A$

Example 4.3.1. *Consider the action of* G_m *on* $\mathbb{A}^n$ *defined in example 2.1.3. The induced* $\mathbb{Z}$*-graded decomposition of* $k[x_1, \dots, x_n]$ *is given by* $k[x_1, \dots, x_n] = \oplus_{m\geqslant 0} k[x_1, \dots, x_n]_m$*, where* $k[x_1, \dots, x_n]_m$ *is the set of homogeneous polynomials of degree* m*. Clearly,* $k[x_1, \dots, x_n]^{G_m} = k[x_1, \dots, x_n]_0 = k$*, and thus* $\pi : \mathbb{A}^n \to \operatorname{Spec} k$ *is the quotient for the* G_m*-action with respect to the trivial linearization*

Let's see what happens when we vary the G_m-linearization on X. Recall that the group of characters of G_m is isomorphic to $\mathbb{Z}$ via

$$\begin{aligned} X(G_m) &\to \mathbb{Z} \\ \chi_r &\mapsto r \end{aligned}$$

where $\chi_r^* : k[t, t^{-1}] \to k[t, t^{-1}]$ is given by $\chi^*(t) = t^r$ (see for example [Hos15, Lemma 3.11]). For every $\alpha \in \mathbb{Z}$, by example 4.1.3 the character χ_α induces a G_m-linearization on $\mathcal{O}_X$ given by multiplying by $t^\alpha \otimes 1$ on $\mathcal{O}_{G_m \times X}$, using that $t^\alpha \otimes 1 \in \mathcal{O}(G_m)^\times \otimes \mathcal{O}(X)^\times \subseteq \mathcal{O}(G_m \times X)^\times$.

Denote by $\mathcal{L}_\alpha$ the G_m-linearization on $\mathcal{O}_X$ induced by χ_α. For every m, this construction induces a dual action of G on $A = H^0(X, \mathcal{L}_\alpha^{\otimes m})$ given by

$$\begin{aligned} A &\overset{\cdot t^{\alpha m}}{\to} k[t, t^{-1}] \otimes A \\ a &\mapsto t^{\alpha m} \otimes a = \sum_{n\in\mathbb{Z}} t^{\alpha m + n} \otimes a_n \end{aligned}$$

and thus

$$H^0(X, \mathcal{L}_\alpha^{\otimes m})^{G_m} = A_{-\alpha m}$$

Suppose now that the action of G_m on X is such that $A = \oplus_{n\geqslant 0} A_n$ and let $\alpha = -1$. Then, we have that

$$\begin{aligned} X^{ss}(\mathcal{L}_{-1}) &= (\text{Definition 4.2.1}) = \\ &= X - (\oplus_{m>0} H^0(X, \mathcal{L}_{-1}^{\otimes m})^{G_m})_0 = \\ &= (\text{Using that } H^0(X, \mathcal{L}_{-1}^{\otimes m})^{G_m} = A_m) = \\ &= X - (\oplus_{m>0} A_m)_0 = \\ &= X - (A_+)_0 \end{aligned}$$

where $A_+ = \oplus_{m>0} A_m$ is the irrelevant ideal of A.

Choose homogeneous elements $f_1, \dots, f_r \in A_+$ such that $A_+ = (f_1, \dots, f_r)$. In general, if $f \in A_+$ is homogeneous of degree d, then $D(f) = \operatorname{Spec} A_f$ is a G_m-invariant open subset of $X^{ss}(\mathcal{L}_{-1})$. Indeed, the dual action of G_m on A restricts to A_f as

$$\begin{aligned} A_f &\to k[t, t^{-1}] \otimes A_f \\ \frac{a}{f^r} &\mapsto \sum_{n \in \mathbb{Z}} t^{n-dr} \otimes \frac{a_n}{f^r} \end{aligned}$$

and thus

$$A_f^{G_m} = [A_f]_0 = \{\frac{a}{f^r} : \ a \text{ is homogeneous of degree } d \cdot r\}$$

from this, by theorem 3.2.1 we have that the GIT quotient of $D(f)$ by the G_m-action is

$$D(f)//G_m = \operatorname{Spec} A_f^{G_m} = D_+(f)$$

so we have that

$$\begin{aligned} X^{ss}(\mathcal{L}_{-1})//G_m &= D(f_1)//G_m \cup \ldots \cup D(f_r)//G_m = \\ &= D_+(f_1) \cup \ldots \cup D_+(f_r) = \\ &= \{\mathfrak{p} \in \operatorname{Spec} A : f_i \notin \mathfrak{p} \text{ for some } i\} = \\ &= (f_1, \dots, f_r \text{ are generators of } A_+) = \\ &= \{\mathfrak{p} \in \operatorname{Spec} A : A_+ \not\subseteq \mathfrak{p}\} = \\ &= \operatorname{Proj} A \end{aligned}$$

Chapter 5

Numerical criterion of stability

In this chapter, we will prove a criterion for deciding if a closed point of a projective scheme is stable.

So far, we have constructed categorical quotients for algebraic group actions by defining a notion of semistable and stable points with respect to a G-linearized invertible sheaf. In general, it's not an easy task easy to determine which points are semistable or stable. This problem can be simplified by using a criterion given in terms of the restriction of the action to what are called one-parameter subgroups. We will start by proving an alternative characterization of stability that goes back to the work of David Hilbert.

Let G be a reductive algebraic group and X a projective G-scheme. Let $\mathcal{L}$ be a very ample G-linearized invertible sheaf on X. $\mathcal{L}$ induces a linear representation $\rho : G \to GL(n+1, k)$ such that the inclusion $i : X \hookrightarrow \mathbb{P}^n = \operatorname{Proj} k[x_0, \dots, x_n]$ is a G-equivariant embedding. For every sufficiently large $m \geqslant 0$, we have that

$$H^0(X, \mathcal{L}^{\otimes m}) \simeq k[x_0, \dots, x_n]/I_{X,m}$$

where $I_X = \oplus_{m \geqslant 0} I_{X,m}$ is an homogeneous ideal of $k[x_0, \dots, x_n]$ such that $X = \operatorname{Proj} k[x_0, \dots, x_n]/I_X$

Proposition 5.0.1 (Hilbert). *In the previous hypotheses, let $x \in X^\bullet(k)$ and $x^* \in \mathbb{A}^{n+1} - \{0\}$ a preimage of x via the quotient map $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$. Then*

$$x \in X^{ss}(\mathcal{L}) \Leftrightarrow 0 \notin \overline{G \cdot x^*}$$

Proof. For every sufficiently large $m \geqslant 0$, we have the isomorphism

$$H^0(X, \mathcal{L}^{\otimes m})^G \simeq (k[x_0, \dots, x_n]/I_{X,m})^G$$

Suppose that $0 \in \overline{G \cdot x^*}$. From the above isomorphism, we that for every $s \in H^0(X, \mathcal{L}^{\otimes m})^G$ there is a homogeneous polynomial $P \in k[x_0, \dots, x_n]_m$ that is G-invariant modulo $I_{X,m}$, and clearly

$$s(x) \neq 0 \Leftrightarrow P(x^*) \neq 0$$

but, since G-invariant functions are constant on the closure of G-orbits

$$\begin{aligned} P(x^*) &= P(\overline{G \cdot x^*}) = \\ &= P(0) = \\ &= 0 \end{aligned}$$

and thus we deduce that

$$0 \in \overline{G \cdot x^*} \Rightarrow x \notin X^{ss}(\mathcal{L}), \text{ or equivalently } x \in X^{ss}(\mathcal{L}) \Rightarrow 0 \notin \overline{G \cdot x^*}$$

Suppose now that $0 \notin \overline{G \cdot x^*}$. Then, $\overline{G \cdot x^*}$ and $\{0\}$ are G-invariant disjoint closed subsets of $\mathbb{A}^{n+1}$. By lemma 3.2.1, there is some G-invariant polynomial $P \in k[x_0, \dots, x_n]^G$ such that $P(x^*) \neq 0$ and $P(0) = 0$. Decomposing P as a sum of G-invariant homogeneous polynomials, we conclude that there is some G-invariant homogeneous polynomial $P \in k[x_0, \dots, x_n]_m^G$ such that $P(x^*) \neq 0$. Since m can be taken to be as large as we want to, the image of P in $(k[x_0, \dots, x_n]_m / I_{X,m})^G$ defines a homogeneous G-invariant section $s \in H^0(X, \mathcal{L}^{\otimes m})^G$ such that $s(x) \neq 0$, and thus $x \in X^{ss}(\mathcal{L})$ □

Using proposition 5.0.1, we can study the stability of a closed point $x \in X$ with respect to a G-linearized invertible sheaf by studying $\overline{G \cdot x^*} \subseteq \mathbb{A}^{n+1}$, where x^* is any point lying over x. For every subgroup $H \subseteq G$, we have the inclusion $\overline{H \cdot x^*} \subseteq \overline{G \cdot x^*}$, and thus $0 \in \overline{H \cdot x^*} \Rightarrow 0 \in \overline{G \cdot x^*}$, so we can check the conditions of proposition 5.0.1 by looking at $\overline{H \cdot x^*}$ for subgroups $H \subseteq G$. We will be considering a particular type of subgroups

Definition 5.0.1. *A one-parameter subgroup of* G *is an algebraic group homomorphism* $\lambda : G_m \to G$

By analogy with the group of characters of G, we will denote

$$X^*(G) = \left\{ \begin{array}{c} \text{one-parameter subgroups} \\ \lambda : G_m \to G \text{ of } G \end{array} \right\}$$

For example, in virtue of proposition 5.0.1, the fact that $0 \notin \overline{\lambda(G_m) \cdot x^*}$ for every $\lambda \in X^*(G)$ is a necessary condition for a point to be semistable. We will later express this condition in terms of the so-called Hilbert-Mumford weights. The surprising fact is that, when G is reductive, this condition is also sufficient. That will be the main result of this chapter.

The main references followed to write this chapter are [MFK94, Chapter 2] and [Dol03, Chapter 9]

5.1 The Hilbert-Mumford weights

Let G be a reductive algebraic group and (X, σ) a G-scheme. Let $x \in X^\bullet(k)$ and let $\lambda : G_m \to G$ be a one-parameter subgroup of G. Let $\sigma_x = \sigma \circ (\mathrm{Id}_G, x) : G \to X$ and consider $\lambda_x := \sigma_x \circ \lambda : G_m \to X$.

Recall that $G_m = \operatorname{Spec} k[t, t^{-1}] = \mathbb{A}^1 - \{0\}$, where $\mathbb{A}^1 = \operatorname{Spec} k[t]$. X is a projective scheme, so in particular it's proper over Spec k, and thus there exists a unique morphism $\widetilde{\lambda}_x : \mathbb{P}^1 \to X$ such that $\widetilde{\lambda}_x \circ j = \lambda_x$ (see [GW10, Corollary 15.10]), where $j : G_m \simeq \mathbb{A}^1 - \{0\} \hookrightarrow \mathbb{P}^1 = \operatorname{Proj} k[x, y]$ is the natural inclusion with $t = \frac{y}{x}$.

We denote

$$\lim\nolimits_{t\to 0}\lambda(t)\cdot x := \widetilde{\lambda}_x([1:0]) \quad \lim\nolimits_{t\to\infty}\lambda(t)\cdot x := \widetilde{\lambda}_x([0:1])$$

Lemma 5.1.1. *Let $\lambda : G_m \to G$ be a one-parameter subgroup of G and $x \in X^\bullet(k)$. Let $\lambda^{-1} = i \circ \lambda \in X^*(G)$, where $i : G \to G$ is the inversion morphism of G. Then*

$$\lim\nolimits_{t\to\infty}\lambda(t)\cdot x = \lim\nolimits_{t\to 0}\lambda^{-1}(t)\cdot x$$

Proof. Consider the automorphism ϕ of $\mathbb{P}^1$ induced by the automorphism of graded k-algebras

$$\begin{aligned} \phi^* : k[x, y] &\to k[x, y] \\ x &\mapsto \phi^*(x) = y \\ y &\mapsto \phi^*(y) = x \end{aligned}$$

i.e., $\phi([x_0, y_0]) = [y_0, x_0]$ for every closed point $[x_0, y_0] \in \mathbb{P}^1$. A direct computation shows that λ_x^{-1} is equal to $\lambda_x \circ \phi$ over G_m, so we conclude by the uniqueness of the extension to $\mathbb{P}^1$ □

Lemma 5.1.2. *Let $\lambda : G_m \to G$ be a one-parameter subgroup of G and $x \in X^\bullet(k)$. Then, $\lim_{t\to 0}\lambda(t)\cdot x$ is a fixed point for the induced action of $\lambda(G_m)$ on X*

Proof. We have to prove that, for every $\widetilde{t} \in G_m^\bullet(k) = k^\times$

$$\lambda(\widetilde{t})\cdot\lim_{t\to 0}\lambda(t)\cdot x = \lim_{t\to 0}\lambda(t)\cdot x$$

Using the notations of section 1.2, we have that

$$\begin{aligned} \lambda(\widetilde{t})\cdot\lim_{t\to 0}\lambda(t)\cdot x &= \lambda(\widetilde{t})\cdot\widetilde{\lambda}_x([1:0]) = \\ &= \widetilde{\alpha}_x([1:0]) \end{aligned}$$

where $\alpha : G_m \to G$ is a one-parameter subgroup of G defined by $\alpha = \lambda \circ L_t$.

The translation $L_t : G_m \to G_m$ admits a unique extension $L_t : \mathbb{P}^1 \to \mathbb{P}^1$ that fixes the closed point $[1 : 0]$. From the uniqueness of extensions, it's easy to check that

$$\widetilde{\alpha}_x = \widetilde{\lambda}_x \circ L_t$$

and thus

$$\begin{aligned} \lambda(\widetilde{t})\cdot\lim_{t\to 0}\lambda(t)\cdot x &= \lambda(\widetilde{t})\cdot\widetilde{\lambda}_x([1:0]) = \\ &= \widetilde{\alpha}_x([1:0]) = \\ &= (\widetilde{\lambda}_x \circ L_t)([1:0]) = \\ &= \widetilde{\lambda}_x([1:0]) = \\ &= \lim_{t\to 0}\lambda(t)\cdot x \end{aligned}$$

so we conclude □

Let $\mathcal{L}$ be a G-linearized invertible sheaf on X and $x \in X^{\bullet}(k)$. Let $\lambda : G_m \to G$ be a non trivial one-parameter subgroup of G. By lemma 5.1.2, $x_0 = \lim_{t \to 0} \lambda(t) \cdot x$ is a fixed point for the action of $\lambda(G_m) \simeq G_m$. The restriction of $\mathcal{L}$ to $\{x_0\}$ admits a natural G_m-linearization given by a character $\chi_r \in X(G_m) \simeq \mathbb{Z}$ (because the pullback of a G-equivariant sheaf is a G-equivariant sheaf)

Definition 5.1.1. *With the previous notations, the Hilbert-Mumford weight of* $x \in X^{\bullet}(k)$ *with respect to a non trivial one-parameter subgroup* $\lambda : G_m \to G$ *and a* G*-linearized invertible sheaf* $\mathcal{L}$ *is*

$$\mu^{\mathcal{L}}(x, \lambda) := -r$$

We will now give an alternative definition of Hilbert-Mumford weights that is easier to compute and work with.

Suppose that $\mathcal{L}$ is a very ample G-linearized invertible sheaf on X and let $i : X \hookrightarrow \mathbb{P}^n$ be the G-equivariant closed embedding associated to $\mathcal{L}$, where G acts on $\mathbb{P}^n$ via the induced linear representation $\rho : G \to GL(n+1, k)$ and $\mathcal{L} = i^* \mathcal{O}_{\mathbb{P}^n}(1)$. Let $\lambda : G_m \to G$ be a non trivial one-parameter subgroup of G.

The linear representation $\rho : G \to GL(n+1, k)$ induces a linear representation of $G_m \simeq \lambda(G_m)$. Since G_m is linearly reductive, this representation splits as a direct sum of irreducible representations of G_m, and thus we can find a basis $\{e_0, \dots, e_n\}$ of $\mathbb{A}^{n+1}$ such that

$$\lambda(t) \cdot e_i := t^{m_i} e_i \ \forall t \in G_m^{\bullet}(k) = k^{\times}$$

where $m_i \in \mathbb{Z}$ for every $i = 0, \dots, n$.

Let $x^* = (x_0, \dots, x_n)$ be a point of $\mathbb{A}^n - \{0\}$ lying over x, i.e., such that $x = [x_0 : \dots : x_n]$

Lemma 5.1.3. *With the previous notations*

$$\mu^{\mathcal{L}}(x, \lambda) = \min_{i=0,\dots,n} \{m_i : x_i \neq 0\}$$

Proof. Denote

$$\alpha = \min_{i=0,\dots,n} \{m_i : x_i \neq 0\}$$

for every $t \in G_m^{\bullet}(k)$, we have that

$$\begin{aligned} \lambda(t) \cdot x = [t^{m_0} x_0 : \dots : t^{m_n} x_n] = \\ = [t^{m_0 - \alpha} x_0 : \dots : t^{m_n - \alpha} x_n] \end{aligned}$$

and, from the definition of α, $m_i - \alpha \geqslant 0$ for every $i = 0, \dots, n$. Let

$$y = \lim_{t \to 0} \lambda(t) \cdot x = [y_0 : \dots : y_n]$$

for each $i = 0, \dots, n$, it's easy to check that

$$y_i \neq 0 \Leftrightarrow x_i \neq 0 \text{ and } m_i = \alpha$$

For any $y^* \in \mathbb{A}^{n+1} - \{0\}$ lying over y, a direct computation shows that

$$\lambda(t) \cdot y^* = t^{\alpha} y^*$$

On the other hand, let $\pi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ be the quotient morphism. We have that

$$\pi^{-1}(y) = \mathcal{O}_{\mathbb{P}^n}(-1)_y - \{0\}$$

where $\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathbb{P}^n$ is the tautological bundle. Clearly, the group $\lambda(G_m)$ acts on $\mathcal{O}_{\mathbb{P}^n}(-1)_y - \{0\}$ by the character α. This implies that $\lambda(G_m)$ acts on $\mathcal{O}_{\mathbb{P}^n}(1)_y - \{0\}$ via the character $-\alpha$. Since $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^n}(-1)$, we have that

$$\mu^{\mathcal{L}}(x, \lambda) = -(-\alpha) = \alpha$$

so we conclude □

The Hilbert-Mumford weights give necessary conditions for the stability of a closed point

Lemma 5.1.4. *Let* G *be a reductive algebraic group and* X *a projective* G*-scheme. Let* $\mathcal{L}$ *be an ample* G*-linearized invertible sheaf on* X *and* $x \in X^\bullet(k)$*. Then*

$$x \in X^{ss}(\mathcal{L}) \Rightarrow \mu^{\mathcal{L}}(x, \lambda) \leqslant 0 \text{ for every } \lambda \in X^*(G)$$
$$x \in X^{s}(\mathcal{L}) \Rightarrow \mu^{\mathcal{L}}(x, \lambda) < 0 \text{ for every } \lambda \in X^*(G)$$

Proof. By observation 4.2.2, we can suppose that $\mathcal{L}$ is very ample. If there were some $\lambda \in X^*(G)$ such that $\mu^{\mathcal{L}}(x, \lambda) \overset{\text{not}}{=} \mu(x, \lambda) > 0$, then clearly $\lim_{t \to 0} \lambda(t) \cdot x^* = 0$ and thus $0 \in \overline{G \cdot x^*}$, so x wouldn't be semistable in virtue of proposition 5.0.1.

Suppose now that $x \in X^s(\mathcal{L})$ and that $\mu(x, \lambda) = 0$. Then $y = \lim_{t \to 0} \lambda(t) \cdot x$ is well defined and belongs to $\overline{G \cdot x} = G \cdot x$ because x is a stable point. But $\lambda(G_m)$ fixes y by lemma 5.1.2, and thus the stabilizer of y is isomorphic to G_m, so x cannot be a stable point □

5.2 The numerical criterion of stability

In this section we will prove the converse to lemma 5.1.4. Let $R = k[[t]]$ be the ring of formal series with coefficients in k and denote by $Q = k((t))$ its field of fractions.

Let G be an algebraic group. Every one-parameter subgroup $\lambda : G_m \to G$ of G defines an element of $G^\bullet(Q)$. Indeed, $G_m = \operatorname{Spec} k[t, t^{-1}]$ and we have the inclusions

$$k[t, t^{-1}] \hookrightarrow k(t) \hookrightarrow Q = k((t))$$

so, composing with $\operatorname{Spec} Q \to \operatorname{Spec} k[t, t^{-1}] = G_m$, every one-parameter subgroup $\lambda : G_m \to G$ defines an element of $G^\bullet(Q)$.

When G is a reductive algebraic group, this interpretation of one-parameter subgroups as Q-valued points is particularly important

Theorem 5.2.1 (Cartan-Iwahori-Matsumoto, [IM65]). *Let* G *be a reductive algebraic group. Every element of the set of double cosets* $G^{\bullet}(R)\diagdown G^{\bullet}(Q)/G^{\bullet}(R)$ *is given by a one-parameter subgroup of* G*, thought as a* Q*-valued point of* G

Proof. We will prove this result for $G = GL(n,k)$ and $k = \mathbb{C}$. In this case

$$GL(n,\mathbb{C})^{\bullet}(A) = \{n \times n \text{ invertible matrices with coefficients in } A\}$$

for any $\mathbb{C}$-algebra A. In particular, for any $M \in GL(n,\mathbb{C})^{\bullet}(Q)$ there is some non negative integer $r \geqslant 0$ such that $M = t^r \cdot \overline{M}$, where $\overline{M} \in GL(n,\mathbb{C})^{\bullet}(R)$. R is a principal ideal domain, so we can diagonalize $\overline{M}$, obtaining a decomposition

$$M = B_1 \cdot D \cdot B_2$$

where $B_1, B_2 \in GL(n,k)^{\bullet}(R)$ and $D = \mathrm{diag}(t^{r_1},\dots,t^{r_n})$ where $r_1,\dots,r_n \in \mathbb{Z}$. The one-parameter subgroup

$$\begin{aligned} \lambda : G_m &\to GL(n,\mathbb{C}) \\ t &\mapsto \lambda(t) := \mathrm{diag}(t^{r_1},\dots,t^{r_n}) \end{aligned}$$

represents the double coset given by $M \in GL(n,\mathbb{C})^{\bullet}(Q)$ □

Theorem 5.2.2 (Hilbert-Mumford, [Dol03], Theorem 9.1, [MFK94], Theorem 2.1). *Let* G *be a reductive algebraic group and let* X *be a projective* G*-scheme. Let* $\mathcal{L}$ *be an ample* G*-linearized invertible sheaf on* X *and* $x \in X^{\bullet}(k)$*. Then*

$$x \in X^{ss}(\mathcal{L}) \Leftrightarrow \mu^{\mathcal{L}}(x,\lambda) \leqslant 0 \text{ for every } \lambda \in X^*(G)$$
$$x \in X^{s}(\mathcal{L}) \Leftrightarrow \mu^{\mathcal{L}}(x,\lambda) < 0 \text{ for every } \lambda \in X^*(G)$$

Proof. The $\Leftarrow$ implications were proven in lemma 5.1.4. By observation 4.2.2, we can replace $\mathcal{L}$ by a sufficiently high power, and we can suppose that we there is a linear representation $\rho : G \to GL(n+1,k)$ with respect to which there is a G-equivariant closed embedding $i : X \hookrightarrow \mathbb{P}^n$ such that $i^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{L}$. We will drop $\mathcal{L}$ by denoting $\mu^{\mathcal{L}}(x,\lambda) = \mu(x,\lambda)$ for any $\lambda \in X^*(G)$.

Suppose that $\mu(x,\lambda) < 0$ for every $\lambda \in X^*(G)$ and that $x \notin X^s(\mathcal{L})$. Let $x^* \in \mathbb{A}^{n+1} - \{0\}$ be a point lying over x. Consider the morphism

$$\begin{aligned} \phi : G &\to \mathbb{A}^{n+1} \\ g &\mapsto \phi(g) := g \cdot x^* \end{aligned}$$

$\phi : G \to \mathbb{A}^{n+1}$ cannot be a proper morphism, because if it were then the restriction $\phi^{-1}(x^*) \to \{x^*\} \simeq \operatorname{Spec} k$ would be proper (because it would be the base change of ϕ by $\{x^*\} \hookrightarrow \mathbb{A}^{n+1}$) and $\phi^{-1}(x^*)$ is an affine scheme, so the dimension of $\phi^{-1}(x^*)$ would be zero. This would imply that $G \cdot x$ is closed and G_x is zero dimensional, so x would be a stable point and we have supposed that $x \notin X^s(\mathcal{L})$.

So $\phi : G \to \mathbb{A}^{n+1}$ is not a proper morphism. Using the valuative criterion of properness (see [GW10, Theorem 15.9]), we deduce that there must be some $g \in$

$G^\bullet(Q)$ such that $g \notin G^\bullet(R)$ and $\phi(g) = g \cdot x^* \in (\mathbb{A}^{n+1})^\bullet(R) = R^{n+1}$.

By theorem 5.2.1, there is a decomposition

$$g = g_1 \cdot \lambda \cdot g_2$$

with $g_1, g_2 \in G^\bullet(R)$ and where λ is the Q-valued point of G defined by some one-parameter subgroup $\lambda : G_m \to G$.

Consider the following homomorphism of k-algebras

$$R \to k$$
$$\sum_{i=0}^{\infty} a_i t^i \mapsto a_0$$

it induces a map $G^\bullet(R) \to G^\bullet(k)$. Let $\overline{g_2} \in G^\bullet(k)$ be the image of $g_2 \in G^\bullet(R)$ via this map. We have that

$$\overline{g_2}^{-1} g_1^{-1} g = \overline{g_2}^{-1} \lambda g_2 = (\overline{g_2}^{-1} \lambda \overline{g}_2) \cdot (\overline{g_2}^{-1} g_2)$$

consider the one-parameter subgroup $\lambda' = \overline{g_2}^{-1} \lambda \overline{g}_2$ of G. We can find a basis $\{e_0, \ldots, e_n\}$ of $\mathbb{A}^{n+1}$ such that

$$\lambda'(t) \cdot e_i = t^{r_i} e_i \ \forall t \in G_m^\bullet(k) = k^\times$$

where $r_i \in \mathbb{Z}$ for every $i = 1, \ldots, r$. Suppose that x^* is expressed as $(x_0, \ldots, x_n)$ in the basis $\{e_0, \ldots, e_n\}$. Then

$$(\overline{g_2}^{-1} g_1^{-1} g \cdot x_i) = t^{r_i} (\overline{g_2}^{-1} g_2 \cdot x_i)$$

but by hypothesis $g \cdot x^* \in R^{n+1}$ and thus

$$\overline{g_2}^{-1} g_2 \cdot x_i \in t^{-r_i} \cdot R$$

so there is then some $a_i \in R$ such that

$$\overline{g_2}^{-1} g_2 \cdot x_i = t^{-r_i} a_i$$

note that $\overline{g_2}^{-1} g_2 = 1 \mod (t)$. If $x_i \neq 0$, then it must be $-r_i \leqslant 0$ so that $x_i \neq 0 \mod (t)$. So necessarily

$$r_i \geqslant 0 \text{ for every } i \text{ such that } x_i \neq 0$$

this is equivalent to saying that $\mu(x, \lambda') \geqslant 0$. But we had supposed that $\mu(x, \alpha) < 0$ for every one-parameter subgroup $\alpha \in X^*(G)$, so we arrive at a contradiction and thus it must be $x \in X^s(\mathcal{L})$. The proof of the other implication is similar and can be found in [Dol03, Theorem 9.1] □

5.3 Example: stability on the Grassmannian

Let k be a field of characteristic zero and let H and V be k-vector spaces. Denote by $\text{Grass}^m(H \otimes V)$ the grassmannian scheme of m-dimensional vector subspaces of $H \otimes V$.

There is a natural representation of $GL(H)$ on $H \otimes V$ given by

$$\begin{aligned} GL(H)^\bullet &\to GL(H \otimes V)^\bullet \\ \rho &\mapsto \rho \otimes \mathrm{Id}_V \end{aligned}$$

for points valued in some scheme.

The inclusion $SL(H) \hookrightarrow GL(H)$ induces a linear representation of $SL(H)$ on $H \otimes V$, that lifts to an action of $SL(H)$ on $\mathrm{Grass}^m(H \otimes V)$ given by

$$\begin{aligned} SL(H)^\bullet \times \mathrm{Grass}^m(H \otimes V)^\bullet &\to \mathrm{Grass}^m(H \otimes V)^\bullet \\ (g, K) &\mapsto g \cdot K := (g \otimes \mathrm{Id}_V)(K) \end{aligned}$$

in valued points. On the other hand, the representation of $SL(H)$ on $H \otimes V$ induces a representation on $\Lambda^m(H \otimes V)$, and thus an action of $SL(H)$ on the projective scheme $\mathbb{P}(\Lambda^m(H \otimes V)) = \mathrm{Proj}\,\mathrm{Sym}^\bullet(\Lambda^m(H \otimes V)^*)$. Consider the Plücker embedding, given over point with values by

$$\begin{aligned} \mathrm{Grass}^m(H \otimes V)^\bullet &\to \mathbb{P}(\Lambda^m(H \otimes V))^\bullet \\ K = < e_1, \dots, e_m > &\mapsto [e_1 \wedge \dots \wedge e_m] \end{aligned}$$

With the actions that we have defined, this is clearly a $SL(H)$-equivariant embedding, so we can use the Hilbert-Mumford numerical criterion to compute the stable and semistable points for the $SL(H)$-action on $\mathrm{Grass}^m(H \otimes V)$ with respect to the $SL(H)$-linearization of the action induced by this embedding

Proposition 5.3.1 ([LP97], § 6.6). *Let $K \in \mathrm{Grass}^m(H \otimes V)^\bullet(k)$, i.e., let K be a m-dimensional subspace of $H \otimes V$. The following statements are equivalent*

- K *is a semistable (resp. stable) point for the $SL(H)$-action with respect to the $SL(H)$-linearization induced by the Plücker embedding $\mathrm{Grass}^m(H \otimes V) \hookrightarrow \mathbb{P}(\Lambda^m(H \otimes V))$*
- *For every non-zero vector subspace $H' \subsetneq H$ we have, denoting $K' := (H' \otimes V) \cap K$*

$$\frac{\dim K'}{\dim H'} \leqslant \frac{\dim K}{\dim H} \quad \left(\textit{resp.}\ \frac{\dim K'}{\dim H'} < \frac{\dim K}{\dim H} \right)$$

Proof. We will use theorem 5.2.2 to compute the stable and semistable points of the $SL(H)$-action.

Let $\lambda : G_m \to SL(H)$ be a one-parameter subgroup. λ induces a representation of G_m on H by elements of $SL(H)$. Since G_m is a reductive algebraic group and $\mathrm{char}(k) = 0$, there is a decomposition

$$H \simeq \oplus_{i=1}^s H_i$$

where each H_i is a direct sum of one-dimensional representations of G_m of weight $r_i \in \mathbb{Z}$ (i.e., G_m acts on each one-dimensional component of H_i as $t \mapsto t^{r_i}$). Since G_m acts on G by elements of $SL(H)$, the determinant of the diagonal matrix $(t^{r_i})_i$ must be

1, and thus $\sum_{i=1}^{s} r_i \dim H_i = 0$. We can suppose that $r_1 > \ldots > r_s$.

Define $F_i := \oplus_{j\leqslant i} H_j$ for each $i = 1, \ldots, s$ and $F_0 = 0$. Then, $\{F_i\}_{i=0}^{s}$ is an ascending filtration of H such that $F_i/F_{i-1} \simeq H_i$ for every $i = 1, \ldots, s$. This filtration induces an ascending filtration $\{K_i := K \cap (F_i \otimes V)\}_{i=1}$ of K.

Let $gr(K) := \oplus_{i=1}^{s}(K_i/K_{i-1})$. $gr(K)$ is a m-dimensional vector subspace of $H \otimes V$. Besides, it can be proven (see [MFK94, Chapter 4, § 4]) that $gr(K) \in \overline{SL(H) \cdot K}$, and thus we can compute $\mu(K, \lambda)$ using $gr(K)$.

Denote $\nu_i := \dim(K_i/K_{i-1})$ for every $i = 1, \ldots, s$. Then, the Plücker embedding yields an inclusion

$$\Lambda^m gr(K) \subseteq \Lambda^{\nu_1}(H_1 \otimes V) \otimes \ldots \otimes \Lambda^{\nu_s}(H_s \otimes V)$$

clearly, $\Lambda^m gr(K)$ is a one-dimensional representation of G_m of weight $\sum_{i=1}^{s} r_i\nu_i$. This proves that

$$\mu(K, \lambda) = \sum_{i=1}^{s} r_i \nu_i$$

We will now apply the Hilbert-Mumford numerical criterion. A direct computation (noting that $K_0 = 0$) shows that we can write

$$\mu(K, \lambda) = \sum_{i=1}^{s} r_i \nu_i = r_s \cdot m + \sum_{i=1}^{s-1} \dim K_i \cdot (r_i - r_{i+1})$$

Suppose that for every non zero vector subspace $H' \subsetneq H$ we have

$$\frac{\dim K'}{\dim H'} \leqslant \frac{\dim K}{\dim H} \quad \left(\text{resp. } \frac{\dim K'}{\dim H'} < \frac{\dim K}{\dim H}\right)$$

where $K' = K \cap (H' \otimes V)$. In particular, for each $F_i \subseteq H$, we have that

$$\dim K_i \leqslant \frac{m}{\dim H} \dim F_i$$

and thus

$$\begin{aligned}\mu(K,\lambda) &= r_s \cdot m + \sum_{i=1}^{s-1} \dim K_i \cdot (r_i - r_{i+1}) = \\ &\leqslant r_S \cdot m + \sum_{i=1}^{s-1} \frac{m}{\dim H} \dim F_i \cdot (r_i - r_{i+1}) = \\ &= \frac{m}{\dim H}(r_s \cdot \dim H + \sum_{i=1}^{s-1} \dim F_i \cdot (r_i - r_{i+1})) = \\ &= \frac{m}{\dim H}(\sum_{i=1}^{s} r_i \dim H_i) = \\ &= (\text{Recall that } \sum_{i=1}^{s} r_i \dim H_i = 0) = \\ &= 0\end{aligned}$$

so we have that $\mu(K,\lambda) \leqslant 0$ (the inequality would be strict in case we had $\frac{\dim K'}{\dim H'} < \frac{\dim K}{\dim H}$). By theorem 5.2.2, K is SL(H)-semistable (resp. stable).

Suppose now that K is a SL(H)-semistable (resp. stable) point. Then, theorem 5.2.2 implies that $\mu(K,\lambda) \leqslant 0$ (resp. $\mu(K,\lambda) < 0$) for every one-parameter subgroup $\lambda : G_m \to SL(H)$. Let $H_1 \subsetneq H$ be a non zero p-dimensional vector subspace of H and let H_2 be a suplementary vector space, i.e., such that $H \simeq H_1 \oplus H_2$.

Consider the one-parameter subgroup $\lambda : G_m \to SL(H)$ given by

$$\lambda(t) = \begin{pmatrix} t^{n-p} \cdot Id_{H_1} & 0 \\ 0 & t^{n-p} \cdot Id_{H_2} \end{pmatrix}$$

where $n = \dim H$. Then, we have that

$$\mu(K,\lambda) = n \cdot \dim K' - p \cdot \dim K$$

where $K' = K \cap (H_1 \otimes V)$. Since $\mu(K,\lambda) \leqslant 0$ (resp. $\mu(K,\lambda) < 0$), then

$$\frac{\dim K'}{\dim H_1} \leqslant \frac{\dim K}{\dim H} \quad \left(\text{resp. } \frac{\dim K'}{\dim H_1} < \frac{\dim K}{\dim H}\right)$$

so we conclude □

Chapter 6

Luna's étale slice theorem

In this chapter we will prove Luna's étale slice theorem, a result about the local structure of GIT quotients. Luna's theorem is a fundamental technical tool when studying the local geometry of moduli spaces. We will work under the hypothesis that $\mathrm{char}(k) = 0$. The main references for this chapter are [Dré04] and [Lun73]

6.1 G-equivariant Zariski's main theorem

Let G be a reductive algebraic group. We are going to prove a version of Zariski's main theorem for G-equivariant morphisms. We start with a lemma

Lemma 6.1.1. *Let X and Y be affine G-schemes, $\phi : X \to Y$ a quasi-finite G-equivariant morphism, and suppose that the induced k-algebra homomorphism $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective. There are affine integral G-schemes X' and Y', containing X and Y respectively as closed subschemes, such that*

- *The action of G on X' induces the action of G on X and the action of G on Y' induces the action of G on Y*
- *There is a dominant quasi-finite G-equivariant morphism $\phi' : X' \to Y'$ such that $\phi'_{|X} = \phi$*
- *X' is a normal scheme*

Proof. Let $\{f_1, \dots, f_n\}$ be a set of generators of $\mathcal{O}(X)$ as a k-algebra. By lemma 3.1.2, there is a G-invariant finite dimensional vector subspace $W \subset \mathcal{O}(X)$ such that $f_1, \dots, f_n \in W$.

The dual action of G on W induces a dual action of G on the symmetric algebra $\mathrm{Sym}^\bullet(W)$ such that the canonical morphism $\rho : \mathrm{Sym}^\bullet(W) \to \mathcal{O}(X)$ is a G-equivariant surjective k-algebra homomorphism.

Let $I = \operatorname{Ker} \rho$. Then, $\mathcal{O}(X) \simeq \mathrm{Sym}^\bullet(W)/I$, and thus X is isomorphic to a closed G-invariant subscheme of $X'' := \operatorname{Spec} \mathrm{Sym}^\bullet(W)$. Again, by lemma 3.1.2, we can find a G-invariant finite dimensional vector subspace $V \subset \mathrm{Sym}^\bullet(W)$ containing a system of generators of I and a set of elements whose images via $\rho : \mathrm{Sym}^\bullet(W) \to \mathcal{O}(X)$ are generators of $\mathcal{O}(Y) \subseteq \mathcal{O}(X)$ as a k-algebra. Let $A = \mathrm{Sym}^\bullet(V) \subseteq \mathrm{Sym}^\bullet(W)$ and denote $Y' := \operatorname{Spec} A$.

By construction, Y is isomorphic to a closed G-invariant subscheme of Y' via the homomorphism $\rho : A \to \mathcal{O}(Y)$. On the other hand, the inclusion $A \subseteq \mathrm{Sym}^\bullet(W)$ induces a dominant G-equivariant morphism $\phi'' : X'' \to Y'$ such that $\phi''_{|X} = \phi$.

For every $y \in Y^\bullet(k)$, let $\mathfrak{m}_y \subset \mathcal{O}(Y)$ be the associated maximal ideal and let $\rho^{-1}(\mathfrak{m}_y) = \mathfrak{m}'_y \subset A$ be the maximal ideal induced by $\rho : A \to \mathcal{O}(Y)$. A direct computation shows that $\rho^{-1}(\mathfrak{m}_y\mathcal{O}(X)) = \mathfrak{m}'_y\mathcal{O}(X'')$, and thus we have that

$$\mathcal{O}(X'')/\mathfrak{m}'_y\mathcal{O}(X'') \simeq \mathcal{O}(X)/\mathfrak{m}_y\mathcal{O}(X)$$

but $\mathcal{O}(X)/\mathfrak{m}_y\mathcal{O}(X)$ is a finite dimensional k-vector space because ϕ is a quasi-finite morphism, so $\mathcal{O}(X'')/\mathfrak{m}'_y\mathcal{O}(X'')$ is also finite dimensional and thus the fiber $(\phi'')^{-1}(y) \simeq \operatorname{Spec} \mathcal{O}(X'')/\mathfrak{m}'_y\mathcal{O}(X'')$ is a finite set.

Consider

$$\widetilde{X} := \{x'' \in X'' : (\phi'')^{-1}(\phi''(x'')) : \text{ is finite}\}$$

By the previous computation, $\widetilde{X}$ is a G-invariant open subset of X'' such that $X \subseteq \widetilde{X}$. By lemma 3.2.1, there is some G-invariant function $h \in \mathcal{O}(X'')^G$ such that $h_{|X''-\widetilde{X}} = 0$ and $h_{|X} \neq 0$. Let $X' := X''_h = \{h \neq 0\}$. X' is a G-invariant affine open subscheme of X'', and the restriction $\phi' := \phi''_{|X'} : X' \to Y'$ is a quasi-finite morphism that satisfies the desired conditions ☐

Theorem 6.1.1 (G-equivariant Zariski's main theorem)**.** *Let X and Y be affine G-schemes and let $\phi : X \to Y$ be a quasi-finite G-equivariant morphism. Then*

(a) There are an affine G-scheme Z, a G-equivariant open immersion $i : X \hookrightarrow Z$ and a G-equivariant finite morphism $\psi : Z \to Y$ such that $\phi = \psi \circ i$

(b) Denote by $\widetilde{\phi} : X//G \to Y//G$ the morphism induced by $\phi : X \to Y$, and suppose that $\widetilde{\phi}$ is finite. If there is some $x \in X^\bullet(k)$ such that $G \cdot x$ and $\phi(G \cdot x)$ are closed, then $\phi : X \to Y$ is a finite morphism

Proof. Let $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ be the k-algebra homomorphism induced by $\phi : X \to Y$, and denote $Y_0 := \operatorname{Spec} \mathcal{O}(Y)/\operatorname{Ker} \phi^*$. The canonical morphism $\mathcal{O}(Y)/\operatorname{Ker} \phi^* \to \mathcal{O}(X)$ is injective, and $\phi(X) \subseteq Y_0$. Besides, $\phi : X \to Y_0$ is also a quasi-finite morphism, and it's G-equivariant because $\operatorname{Ker} \phi^* \subseteq \mathcal{O}(Y)$ is invariant by the dual action of G on $\mathcal{O}(Y)$. This proves that we can suppose that $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective, so we are in the hypotheses of lemma 6.1.1.

Let X' and Y' as in lemma 6.1.1; there is a quasi-finite dominant G-equivariant morphism $\phi' : X' \to Y'$ such that $\phi'_{|X} = \phi$. Let B' be the integral closure of $\mathcal{O}(Y')$ in $\mathcal{O}(X')$. B' is then a finitely generated $\mathcal{O}(Y')$-module, and thus a k-algebra of finite type.

Let B be the image of B' in $\mathcal{O}(X)$ via the homomorphism induced by the inclusion $X \hookrightarrow X'$. Denote $Z = \operatorname{Spec} B$ and $Z' = \operatorname{Spec} B'$. There are inclusions

$$\mathcal{O}(Y') \subseteq B' \subseteq \mathcal{O}(X')$$
$$\mathcal{O}(Y) \subseteq B \subseteq \mathcal{O}(X)$$

that induce G-equivariant morphisms

$$i' : X' \to Z' \qquad \psi' : Z' \to Y'$$
$$i : X \to Z \qquad \psi : Z \to Y$$

if we prove that i' is an open immersion and that ψ' is a finite morphism, then, by restriction, i and ψ would also be an open immersion and a finite morphism respectively, so we would conclude.

Let's start proving that $i' : X' \to Z'$ is an open immersion. X' and Z' are normal varieties, so by Zariski's main theorem (see [Sta19, Tag 02LQ]), we just have to prove that $i' : X' \to Z'$ is birational and quasi-finite. Since $\phi' = \psi' \circ i'$ and ϕ' is quasi-finite, then i' is also quasi-finite. If we prove that $k(Z') = k(X')$, then i' would be birational.

In general, we have the inclusion $\mathcal{O}(Z') \hookrightarrow \mathcal{O}(X')$, so $k(Z') \subseteq k(X')$. If we prove that $\mathcal{O}(X') \subseteq k(Z')$, then we conclude. Let $\alpha \in \mathcal{O}(X')$. Since $\phi' : X' \to Y'$ is dominant and quasi-finite, then $k(X')$ and $k(Y')$ have the same trascendence degree, so α must be integral over $k(Y')$. This proves that there are $f_1, \dots, f_n \in k(Y')$ such that

$$\alpha^n + f_1\alpha^{n-1} + \ldots + f_1\alpha + f_n = 0$$

If we write $f_i = \frac{a_i}{b_i}$ for each $i = 1, \dots, n$ and denote $b := b_1 \cdot \ldots \cdot b_n$, then $b \cdot \alpha$ is clearly integral over $\mathcal{O}(Y')$, but B' is the integral closure of $\mathcal{O}(Y')$ in $\mathcal{O}(X')$, so $b \cdot \alpha \in B'$, and thus $\alpha \in k(Z')$, so $i' : X' \to Z'$ is indeed birational and thus an open immersion.

On the other hand, $\psi' : Z' \to Y'$ is a finite morphism, since it's induced by the inclusion $\mathcal{O}(Y') \hookrightarrow B'$ of $\mathcal{O}(Y')$ in its integral closure, and B' is a finitely generated $\mathcal{O}(Y')$-module.

The proof of (b) uses the decomposition given in (a) and the topological properties of G-invariant sets. For a proof, see [Lun73, p.89] □

Observation 6.1.1. *If X is an integral scheme, it follows immediately from the proof of theorem 6.1.1 that Z can be taken to be the spectrum of the integral closure of $\phi^*(\mathcal{O}(Y))$ in $\mathcal{O}(X)$*

6.2 G-equivariant étale morphisms

In this section, we will prove some technical results about G-equivariant étale morphisms used to prove Luna's theorem.

We will start by recalling some relations between étale morphisms and quotients by finite group actions.

Let G be a finite reductive algebraic group and X an affine normal G-scheme. Since G is finite, every G-orbit in X is closed, and thus the GIT quotient $\pi : X \to X//G$ is a good geometric quotient. Besides, $X//G$ is also a normal scheme by proposition 2.1.1. We state without proof the following

Proposition 6.2.1 ([Dré04] Propositions 4.10-4.12)**.** *Suppose that the action of* G *on* X *is faithful, i.e. the only element of* G *that acts via the identity is the identity element* $e \in G^{\bullet}(k)$*. Then,*

(a) The field extension $k(X//G) \hookrightarrow k(X)$ *is a finite Galois extension, and its Galois group is isomorphic to* G

(b) For every $x \in X^{\bullet}(k)$*, the quotient morphism* $\pi : X \to X//G$ *is étale on* x *if and only if* $G_x = \{e\}$

(c) For every $x \in X^{\bullet}(k)$ *and every subgroup* $H \subseteq G$*, the induced morphism* $X//H \to X//G$ *is étale on the image of* x *in* $X//H$ *if and only if* $G_x \subseteq H$

Let G be a reductive algebraic group (not necessarily finite) and let X and Y be affine G-schemes. Denote by $\pi_X : X \to X//G$ and $\pi_Y : Y \to Y//G$ the associated GIT quotients

Definition 6.2.1. *Let* $\phi : X \to Y$ *be a* G*-equivariant morphism.* ϕ *is strongly étale if*

- *The induced morphism* $\widetilde{\phi} : X//G \to Y//G$ *is étale and surjective*
- *The square*

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X//G & \xrightarrow[\ \widetilde{\phi}\ ]{} & Y//G \end{array}$$

is cartesian. In other words, $(\phi, \pi_X) : X \to Y \times_{Y//G} X//G$ *is an isomorphism of* G*-schemes*

Lemma 6.2.1. *Let* $\phi : X \to Y$ *a* G*-equivariant strongly étale morphism. The following properties are true*

- $\phi : X \to Y$ *is étale and surjective*
- *For every* $u \in (X//G)^{\bullet}(k)$*,* ϕ *induces a* G*-equivariant isomorphism*

$$\phi : \pi_X^{-1}(u) \xrightarrow{\simeq} \pi_Y^{-1}(\widetilde{\phi}(u))$$

- *For every* $x \in X^{\bullet}(k)$*, the restriction* $\phi_{|G \cdot x} : G \cdot x \to \phi(G \cdot x)$ *is injective. Besides,* $G \cdot x \subseteq X$ *is closed if and only if* $\phi(G \cdot x) = G \cdot \phi(x) \subseteq Y$ *is closed*

Proof. Since $\phi : X \to Y$ is a G-equivariant strongly étale morphism, the following diagram is cartesian

$$\begin{array}{ccc} X & \xrightarrow{\pi_X} & X//G \\ \phi \downarrow & & \downarrow \widetilde{\phi} \\ Y & \xrightarrow[\pi_Y]{} & Y//G \end{array}$$

By definition, $\widetilde{\phi} : X//G \to Y//G$ is a surjetive étale morphism, so $\phi : X \to Y$, being the base change of $\widetilde{\phi}$ via $\pi_Y : Y \to Y//G$, is also surjective and étale.

For every $u \in (X//G)^\bullet(k)$, we have that

$$\begin{aligned} \pi_X^{-1}(u) &= X \times_{X//G} \{u\} \simeq \\ &\simeq (Y \times_{Y//G} X//G) \times_{X//G} \{u\} \simeq \\ &\simeq Y \times_{Y//G} \{\widetilde{\phi}(u)\} \simeq \\ &\simeq \pi_Y^{-1}(\widetilde{\phi}(u)) \end{aligned}$$

so $\phi : \pi_X^{-1}(u) \to \pi_Y^{-1}(\widetilde{\phi}(u))$ is a G-equivariant isomorphism.

Finally, for every $x \in X^\bullet(k)$, we have that $G \cdot x \subseteq \pi_X^{-1}(\pi_X(x))$, so the restriction to $G \cdot x$ of the isomorphism

$$\phi : \pi_X^{-1}(\pi_X(x)) \xrightarrow{\simeq} \pi_Y^{-1}(\widetilde{\phi}(\pi_X(x))) \simeq \pi_Y^{-1}(\pi_Y(\phi(x))$$

is injective. In particular, $G \cdot x \subseteq \pi_X^{-1}(\pi_X(x))$ is closed if and only if $\phi(G \cdot x) = G \cdot \phi(x) \subseteq \pi_Y^{-1}(\pi_Y(\phi(x)))$ is closed □

We will now study some important technical conditions under which we can construct strongly étale morphisms

Lemma 6.2.2. *Let X and Y be normal affine G-schemes and let $\theta : X \to Y$ be a finite G-equivariant morphism. Let $u \in (X//G)^\bullet(k)$, and let $G \cdot x$ be the unique closed G-orbit contained in $\pi_X^{-1}(u)$ (recall proposition 2.1.3). Suppose that*

- $\theta : X \to Y$ *is étale on* x
- $\theta_{|G \cdot x} : G \cdot x \to G \cdot \theta(x)$ *is injective*

then, the induced morphism $\widetilde{\theta} : X//G \to Y//G$ is étale on u

Proof. We will suppose that G is connected (see [Lun73, § II, Lemme 1] for the general case).

X and Y are normal schemes and $\theta : X \to Y$ is étale on x, so the induced k-algebra homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective. Besides, being étale is an open and G-invariant property and the conditions of the lemma are local, so we can suppose that $\theta : X \to Y$ is étale and that $\mathcal{O}(Y)$ is a subalgebra of $\mathcal{O}(X)$.

$\theta : X \to Y$ is a finite morphism, so $k(Y) \hookrightarrow k(Y)$ is a finite field extension and we can find a finite Galois extension $k(Y) \hookrightarrow K$ such that $k(X) \subseteq K$. Let $\mathcal{G}$ be the Galois group of the extension $k(Y) \hookrightarrow K$, and let $\mathcal{H}$ be a subgroup of $\mathcal{G}$ such that $K^{\mathcal{H}} = k(X)$.

Let C be the integral closure of $\mathcal{O}(Y)$ in K, and let C' be the integral closure of $\mathcal{O}(Y)^G$ in K. C and C' are finite type $\mathcal{G}$-invariant k-algebras. Denote $Z := \operatorname{Spec} C$ and $Z' := \operatorname{Spec} C'$. Since $\mathcal{G}$ is a finite group, it's reductive, so the quotients $\pi_{\mathcal{G}} : Z \to Z//\mathcal{G}$ and $\pi'_{\mathcal{G}} : Z' \to Z'//\mathcal{G}$ exist by theorem 3.2.1. We have that

(a) $Z//\mathcal{G} \simeq Y$

Indeed, we have that

$$\begin{aligned} Z//\mathcal{G} &= \operatorname{Spec} C^{\mathcal{G}} = \\ &= (\text{C is the integral closure of } \mathcal{O}(Y) \text{ in the Galois extension } k(Y) \hookrightarrow K) = \\ &= \operatorname{Spec} \mathcal{O}(Y) = \\ &= Y \end{aligned}$$

(b) $Z'//\mathcal{G} \simeq Y//G$

By theorem 3.2.1, we just have to prove that $\mathcal{O}(Y)^G \simeq (C')^{\mathcal{G}}$. We have that $\mathcal{O}(Y)^G \subseteq C'$ because C' is the integral closure of $\mathcal{O}(Y)^G$ in K. Besides, the elements of $\mathcal{O}(Y)^G$ are $\mathcal{G}$-invariant, since

$$\mathcal{O}(Y)^G \subseteq \mathcal{O}(Y) = C^{\mathcal{G}}$$

and thus $\mathcal{O}(Y)^G \subseteq (C')^{\mathcal{G}}$.

On the other hand, C' is integral over $\mathcal{O}(Y)^G$, so $(C')^{\mathcal{G}}$ is also integral over $\mathcal{O}(Y)^G$. Besides, $(C')^{\mathcal{G}} \subset K^{\mathcal{G}} = k(Y)$. Since Y is a normal scheme, $\mathcal{O}(Y)$ is integrally closed in $k(Y)$ and thus $(C')^{\mathcal{G}} \subseteq \mathcal{O}(Y)$. If we prove that $\mathcal{O}(Y)^G$ is integrally closed in $\mathcal{O}(Y)$, we conclude.

Let $f \in \mathcal{O}(Y)$ be such that there exist $a_1, \ldots, a_n \in \mathcal{O}(Y)^G$ with

$$f^n + a_1 f^{n-1} + \ldots + a_n = 0$$

for every $g \in G$ we have that

$$(g \cdot f)^n + a_1 (g \cdot f)^{n-1} + \ldots + a_n = 0$$

but the polynomial $t^n + a_1 t^{n-1} + \ldots + a_n$ has at most a finite number of roots in K, so $G \cdot f$ must be a finite set. Since G is connected, it must be $f \in \mathcal{O}(Y)^G$, so we conclude

(c) $Z//\mathcal{H} \simeq X$ and $Z'//\mathcal{H} \simeq X//G$

These isomorphisms can be proven in a completely analogous way to (a) and (b)

We have thus proven that there is a commutative diagram

$$\begin{array}{ccc}
Z & \longrightarrow & Z' \\
{\scriptstyle \pi_{\mathcal{H}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi'_{\mathcal{H}}} \\
Z//\mathcal{H} = X & \xrightarrow{\pi_X} & Z'//\mathcal{H} = X//G \\
{\scriptstyle \theta}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\theta}} \\
Z//\mathcal{G} = Y & \xrightarrow[\pi_Y]{} & Z'//\mathcal{G} = Y//G
\end{array}$$

such that $\theta \circ \pi_{\mathcal{H}} = \pi_{\mathcal{G}}$ and $\widetilde{\theta} \circ \pi'_{\mathcal{H}} = \pi'_{\mathcal{G}}$. Let $z \in \pi_{\mathcal{H}}^{-1}(x)$ and denote by z' the image of z via $Z \to Z'$. Clearly, $\pi'_{\mathcal{H}}(z') = u \in X//G$.

$\theta : X = Z//\mathcal{H} \to Y = Z//\mathcal{G}$ is étale on $\pi_{\mathcal{H}}(z) = x$, so by proposition 6.2.1 we deduce that $\mathcal{G}_z \subseteq \mathcal{H}$. If we prove that

$$\widetilde{\theta} : X//G = Z'//\mathcal{H} \to Y = Z//\mathcal{G}$$

is étale on u, we conclude. Again, by proposition 6.2.1, it suffices to prove that $\mathcal{G}_{z'} \subseteq \mathcal{H}$.

Let $\sigma \in \mathcal{G}_{z'}$. Then, if $y = \theta(x)$, we have that $\theta(\pi_{\mathcal{H}}(\sigma \cdot z)) = \pi_{\mathcal{G}}(\sigma \cdot z) = \pi_{\mathcal{G}}(z) = y$. Analogously, we have that $\pi_X(\pi_{\mathcal{H}}(\sigma \cdot z)) = \pi'_{\mathcal{H}}(\sigma \cdot z') = \pi'_{\mathcal{H}}(z') = u$.

θ is a finite morphism and $G \cdot x$ is a closed orbit by hypothesis, so $G \cdot y$ and $G \cdot \pi_{\mathcal{H}}(\sigma \cdot z)$ are also closed. On the other hand, $G \cdot x$ is the unique closed orbit in $\pi_X^{-1}(u)$, so $\pi_{\mathcal{H}}(\sigma \cdot z) \in G \cdot x$. The restriction $\theta_{|G \cdot x}$ is injective, so $\pi_{\mathcal{H}}(\sigma \cdot z) = x = \pi_{\mathcal{H}}(z)$ and there must be some $\tau \in \mathcal{H}$ such that $\tau \cdot z = \sigma \cdot z$ and thus $\tau^{-1} \cdot \sigma \in \mathcal{G}_z \subseteq \mathcal{H}$, so $\sigma \in \mathcal{H}$ and we conclude $\square$

Lemma 6.2.3. *Let X and Y be affine G-schemes and $\theta : X \to Y$ a G-equivariant morphism. Suppose that X is normal at $x \in X^{\bullet}(k)$ and that*

- $\theta : X \to Y$ *is étale on* x
- $G \cdot \theta(x) \subseteq Y$ *is closed*
- $\theta_{|G \cdot x} : G \cdot x \to G \cdot \theta(x)$ *is injective*

then, there is an affine open subset $\widetilde{U} \subseteq X//G$ containing $\pi_X(x)$ such that, if $U := \pi_X^{-1}(\widetilde{U})$,

- $\theta_{|U} : U \to \theta(U)$ *and the induced morphism* $\widetilde{\theta}_{|\widetilde{U}} : \widetilde{U} \to \widetilde{\theta}(\widetilde{U})$ *are étale*
- $V := \theta(U)$ *and* $\widetilde{V} := \widetilde{\theta}(\widetilde{U})$ *are affine open subsets and* $V = \pi_Y^{-1}(\widetilde{V})$
- *The image by* θ *of every closed orbit in* U *is closed*

Proof. Being normal is an open and G-invariant property. Besides, the image of a normal point by an étale morphism is a normal point, hence Y is normal at $\theta(x)$.

By lemma 3.2.1, there are $f \in \mathcal{O}(X)^G$ and $h \in \mathcal{O}(Y)^G$ such that X_f and Y_h are normal affine schemes containing x and $\theta(x)$, respectively. Since f and h are G-invariant functions, $X_f \cap \theta^{-1}(Y_h) = X_{f\cdot\theta^*(h)}$ is a normal, saturated affine open subset of X such that $\theta(X_f \cap \theta^{-1}(Y_h)) \subseteq Y_h$. Hence, we can suppose that X and Y are normal schemes. By a similar argument, we can suppose that $X//G$ and $Y//G$ are irreducible schemes and that $\theta : X \to Y$ is a quasi-finite morphism.

By theorem 6.1.1, there is a normal scheme Z (recall observation 6.1.1), a G-equivariant open immersion $i : X \hookrightarrow Z$ and a finite G-equivariant morphism $\psi : Z \to Y$ such that $\theta = \psi \circ i$.

Since ψ is a finite morphism, $\psi^{-1}(G \cdot \theta(x))$ contains only a finite number of orbits of the same dimension as $G \cdot x$, hence $G \cdot x$ is closed in Z. Besides, ψ is étale on x. By lemma 6.2.2, the induced morphism $\widetilde{\psi} : Z//G \to Y//G$ is étale on $\pi_Z(x) = \widetilde{i}(\pi_X(x))$.

Let $f \in \mathcal{O}(Z)^G$. The set $\{z \in Z : z \notin X_f \text{ and } \psi \text{ is not étale on } z\}$ is a closed G-invariant subset of Z. Besides, it does not contain the closed orbit $G \cdot x$. By lemma 3.2.1, we can suppose that $Z_f = X_f$ and that ψ and $\widetilde{\psi}$ are étale on X_f and $X_f//G = (X//G)_f$, respectively.

Since $\theta_{|X_f} = \psi_{X_f} \circ i_{X_f}$ is a composition of an open immersion and an étale morphism, it's étale, and thus $\theta(X_f)$ is open. By the same arguments as before, there is some $h \in \mathcal{O}(Y)^G$ such that $G \cdot \theta(x) \subseteq Y_h \subseteq \theta(X_f)$.

Define $\widetilde{U} := (X//G)_f \cap \widetilde{\theta}^{-1}((Y//G)_h)$. We have that $\widetilde{V} = \widetilde{\theta}(\widetilde{U}) = (Y//G)_h$, because $Y_h \subseteq \theta(X_f)$. Clearly, $U = \pi_X^{-1}(\widetilde{U}) = X_f \cap \theta^{-1}(Y_h)$ and $V = \theta(U) = Y_h$.

Finally, we have seen that θ is étale on U. Similarly, $\widetilde{\theta}$ is étale on $\widetilde{U}$ because $\widetilde{\theta} = \widetilde{\psi} \circ \widetilde{i}$ and $\widetilde{\psi}$ is étale on $Z_f = X_f$. The image of every closed orbit in U by θ is closed, because every closed orbit of U is closed in Z and $\psi : Z \to Y$ is finite □

Proposition 6.2.2. *Let* X *and* Y *be affine* G*-schemes and suppose that* X *is normal at* $x \in X^\bullet(k)$. *Let* $\theta : X \to Y$ *be a* G*-equivariant morphism such that*

- θ *is étale on* x
- $\theta(G \cdot x) = G \cdot \theta(x)$ *is closed*
- *The restriction* $\theta_{|G\cdot x} : G \cdot x \to G \cdot \theta(x)$ *is injective*

Then, there is an affine G*-invariant open subset* $U \subset X$ *such that* $x \in U$ *and*

(a) U *is saturated with respect to the* G*-action*

(b) $V = \theta(U)$ *is an affine* G*-invariant saturated open subset of* Y *with respect to the* G*-action*

(c) The restriction $\theta_{|U} : U \to V$ is strongly étale

Proof. Properties (a) and (b) follow directly from lemma 6.2.3. Besides, also from lemma 6.2.3, we have that the induced morphism $\widetilde{\theta} : U//G \to V//G$ is étale. It only remains to prove that the square

$$\begin{array}{ccc} U & \xrightarrow{\theta} & V \\ {\scriptstyle \pi_U}\downarrow & & \downarrow{\scriptstyle \pi_V} \\ U//G & \xrightarrow[\widetilde{\theta}]{} & V//G \end{array}$$

is cartesian, and this is a consequence of the fact that the GIT quotient is uniform (recall theorem 3.2.1) □

In the proof of Luna's theorem we will need a way to restrict the previous constructions to G-invariant subschemes. This is given by the following

Lemma 6.2.4. *With the previous notations, let $Y' \subseteq Y$ be a G-invariant affine subscheme of Y and $X' := Y' \times_Y X$. Let $\theta' = \mathrm{Id}_{Y'} \times \theta : X' = Y' \times_Y X \to Y' \times_Y Y \simeq Y'$. Suppose that $\theta : X \to Y$ and $\widetilde{\theta} : X//G \to Y//G$ are G-equivariant étale morphisms, and suppose that $(\pi_X, \theta) : X \to X//G \times_{Y//G} Y$ is a G-equivariant isomorphism (locally, this is the situation of proposition 6.2.2). Then, $\theta' : X' \to Y'$ and $\widetilde{\theta'} : X'//G \to Y'//G$ are étale morphisms, and besides $(\theta', \pi_{X'}) : X' \to Y' \times_{Y'//G} X'//G$ is a G-equivariant isomorphism*

Proof. $\theta' : X' \to Y'$ is an étale morphism, because it's the base change of the étale morphism $\theta : X \to Y$ by the inclusion $Y' \hookrightarrow Y$.

On the other hand, we have that

$$\begin{aligned} X' = Y' \times_Y X &\simeq \\ &\simeq Y' \times_Y (X//G \times_{Y//G} Y) \simeq \\ &\simeq Y' \times_{Y//G} X//G \end{aligned}$$

and thus

$$X'//G \simeq Y'//G \times_{Y//G} X//G$$

so $\widetilde{\theta'} : X'//G \to Y'//G$ is an étale morphism, since it's the base change of the étale morphism $\widetilde{\theta} : X//G \to Y//G$ by $Y'//G \hookrightarrow Y//G$.

Finally, we have that

$$\begin{aligned} X'//G \times_{Y'//G} Y' &\simeq (X//G \times_{Y//G} Y'//G) \times_{Y'//G} Y' \simeq \\ &\simeq X//G \times_{Y//G} Y' \simeq \\ &\simeq X' \end{aligned}$$

and we conclude □

6.3 Étale slice theorem

Let G be a reductive algebraic group and X an affine G-scheme

Lemma 6.3.1. *Let* $x \in X^\bullet(k)$ *be a regular point such that* G_x *is a reductive algebraic group. Denote by* $\mathbb{T}_xX = \operatorname{Spec}\operatorname{Sym}^\bullet(T_x^*X)$ *the scheme associated to the tangent space* $T_xX = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ *to* X *at* x. *There is a morphism* $\phi : X \to \mathbb{T}_xX$ *such that*

- $\phi : X \to \mathbb{T}_xX$ *is* G_x*-equivariant*
- ϕ *is étale at* x
- $\phi(x) = 0 \in (\mathbb{T}_xX)^\bullet(k) = T_xX$

Proof. Let $\mathfrak{m}_x \subset \mathcal{O}(X)$ be the maximal ideal given by $x \in X^\bullet(k)$. X is regular at x, so $\mathfrak{m}_x/\mathfrak{m}_x^2 \simeq T_x^*X$ as vector spaces. A direct computation shows that the quotient morphism

$$d_x : \mathfrak{m}_x \to \mathfrak{m}_x/\mathfrak{m}_x^2 \simeq T_x^*X$$

is G_x-equivariant. Since G_x is reductive, we can use Schur's lemma to find a finite dimensional G_x-invariant subspace V of $\mathfrak{m}_x$ containing a system of generators of $\mathfrak{m}_x$ such that the restriction $d_x : V \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is a G_x-equivariant isomorphism. Denote by $\alpha : \mathfrak{m}_x/\mathfrak{m}_x^2 \to V$ its inverse.

Composing the extension $\alpha : \operatorname{Sym}^\bullet(\mathfrak{m}_x/\mathfrak{m}_x^2) \to \operatorname{Sym}^\bullet(V)$ with the canonical morphism $\operatorname{Sym}^\bullet(V) \to \mathcal{O}(X)$, we obtain a morphism $\operatorname{Sym}^\bullet(T_x^*X) \to \mathcal{O}(X)$.

Let $\phi : X \to \mathbb{T}_xX$ be the morphism of schemes induced by $\operatorname{Sym}^\bullet(T_x^*X) \to \mathcal{O}(X)$. ϕ is étale at x because, by construction, the tangent map at x is the transpose to the linear isomorphism $d_x : V \xrightarrow{\simeq} \mathfrak{m}_x/\mathfrak{m}_x^2$. The rest of the conditions of the lemma follow immediately ☐

Before proving Luna's theorem, we will recall some useful results about associated bundles.

Let H be a reductive algebraic subgroup of G. There is a natural left action

$$\begin{aligned} H^\bullet \times G^\bullet &\to G^\bullet \\ (h, g) &\mapsto g \cdot h^{-1} \end{aligned}$$

defined at points of G and H with values in some scheme. Clearly, $G \to G/H$ is a principal H-bundle.

Let Y be an affine H-scheme. There is a natural H-action on $G \times Y$ given by

$$\begin{aligned} H^\bullet \times (G \times Y)^\bullet &\to (G \times Y)^\bullet = G^\bullet \times Y^\bullet \\ (h, (g, y)) &\mapsto h \cdot (g, y) := (g \cdot h^{-1}, h \cdot y) \end{aligned}$$

defined at points with values.

The bundle associated to the principal H-bundle $G \to G/H$ and the H-scheme Y is

$$G \times_H Y := (G \times Y)//H$$

where the quotient is taken with respect to the H-action on $G \times Y$ that we have just defined, and exists by theorem 3.2.1

Proposition 6.3.1 (Properties of the associated bundle)**.** *With the previous notations, let* $X = G \times_H Y$*. The following properties are true*

(a) Consider the G*-action on* $X = Y \times_H Y$ *by multiplication on the* G*-component. The quotient* $X//G$ *exists, and it's isomorphic to* $Y//H$

(b) For every $u \in (X//G)^\bullet(k)$*, we have a* G*-equivariant isomorphism*

$$\pi_X^{-1}(u) \simeq G \times_H \pi_Y^{-1}(u)$$

(c) For every $y \in Y^\bullet(k)$ *and* $g \in G^\bullet(k)$*, if we denote by* $u = \overline{(g, y)}$ *the image of* $(g, y) \in (G \times Y)^\bullet(k)$ *in* X*, then* $G_u = g \cdot H_y \cdot g^{-1}$

(d) Let $X' \subseteq X$ *be a* G*-invariant affine subscheme of* X*. There is a* H*-invariant affine subscheme* $Y' \subseteq Y$ *such that*

$$X' \simeq G \times_H Y'$$

(e) For every $y \in Y^\bullet(k)$*, let* $\overline{y} := \overline{(e, y)} \in X^\bullet(k)$*. There is an isomorphism of vector spaces*

$$T_{\overline{y}}X \simeq (T_e G \oplus T_y Y)/T_e H$$

Proof. We will prove each property separately

(a) Consider the inclusion of Y in $G \times Y$ via the identity element $e \in G^\bullet(k)$

$$Y^\bullet \to G^\bullet \times Y^\bullet$$
$$y \mapsto (e, y)$$

defined at points with values. The induced morphism $Y \to X//G$ is clearly H-invariant, and thus induces a morphism $Y//H \to X//G$.

On the other hand, consider the projection on the Y-component $G \times Y \to Y$. The composition $G \times Y \to Y//H$ is H-invariant and induces a morphism $X \to Y//H$. It's easy to prove that $X \to Y//H$ is G-invariant and induces a morphism $X//G \to Y//H$. A direct computation shows that the morphisms that we have constructed are mutually inverse

(b) For every $u \in (X//G)^\bullet(k)$, we have that

$$\begin{aligned}\pi_X^{-1}(u) &\simeq X \times_{X//G} \{u\} \simeq \\ &\simeq (G \times_H Y) \times_{X//G} \{u\} \simeq \\ &\simeq \text{(By (a) we have } X//G \simeq Y//H) \simeq \\ &\simeq ((G \times Y)//H) \times_{Y//H} \{u\} \simeq \\ &\simeq G \times_H \pi_Y^{-1}(u)\end{aligned}$$

(c) Let $h \in G^\bullet(k)$ be such that $h \cdot \overline{(g, y)} = \overline{(g, y)}$. By definition, there is some $\widehat{h} \in H^\bullet(k)$ such that

$$(h \cdot g, y) = \widehat{h} \cdot (g, y) = (g \cdot \widehat{h}^{-1}, \widehat{h} \cdot y)$$

and thus $\widehat{h} \cdot y = y$ and $h = g \cdot \widehat{h}^{-1} \cdot g^{-1}$, so $h \in g \cdot H_y \cdot g^{-1}$

(d) Let $\pi_{G\times Y} : G \times Y \to X$ be the quotient morphism. Since GIT quotients are universal quotients, if we denote $Z := \pi_{G\times Y}^{-1}(X') = (G \times Y) \times_X X'$, then the induced morphism $Z \to X'$ is a quotient for the H-action on Z (note that Z is an affine scheme since X' is affine and GIT quotients are affine morphisms). On the other hand, $X' \subseteq X$ is a G-invariant subscheme, so $Z \subseteq G \times Y$ is H-invariant and G-invariant with respect to the G-action by multiplication on the G-component. This means that necessarily $Z = G \times Y'$, where Y' is a H-invariant affine subscheme of Y. Since $X' = Z//H$, then $X' \simeq G \times_H Y'$

□

Theorem 6.3.1 ([Lun73], § 3)**.** *Let* G *be a reductive algebraic group,* (X, σ) *an affine* G-*scheme, and* $x \in X^\bullet(k)$. *If* $G \cdot x$ *is closed, there is a locally closed subscheme* $U \subseteq X$ *such that*

- U *is affine and* $x \in U$
- U *is* G_x*-invariant*
- *The restriction of the* G*-action to* U *is a* G_x*-invariant morphism* $G \times U \to X$, *and the induced morphism* $\psi : G \times_{G_x} U \to X$ *has an open saturated affine image* $V := \psi(U)$ *with respect to the* G*-action*
- $\psi : G \times_{G_x} U \to V$ *is a* G*-equivariant strongly étale morphism*

Proof. By lemma 3.1.2, we can find a finite dimensional G-invariant subspace $W \subseteq \mathcal{O}(X)$ containing a system of generators of $\mathcal{O}(X)$ such that the induced k-algebra homomorphism $\mathrm{Sym}^\bullet(W) \to \mathcal{O}(X)$ is G-equivariant and surjective, so X is isomorphic to a G-invariant closed subscheme of the smooth affine scheme $\operatorname{Spec}\mathrm{Sym}^\bullet(W)$. By lemma 6.2.4 and proposition 6.3.1, we see that it suffices to prove the result for the case of a smooth scheme.

We will suppose then that X is a smooth scheme at x. Since the G-orbit $G \cdot x$ is closed, the stabilizer subgroup G_x is a reductive algebraic group (see [Mat60]), and thus, by lemma 6.3.1, there is a G_x-equivariant morphism $\phi : X \to \mathbb{T}_x X$ that is étale at x and that satisfies $\phi(x) = 0$.

G_x is a reductive algebraic group, so there is a G_x-invariant vector subspace $N \subseteq T_x X$ such that (recall that $\operatorname{char}(k) = 0$)

$$T_x X = T_x(G \cdot x) \oplus N$$

If we also denote by N the subscheme of $\mathbb{T}_x X$ induced by the subspace N, then we can take $Y := \phi^{-1}(N)$, a closed G_x-invariant subscheme of X that is smooth at x (since $\phi : Y \to N$ is étale at x and N is smooth at x).

Consider the associated bundle $G \times_{G_x} Y$ and the restriction $\widetilde{\sigma} : G \times Y \to X$ of the G-action on X to Y. $\widetilde{\sigma}$ is clearly a G_x-invariant morphism, because for every scheme S, $h \in G_x^\bullet(S)$ and every $(g, y) \in (G \times Y)^\bullet(S)$, we have that

$$\begin{aligned}
\widetilde{\sigma}(h \cdot (g, y)) &= \widetilde{\sigma}((g \cdot h^{-1}, h \cdot y)) = \\
&= (g \cdot h^{-1}) \cdot (h \cdot y) = \\
&= g \cdot y = \\
&= \widetilde{\sigma}(g, y)
\end{aligned}$$

so it induces a morphism $\psi : G \times_{G_x} Y \to X$. Besides, by proposition 6.3.1, the associated bundle $G \times_{G_x} Y$ is smooth at $\overline{x} = \overline{(e, x)}$, and we have that

$$\begin{aligned}
T_{\overline{x}}(G \times_{G_x} Y) &= (T_e G \oplus T_x Y)/T_e G_x \simeq \\
&\simeq (\text{Since } Y = N \times_{\mathbb{T}_x X} T_x X \text{ its tangent space at } x \text{ is } N) \simeq \\
&\simeq (T_e G \oplus N)/T_e G_x \simeq \\
&\simeq T_e(G/G_x) \oplus N \simeq \\
&\simeq T_x(G \cdot x) \oplus N \simeq \\
&\simeq T_x X
\end{aligned}$$

We have obtained a morphism $\psi : G \times_{G_x} Y \to X$ such that

- ψ is G-equivariant with respect to the action of G on $G \times_{G_x} Y$ given by multiplication on the G-component
- ψ is étale at $\overline{x}$
- $G \times_{G_x} Y$ is smooth at $\overline{x}$ (in particular, it's a normal scheme at $\overline{x}$)
- $\psi(\overline{x}) = x$, and $\psi(G \cdot \overline{x}) = G \cdot x$ is closed by hypothesis
- By proposition 6.3.1, there is an isomorphism $G \cdot \overline{x} \simeq G \times_{G_x} (G_x \cdot x) \simeq G \times_{G_x} \{x\}$, and thus $\psi_{|G \cdot \overline{x}}$ is injective

by proposition 6.2.2, there is a G-invariant saturated affine open subset $U' \subseteq G \times_{G_x} Y$ such that $V := \psi(U')$ is a saturated affine open subset of X and the restriction $\psi : U \to V$ is a G-equivariant strongly étale morphism. By proposition 6.3.1, there is a G_x-invariant affine open set $U \subseteq Y$ and a G-equivariant isomorphism $U' \simeq G \times_{G_x} U$. By definition, U is an open subset of the closed subscheme $Y \subseteq X$ and thus it's a locally closed subscheme of X satisfying the conditions of the theorem □

Corollary 6.3.1. *Let* G *be a reductive algebraic group and let* X *be a* G*-scheme. Let* $\mathcal{L}$ *be a* G*-linearized invertible sheaf on* X*. Then,* $\pi : X^s(\mathcal{L}) \to X^s(\mathcal{L})//G$ *is a principal* G*-bundle. In particular, if* $x \in X^s(\mathcal{L})^\bullet(k)$ *is smooth, then* $\pi(x) \in X^s(\mathcal{L})//G$ *is also smooth*

Proof. By definition, for every $x \in X^s(\mathcal{L})^\bullet(k)$ we have that $G \cdot x$ is closed and $G_x = \{e\}$. By theorem 6.3.1, there is a locally closed affine subscheme $U \subseteq X$ containing x such that the restriction of the action $G \times_{\{e\}} U \simeq G \times U \to V$ is a strongly étale morphism, where V is an affine saturated open subset of $X^s(\mathcal{L})$.

Since $G \times U \to V$ is strongly étale, the induced morphism $(G \times U)//G \simeq U \to V//G$ is a surjective étale morphism, and $V//G \subseteq X//G$ is an affine open subset of $X^s(\mathcal{L})//G$ by lemma 3.2.5. Besides, we have a G-equivariant isomorphism $V//G \times_{X//G} U \simeq G \times U$. This proves that $X^s(\mathcal{L}) \to X^s(\mathcal{L})//G$ is a principal G-bundle □

Part II

Moduli of vector bundles over a curve

Chapter 7

Riemann-Roch theorem for algebraic curves

In this chapter, we will extend the Riemann-Roch formula for invertible sheaves on a smooth projective algebraic curve to arbitrary coherent sheaves.

Let X be a smooth projective algebraic curve of genus g, and let $\mathcal{L}$ be an invertible sheaf on X. The Riemann-Roch formula is a well known result that computes the Euler characteristic of $\mathcal{L}$ in terms of its degree and the genus of X

Theorem 7.0.1 (Riemann-Roch, [Har77], Chapter IV, Theorem 1.3)**.** *Denote by $\chi(\mathcal{L})$ the Euler characteristic of $\mathcal{L}$. Then*

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \deg(\mathcal{L}) = 1 - g + \deg(\mathcal{L})$$

In this chapter, we will extend this formula to an arbitrary coherent sheaf $\mathcal{M}$ on X. This will allow us to introduce several important notions that will be useful for the construction of moduli spaces of vector bundles over curves

7.1 Rank of a coherent sheaf

In this section, X will denote an integral algebraic variety

Lemma 7.1.1. *Let $\mathcal{M}$ be a coherent sheaf on X. There is an open subset $U \subseteq X$ such that $\mathcal{M}_{|U}$ is locally free*

Proof. The condition of being a locally free $\mathcal{O}_X$-module is open. Let $x_0 \in X$ be the generic point of X. If $\mathcal{M}_{x_0} \neq 0$, then $\mathcal{M}_{x_0}$ is a free $k(X)$-vector space, and hence $\mathcal{M}$ is locally free on an open neighbourhood of x_0. The result follows □

Observation 7.1.1. *The open subset U in lemma 7.1.1 is empty if $\mathcal{M}_{x_0} = 0$*

Definition 7.1.1. *Let X be an integral algebraic variety and let $\mathcal{M}$ be a coherent sheaf on X. Let $U \subseteq X$ be an open subset such that $\mathcal{M}_{|U}$ is locally free. The rank of $\mathcal{M}$ is*

$$\mathrm{rk}(\mathcal{M}) := \mathrm{rk}(\mathcal{M}_{|U})$$

Observation 7.1.2. *The rank of* $\mathcal{M}$ *is independent of the open subset chosen in lemma 7.1.1. Indeed, if* $x_0 \in X$ *is the generic point of* X*, then* x_0 *is contained in every non-empty open subset* $U \subseteq X$ *and* $\mathrm{rk}(\mathcal{M}_{|U}) = \mathrm{rk}(\mathcal{M}_{x_0})$*. Note that the rank of* $\mathcal{M}$ *will be zero if* $\mathcal{M}_{x_0} = 0$

Recall that the support of a coherent sheaf $\mathcal{M}$ on X is defined as

$$\mathrm{Supp}(\mathcal{M}) := \{x \in X : \mathcal{M}_x \neq 0\}$$

$\mathrm{Supp}(\mathcal{M})$ is a closed subset of X (see [GW10, Corollary 7.31])

Lemma 7.1.2. *The following statements are equivalent*

(a) $\mathrm{rk}(\mathcal{M}) = 0$

(b) $\mathrm{Supp}(\mathcal{M}) \neq X$

(c) For every $x \in X$*,* $\mathcal{M}_x$ *is either zero or a torsion* $\mathcal{O}_{X,x_0}$*-module*

Proof. Let $x_0 \in X$ be the generic point of X. Then, $\overline{\{x_0\}} = X$ and we have

$$\mathrm{rk}(\mathcal{M}) \neq 0 \Leftrightarrow x_0 \in \mathrm{Supp}(\mathcal{M}) \Leftrightarrow \mathrm{Supp}(\mathcal{M}) = X$$

because $\mathrm{Supp}(\mathcal{M}) \subseteq X$ is a closed subset. This proves the equivalence between (a) and (b).

Suppose now that $\mathrm{rk}(\mathcal{M}) = 0$, i.e., that $\mathcal{M}_{x_0} = 0$. Moreover, for every $x \in X$ there is a natural isomorphism

$$\mathcal{M}_{x_0} \simeq \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x_0}$$

and thus $\mathcal{M}_x$ is a torsion $\mathcal{O}_{X,x_0}$-module. On the other hand, if $\mathcal{M}_x$ is a torsion $\mathcal{O}_{X,x_0}$-module for any $x \in X$, then $\mathcal{M}_{x_0} = \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x_0} = 0$ and thus $\mathcal{M}_{x_0} = 0$, so (a) and (c) are equivalent □

We will denote by $\mathrm{Coh}(X)$ the category of coherent sheaves on X. We will write $\mathcal{M} \in \mathrm{Coh}(X)$ to say that $\mathcal{M}$ is an object of $\mathrm{Coh}(X)$, i.e., a coherent sheaf on X

Lemma 7.1.3. *Let* $\mathcal{M}', \mathcal{M}, \mathcal{M}'' \in \mathrm{Coh}(X)$ *and suppose that we have an exact sequence*

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

then

$$\mathrm{rk}(\mathcal{M}) = \mathrm{rk}(\mathcal{M}') + \mathrm{rk}(\mathcal{M}'')$$

Proof. Let $x_0 \in X$ be the generic point of X. There is an induced exact sequence of $\mathcal{O}_{X,x_0}$-vector spaces:

$$0 \to \mathcal{M}'_{x_0} \to \mathcal{M}_{x_0} \to \mathcal{M}''_{x_0} \to 0$$

so the result follows from the fact that the dimension of a vector space is an additive function □

7.2 The Grothendieck group of a curve

Let X be a smooth projective algebraic curve and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X.

We will start this section by proving that every coherent sheaf on X has a finite locally free resolution

Lemma 7.2.1. *Let $(A, \mathfrak{m})$ be a local ring and let M be a finitely generated A-module. Then*

$$M \textit{ is a free } A\textit{-module} \Leftrightarrow \operatorname{Tor}^1(M, A/\mathfrak{m}) = 0$$

Proof. If M is a free A-module, then M is flat and thus $\operatorname{Tor}^1(M, -) = L^1(M \otimes_A -) = 0$, so in particular $\operatorname{Tor}^1(M, A/\mathfrak{m}) = 0$.

Suppose now that $\operatorname{Tor}^1(M, A/\mathfrak{m}) = 0$. Let $\{m_1, \dots, m_r\}$ be a minimal system of generators of M. There is an exact sequence

$$0 \longrightarrow N \longrightarrow A^{\oplus r} \xrightarrow{\phi} M \longrightarrow 0$$

where $\phi = (m_1, \dots, m_r)$. Applying $A/\mathfrak{m} \otimes_A -$, we obtain the long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}^1(M, A/\mathfrak{m}) = 0 \longrightarrow N/\mathfrak{m}N \longrightarrow (A/\mathfrak{m})^{\oplus r} \xrightarrow{\simeq} M/\mathfrak{m}M \longrightarrow 0$$

where $(A/\mathfrak{m})^{\oplus r} \simeq M/\mathfrak{m}M$ by Nakayama's lemma and the choice of $\{m_1, \dots, m_r\}$ as a minimal system of generators. From this, we deduce that $N/\mathfrak{m}N = 0$, so $N = 0$ by Nakayama's lemma and thus $A^{\oplus r} \simeq M$ □

Lemma 7.2.2. *Let $(A, \mathfrak{m})$ be a one-dimensional regular local ring and let M be a finitely generated A-module. Then*

$$\operatorname{Tor}^p(M, A/\mathfrak{m}) = 0 \textit{ for every } p > 1$$

Proof. A is a one-dimensional regular local ring, so $\mathfrak{m}$ is generated by a single element and thus $\mathfrak{m}$ is a free A-module. We obtain a finite free resolution of $A/\mathfrak{m}$

$$0 \longrightarrow A \simeq \mathfrak{m} \longrightarrow A \longrightarrow A/\mathfrak{m} \longrightarrow 0$$

and hence $\operatorname{Tor}^p(M, A/\mathfrak{m}) = 0$ for every $p > 1$ □

Proposition 7.2.1. *Let $\mathcal{M} \in \operatorname{Coh}(X)$. There is an exact sequence*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{M} \to 0$$

where $\mathcal{E}$ and $\mathcal{E}'$ are locally free sheaves on X of finite rank

Proof. By Serre's theorems (see [Har77, Chapter III, Theorem. 5.2]), there is a sufficiently big $N \geqslant 0$ such that $\mathcal{M}(N) = \mathcal{M} \otimes \mathcal{O}_X(N)$ is generated by its global sections, and thus there is some $r \geqslant 0$ and a surjective homomorphism

$$\mathcal{O}_X^{\oplus r} \to \mathcal{M}(N) \to 0$$

tensoring with $\mathcal{O}_X(-N)$, we obtain

$$\mathcal{O}_X(-N)^{\oplus r} \to \mathcal{M} \to 0$$

and $\mathcal{E} := \mathcal{O}_X(-N)^{\oplus r}$ is a locally free sheaf on X of finite rank r. We have an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{M} \to 0$$

where $\mathcal{E}'$ is a coherent sheaf on X because it's the kernel of a morphism between coherent sheaves. Let's see that $\mathcal{E}'$ is also locally free. It suffices to show that $\mathcal{E}'_x$ is a free $\mathcal{O}_{X,x}$-module for every $x \in X$. We have an exact sequence of $\mathcal{O}_{X,x}$-modules

$$0 \to \mathcal{E}'_x \to \mathcal{E}_x \to \mathcal{M}_x \to 0$$

By lemma 7.2.1, we just have to check that $\mathrm{Tor}^1(\mathcal{E}'_x, k(x)) = 0$, where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. Tensoring with $k(x)$ and using lemma 7.2.2, we get an exact sequence

$$\cdots \to \mathrm{Tor}^2(\mathcal{M}_x, k(x)) = 0 \to \mathrm{Tor}^1(\mathcal{E}'_x, k(x)) \to \mathrm{Tor}^1(\mathcal{E}_x, k(x)) = 0 \to \cdots$$

and thus $\mathrm{Tor}^1(\mathcal{E}'_x, k(x)) = 0$, so we conclude $\square$

Observation 7.2.1. *Proposition 7.2.1 extends to arbitrary quasiprojective schemes over a noetherian ring. See [Har77, Chapter III, Example 6.5.1]*

We will now introduce the Grothendieck group of X

Definition 7.2.1. *Let* G *be the free abelian group generated by coherent sheaves on* X*. The Grothendieck group of* X *is the quotient group*

$$K(X) := G \Big/ \left\{ \begin{array}{c} \mathcal{M} - \mathcal{M}' - \mathcal{M}'' \textit{ where } \mathcal{M}, \mathcal{M}', \mathcal{M}'' \in \mathrm{Coh}(X) \textit{ and} \\ \textit{there is an exact sequence } 0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0 \end{array} \right\}$$

Denote by G_0 the subgroup of G given by locally free sheaves of finite rank on X and let

$$K_0(X) := G_0/(G_0 \cap H)$$

where $H = \{\mathcal{M} - \mathcal{M}' - \mathcal{M}'' : \text{ there is an exact sequence } 0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0\}$.

The groups $K(X)$ and $K_0(X)$ are fundamentally related by the following proposition

Proposition 7.2.2 ([LP97], Proposition 2.6.6)**.** *The canonical homomorphism* $i : K_0(X) \to K(X)$ *is a group isomorphism*

We will denote by $j : K(X) \to K_0(X)$ the inverse $j = i^{-1}$

Corollary 7.2.1. *Via the identification* $j : K(X) \to K_0(X)$*, the Grothendieck group of* X *is a ring with the tensor product of locally free sheaves on* X

The isomorphism $j : K(X) \to K_0(X)$ will allow us to define a notion of degree for arbitrary coherent sheaves. First recall the following easy result

Lemma 7.2.3 ([Sta19], Tag 0B37). *Let*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

be an exact sequence of locally free sheaves. Then

$$\det(\mathcal{E}) = \det(\mathcal{E}') \otimes \det(\mathcal{E}'')$$

From lemma 7.2.3 and the universal property of the Grothendieck group, we have that

$$\begin{aligned} \det : K_0(X) &\to \mathrm{Pic}(X) \\ \mathcal{E} &\mapsto \det(\mathcal{E}) \end{aligned}$$

is a group homomorphism.

Definition 7.2.2. *The determinant of a coherent sheaf* $\mathcal{M} \in \mathrm{Coh}(X)$ *is the invertible sheaf*

$$\det(\mathcal{M}) := \Lambda^{\max}(j(\mathcal{M})) = \det(j(\mathcal{M})) \in \mathrm{Pic}(X)$$

Definition 7.2.3. *The degree of a coherent sheaf* $\mathcal{M} \in \mathrm{Coh}(X)$ *is*

$$\deg(\mathcal{M}) := \deg(\det(\mathcal{M})) \in \mathbb{Z}$$

From this, we deduce immediately the following proposition

Proposition 7.2.3. *The degree*

$$\begin{aligned} \deg : K(X) &\to \mathbb{Z} \\ \mathcal{M} &\mapsto \deg(\mathcal{M}) \end{aligned}$$

is a group homomorphism

Observation 7.2.2. *In general,* $\deg(\mathcal{M} \otimes \mathcal{N}) \neq \deg(\mathcal{M}) \cdot \deg(\mathcal{N})$*, so the degree is not a ring homomorphism*

7.3 Riemann-Roch formula for coherent sheaves on a curve

Let X be a smooth projective algebraic curve. From now on, every locally free sheaf that we consider will be of finite rank

Lemma 7.3.1. *Let* $\mathcal{M} \in \mathrm{Coh}(X)$*. Then*

$$\mathcal{M} \textit{ is torsion-free} \Leftrightarrow \mathcal{M} \textit{ is locally free}$$

Proof. Clearly, every locally free sheaf is torsion-free.

Suppose that $\mathcal{M}$ is torsion-free. For every $x \in X$, $\mathcal{M}_x$ is a torsion-free finitely generated module over the principal ideal domain $\mathcal{O}_{X,x}$, so $\mathcal{M}_x$ is free for every $x \in X$ and thus we conclude □

Let $\mathcal{M} \in \mathrm{Coh}(X)$. The double dual $\mathcal{M}^{\vee\vee} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}^{\vee}, \mathcal{O}_X)$ of $\mathcal{M}$ is a coherent sheaf on X and we have the canonical exact sequence

$$0 \to \mathcal{T} \to \mathcal{M} \to \mathcal{M}^{\vee\vee}$$

Let $W := \{x \in X : \mathcal{M}_x \text{ is a free } \mathcal{O}_{X,x}\text{-module}\}$. W is an open subset of X on which $\mathcal{T}$ vanishes. This means that

- By lemma 7.1.1, $\mathcal{T}$ is a torsion sheaf on X, because we have that $\mathrm{Supp}(\mathcal{T}) \neq X$. In particular, $\mathrm{Supp}(\mathcal{T})$ is a finite subset of X and thus $H^1(X, \mathcal{T}) = 0$
- $\mathcal{E} := \mathcal{M}/\mathcal{T}$ is a torsion-free sheaf on $\mathcal{M}$, so it's locally free by lemma 7.3.1

Every finitely generated module over a principal ideal domain splits as a direct sum of a free module and a torsion module. The following proposition generalizes this well-known fact to coherent sheaves on smooth projective algebraic curves

Proposition 7.3.1. *Let $\mathcal{M} \in \mathrm{Coh}(X)$. There is an exact sequence*

$$0 \to \mathcal{T} \to \mathcal{M} \to \mathcal{E} \to 0$$

where $\mathcal{T}$ is a torsion sheaf and $\mathcal{E}$ is locally free. Furthermore, this sequence splits, and we have that

$$\mathcal{M} \simeq \mathcal{T} \oplus \mathcal{E}$$

Proof. The exact sequence of the statement of the proposition is constructed from the double dual of $\mathcal{M}$ as above. We just have to check that it splits. Applying $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E}, -) = \mathcal{E}^{\vee} \otimes_{\mathcal{O}_X} (-)$ and using that $\mathcal{E}^{\vee}$ is locally free (and thus flat), we have the exact sequence

$$0 \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{T}) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M}) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to 0$$

taking cohomology, we obtain

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{T}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to H^1(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{T})) \to \cdots$$

but $H^1(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{T})) = 0$ because $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{T}) = \mathcal{E}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{T}$ has finite support □

We are now ready to prove the general form of the Riemann-Roch formula for coherent sheaves over X. We will prove first some auxiliary results

Lemma 7.3.2. *Every coherent subsheaf of a locally free sheaf $\mathcal{E}$ on X is also locally free*

Proof. Let $\mathcal{M} \subseteq \mathcal{E}$ be a coherent subsheaf. In virtue of lemma 7.2.1, we just have to check that for every $x \in X$ we have $\mathrm{Tor}^1(\mathcal{M}_x, k(x)) = 0$.

Consider the exact sequence

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{N} \to 0$$

$\mathcal{N}$ is a coherent sheaf because it's the cokernel of a morphism between coherent sheaves. We have the exact sequence

$$\cdots \to \mathrm{Tor}^2(\mathcal{N}_x, k(x)) = 0 \to \mathrm{Tor}^1(\mathcal{M}_x, k(x)) \to \mathrm{Tor}^1(\mathcal{E}_x, k(x)) = 0 \to \cdots$$

(where $\mathrm{Tor}^2(\mathcal{N}_x, k(x)) = 0$ by lemma 7.2.2) so $\mathrm{Tor}^1(\mathcal{M}_x, k(x)) = 0$, as we wanted to see □

Lemma 7.3.3. *Let $\mathcal{M} \in \mathrm{Coh}(X)$. The degree of the invertible subsheaves of $\mathcal{M}$ is bounded above by parameters depending only on the genus of X and $\mathcal{M}$*

Proof. Let $\mathcal{L}$ be an invertible subsheaf of $\mathcal{M}$. Using the Riemann-Roch formula for invertible sheaves, we obtain

$$\deg(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) - \chi(\mathcal{O}_X) \leqslant h^0(\mathcal{M}) - \chi(\mathcal{O}_X) = h^0(\mathcal{M}) - 1 + g$$

□

Lemma 7.3.4. *Let $\mathcal{E}$ be a locally free sheaf on X and let $\mathcal{L}$ be an invertible sheaf. Every homomorphism $\mathcal{L} \to \mathcal{E}$ is either zero or injective*

Proof. Let $\varphi : \mathcal{L} \to \mathcal{E}$ be a homomorphism. Consider the associated exact sequence

$$0 \to \mathrm{Ker}(\varphi) \to \mathcal{L} \to \mathrm{Im}(\varphi) \to 0$$

clearly, $\mathrm{Ker}(\varphi)$ and $\mathrm{Im}(\varphi)$ are coherent sheaves on X and we have, by lemma 7.1.3

$$1 = \mathrm{rk}(\mathcal{L}) = \mathrm{rk}(\mathrm{Ker}(\varphi)) + \mathrm{rk}(\mathrm{Im}(\varphi))$$

so there are two possibilities

(a) If $\mathrm{rk}(\mathrm{Ker}(\varphi)) = 0$, then $\mathrm{Ker}(\varphi)$ is a torsion subsheaf of $\mathcal{L}$. By lemma 7.3.1, that's impossible, so $\mathrm{Ker}(\varphi) = 0$

(b) If $\mathrm{rk}(\mathrm{Ker}(\varphi)) = 1$, then $\mathrm{Im}(\varphi)$ is a torsion subsheaf of $\mathcal{E}$, which is again impossible by lemma 7.3.1 and thus $\mathrm{Ker}(\varphi) = \mathcal{L}$

□

Lemma 7.3.5. *Let $0 \to \mathcal{L}' \to \mathcal{L}$ be an injective homomorphism between invertible sheaves on X. Then, $\deg(\mathcal{L}') \leqslant \deg(\mathcal{L})$. Furthermore, if $\deg(\mathcal{L}) = \deg(\mathcal{L}')$, then it is an isomorphism*

Proof. We have the exact sequence

$$0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{T} \to 0$$

where $\mathcal{T}$ is a torsion sheaf because $\mathrm{rk}(\mathcal{T}) = \mathrm{rk}(\mathcal{L}) - \mathrm{rk}(\mathcal{L}') = 1 - 1 = 0$.

Taking global sections and noting that $H^1(X, \mathcal{T}) = 0$, we deduce that

$$h^0(\mathcal{L}') \leqslant h^0(\mathcal{L})$$
$$h^1(\mathcal{L}) \leqslant h^1(\mathcal{L}')$$

and thus, by the Riemann-Roch formula for invertible sheaves, we obtain

$$\begin{aligned}\deg(\mathcal{L}') &= \chi(\mathcal{L}') - \chi(\mathcal{O}_X) = \\ &= h^0(\mathcal{L}') - h^1(\mathcal{L}') - \chi(\mathcal{O}_X) \leqslant \\ &\leqslant h^0(\mathcal{L}) - h^1(\mathcal{L}) - \chi(\mathcal{O}_X) = \\ &= \deg(\mathcal{L})\end{aligned}$$

If, besides, we have that $\deg(\mathcal{L}) = \deg(\mathcal{L}')$, then

$$\chi(\mathcal{T}) = \chi(\mathcal{L}) - \chi(\mathcal{L}') = 0 = h^0(\mathcal{T}) = \sum_{x \in \mathrm{Supp}(\mathcal{T})} \dim_k(\mathcal{T}_x)$$

and thus $\mathcal{T} = 0$, so we conclude □

Observation 7.3.1. *The previous results show that every locally free sheaf $\mathcal{E}$ on X has invertible subsheaves. Indeed, by Serre's theorems, there is a sufficiently large n such that $\mathcal{E}(n)$ is generated by its global sections, and thus there is a surjective homomorphism*

$$\mathcal{O}_X(-n) \oplus \cdots \oplus \mathcal{O}_X(-n) \to \mathcal{E} \to 0$$

In particular, there exists a nonzero homomorphism

$$\mathcal{O}_X(-n) \to \mathcal{E}$$

which, by lemma 7.3.4, must be injective. So $\mathcal{E}$ admits invertible subsheaves.

Combining this result with lemmas 7.3.3 and 7.3.5, we conclude that $\mathcal{E}$ admits invertible subsheaves of maximal degree

Proposition 7.3.2. *Let $\mathcal{E}$ be a locally free sheaf on X. Then*

$$\chi(\mathcal{E}) = \mathrm{rk}(\mathcal{E})\chi(\mathcal{O}_X) + \deg(\mathcal{E})$$

Proof. We proceed by induction on $\mathrm{rk}(\mathcal{E})$. The case of $\mathrm{rk}(\mathcal{E}) = 1$ is theorem 7.0.1.

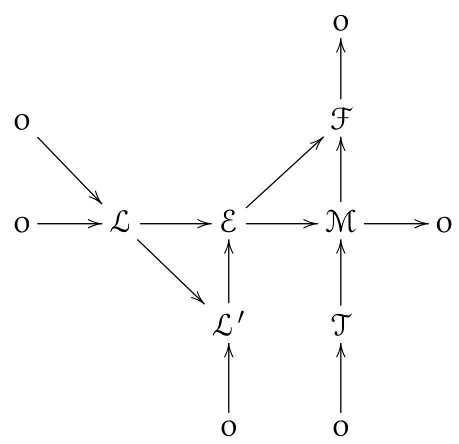
Suppose that the result holds for every locally free sheaf on X of rank less than $\mathrm{rk}(\mathcal{E})$. By observation 7.3.1, there is an invertible subsheaf $\mathcal{L}$ of $\mathcal{E}$ with maximal degree. Consider the associated exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$$

$\mathcal{M}$ is a coherent sheaf on X. Let's see that it is locally free. By proposition 7.3.1, there is a split exact sequence

$$0 \to \mathcal{T} \to \mathcal{M} \to \mathcal{F} \to 0$$

where $\mathcal{T}$ is a torsion sheaf and $\mathcal{F}$ is locally free. Let $\mathcal{L}' = \mathrm{Ker}(\mathcal{E} \to \mathcal{F})$; we have the commutative diagram



$\mathcal{L}'$ is an invertible subsheaf of $\mathcal{E}$ because

$$\begin{aligned}\mathrm{rk}(\mathcal{L}') &= \mathrm{rk}(\mathcal{E}) - \mathrm{rk}(\mathcal{F}) = \\ &= \mathrm{rk}(\mathcal{E}) - (\mathrm{rk}(\mathcal{M}) - \mathrm{rk}(\mathcal{T})) = \\ &= \mathrm{rk}(\mathcal{E}) - \mathrm{rk}(\mathcal{M}) = \\ &= \mathrm{rk}(\mathcal{L}) = 1\end{aligned}$$

Besides, there is a natural morphism $\mathcal{L} \to \mathcal{L}'$ that is injective by lemma 7.3.4, and furthermore $\deg(\mathcal{L}) = \deg(\mathcal{L}')$ by the construction of $\mathcal{L}$ and lemma 7.3.5, so $\mathcal{L} \simeq \mathcal{L}'$ and thus $\mathcal{F} \simeq \mathcal{E}/\mathcal{L} \simeq \mathcal{M}$, so $\mathcal{M}$ is a locally free sheaf of rank $\mathrm{rk}(\mathcal{M}) = \mathrm{rk}(\mathcal{E}) - 1 < \mathrm{rk}(\mathcal{E})$. Applying the induction hypothesis and lemmas 7.2.3 and 7.1.3, we obtain

$$\begin{aligned}\chi(\mathcal{E}) &= \chi(\mathcal{L}) + \chi(\mathcal{M}) = \\ &= \chi(\mathcal{O}_X) + \deg(\mathcal{L}) + \mathrm{rk}(\mathcal{M})\chi(\mathcal{O}_X) + \deg(\mathcal{M}) = \\ &= \mathrm{rk}(\mathcal{E})\chi(\mathcal{O}_X) + \deg(\mathcal{E})\end{aligned}$$

□

Lemma 7.3.6. *Let $\mathcal{T}$ be a torsion sheaf on X. Then*

$$\deg(\mathcal{T}) = h^0(\mathcal{T})$$

In particular, the degree of a torsion sheaf is always non negative

Proof. By proposition 7.2.1 there is a resolution of $\mathcal{T}$ by locally free sheaves

$$0 \to \mathcal{L}_0 \to \mathcal{L}_1 \to \mathcal{T} \to 0$$

and thus, applying proposition 7.3.2 and using that $\mathrm{rk}(\mathcal{L}_0) = \mathrm{rk}(\mathcal{L}_1) - \mathrm{rk}(\mathcal{T}) = \mathrm{rk}(\mathcal{L}_1)$, we obtain

$$\begin{aligned}h^0(\mathcal{T}) &= \chi(\mathcal{T}) = \\ &= \chi(\mathcal{L}_0) - \chi(\mathcal{L}_1) = \\ &= \mathrm{rk}(\mathcal{L}_1)\chi(\mathcal{O}_X) + \deg(\mathcal{L}_1) - (\mathrm{rk}(\mathcal{L}_0)\chi(\mathcal{O}_X) + \deg(\mathcal{L}_0)) = \\ &= \deg(\mathcal{L}_1) - \deg(\mathcal{L}_0) = \\ &= \deg(\mathcal{T})\end{aligned}$$

□

Proposition 7.3.3 (Riemann-Roch formula for coherent sheaves)**.** *Let X be a smooth projective algebraic curve and let $\mathcal{M}$ be a coherent sheaf on X. Then*

$$\chi(\mathcal{M}) = \mathrm{rk}(\mathcal{M})\chi(\mathcal{O}_X) + \deg(\mathcal{M})$$

Proof. Consider the split exact sequence associated to $\mathcal{M}$ by proposition 7.3.1

$$0 \to \mathcal{T} \to \mathcal{M} \to \mathcal{E} \to 0$$

where $\mathcal{T}$ is a torsion sheaf on X and $\mathcal{E}$ is locally free. Applying proposition 7.3.2 and lemma 7.3.6, we obtain

$$\chi(\mathcal{M}) = \chi(\mathcal{T}) + \chi(\mathcal{E}) = \deg(\mathcal{T}) + \mathrm{rk}(\mathcal{E})\chi(\mathcal{O}_X) + \deg(\mathcal{E}) = \mathrm{rk}(\mathcal{M})\chi(\mathcal{O}_X) + \deg(\mathcal{M})$$

□

7.4 Applications

We are going to give some useful consequences of this generalization of the Riemann-Roch formula. Let's start by extending lemma 7.3.3 to arbitrary coherent subsheaves

Proposition 7.4.1. *Let* X *be a genus* g *smooth projective algebraic curve and* $\mathcal{M} \in \mathrm{Coh}(X)$*. Let* $\mathcal{N}$ *be a coherent subsheaf of* $\mathcal{M}$

- *If* $g = 0$*, then*
$$\deg(\mathcal{N}) \leqslant \deg(\mathcal{M}) + h^1(\mathcal{M}) + \mathrm{rk}(\mathcal{M}) = h^0(\mathcal{M})$$
- *If* $g > 0$*, then*
$$\deg(\mathcal{N}) \leqslant \deg(M) + h^1(\mathcal{M})$$

Proof. Suppose that $g = 0, 1$. Then, $\chi(\mathcal{O}_X) = 1 - g \geqslant 0$, and thus, by proposition 7.3.3,

$$\begin{aligned}\deg(\mathcal{N}) &= \chi(\mathcal{N}) - \mathrm{rk}(\mathcal{N})\chi(\mathcal{O}_X) \leqslant \\ &\leqslant h^0(\mathcal{M}) - \mathrm{rk}(\mathcal{N})\chi(\mathcal{O}_X) \leqslant \\ &\leqslant h^0(\mathcal{M})\end{aligned}$$

because $-\mathrm{rk}(\mathcal{N})\chi(\mathcal{O}_X) \leqslant 0$. Again, by proposition 7.3.3, we have that

$$h^0(\mathcal{M}) = \deg(\mathcal{M}) + h^1(\mathcal{M}) + \mathrm{rk}(\mathcal{M})(1 - g)$$

so, plugging in the values $g = 0, 1$, we obtain the desired bounds.

Suppose now that $g > 1$. Then $\chi(\mathcal{O}_X) = 1 - g < 0$, and thus $-\mathrm{rk}(\mathcal{N})\chi(\mathcal{O}_X) \leqslant -\mathrm{rk}(\mathcal{M})\chi(\mathcal{O}_X)$ so

$$\begin{aligned}\deg(\mathcal{N}) &\leqslant h^0(\mathcal{M}) - \mathrm{rk}(\mathcal{M})\chi(\mathcal{O}_X) = \\ &= h^0(\mathcal{M}) - (\chi(\mathcal{M}) - \deg(\mathcal{M})) = \\ &= \deg(\mathcal{M}) + h^1(\mathcal{M})\end{aligned}$$

□

Observation 7.4.1. *Note that, although we have arrived at the same bound, the cases* $g = 1$ *and* $g > 1$ *in proposition 7.4.1 have been proven differently*

Lemma 7.4.1. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be coherent sheaves on* X *of the same rank such that* $\mathcal{N} \subseteq \mathcal{M}$*. Then,* $\deg(\mathcal{N}) \leqslant \deg(\mathcal{M})$*. Furthermore,* $\deg(\mathcal{M}) = \deg(\mathcal{N})$ *if and only if* $\mathcal{M} = \mathcal{N}$

Proof. Consider the associated exact sequence

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{T} \to 0$$

$\mathcal{T}$ is a torsion sheaf because $\mathrm{rk}(\mathcal{T}) = 0$. We have that

$$\deg(\mathcal{M}) = \deg(\mathcal{N}) + \deg(\mathcal{T}) \geqslant \deg(\mathcal{N})$$

because $\deg(\mathcal{T}) \geqslant 0$ by lemma 7.3.6. Besides

$$\deg(\mathcal{M}) = \deg(\mathcal{N}) \Leftrightarrow \deg(\mathcal{T}) = 0 \Leftrightarrow \mathcal{T} = 0 \Leftrightarrow \mathcal{M} = \mathcal{N}$$

□

Corollary 7.4.1. *Let $\mathcal{M}$ be a coherent sheaf on X. Then, for each $0 < r \leqslant \mathrm{rk}(\mathcal{M})$, $\mathcal{M}$ contains coherent subsheaves of rank r and maximal degree*

Proof. This follows easily from proposition 7.4.1 and lemma 7.4.1 □

Definition 7.4.1. *Let $\mathcal{E}$ be a locally free sheaf on X. A coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$ is called a sub-bundle of $\mathcal{E}$ if, for every $x \in X$, the canonical homomorphism*

$$\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x \to \mathcal{E}_x/\mathfrak{m}_x\mathcal{E}_x$$

is injective

Lemma 7.4.2. *Let $\mathcal{E}$ be a locally free sheaf on X and $\mathcal{F} \subseteq \mathcal{E}$ a coherent subsheaf. Then, $\mathcal{F}$ is a sub-bundle of $\mathcal{E}$ if and only if the quotient sheaf $\mathcal{E}/\mathcal{F}$ is locally free*

Proof. It's easy to deduce from the definition that $\mathcal{F}$ is a sub-bundle of $\mathcal{E}$ if and only if, for every $x \in X$, we have

$$\mathrm{Tor}^1((\mathcal{E}/\mathcal{F})_x, k(x)) = 0$$

we conclude by lemma 7.2.1 □

Corollary 7.4.2. *Let $\mathcal{E}$ be a locally free sheaf and let $\mathcal{F}$ be a coherent subsheaf of $\mathcal{E}$ with maximal degree. Then, $\mathcal{F}$ is a sub-bundle of $\mathcal{E}$*

Proof. Consider the exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{M} \to 0$$

By lemma 7.4.2, it suffices to show that $\mathcal{M}$ is locally free. Consider the splitting $\mathcal{M} \simeq \mathcal{T} \oplus \mathcal{G}$, where $\mathcal{T}$ is a torsion sheaf and $\mathcal{G}$ is locally free (proposition 7.3.1). let $\mathcal{F}' := \mathrm{Ker}(\mathcal{E} \to \mathcal{G})$. By similar arguments as in proposition 7.3.2, we have that $\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{F}')$. Besides, there is an injective homomorphism $\mathcal{F} \to \mathcal{F}'$. This means that $\deg(\mathcal{F}) \leqslant \deg(\mathcal{F}')$ by lemma 7.4.1. But $\deg(\mathcal{F}) \geqslant \deg(\mathcal{F}')$ by hypothesis so $\mathcal{F} \simeq \mathcal{F}'$ again by lemma 7.4.1 and thus

$$\mathcal{M} \simeq \mathcal{E}/\mathcal{F} \simeq \mathcal{E}/\mathcal{F}' \simeq \mathcal{G}$$

so $\mathcal{M}$ is locally free and we conclude □

Corollary 7.4.3. *Let $\mathcal{E}$ be a locally free sheaf on X. Every coherent subsheaf of $\mathcal{E}$ is contained in some subbundle of $\mathcal{E}$*

Proof. This follows easily from corollaries 7.4.1 and 7.4.2 □

We will end this chapter giving an explicit computation of the Hilbert polynomial of a coherent sheaf on a smooth projective algebraic curve

Proposition 7.4.2. *Let $\mathcal{M} \in \mathrm{Coh}(X)$ and $\mathcal{E}$ a locally free sheaf. Then*

$$\deg(\mathcal{M} \otimes \mathcal{E}) = \deg(\mathcal{M})\mathrm{rk}(\mathcal{E}) + \mathrm{rk}(\mathcal{M})\deg(\mathcal{E})$$

Proof. First note that a direct computation shows that, for every torsion sheaf $\mathcal{T}$ on X, we have that

$$\deg(\mathcal{T}\otimes\mathcal{E}) = h^0(\mathcal{T}\otimes\mathcal{L}) = \mathrm{rk}(\mathcal{E})h^0(\mathcal{T}) = \mathrm{rk}(\mathcal{E})\deg(\mathcal{T}) = \deg(\mathcal{T})\mathrm{rk}(\mathcal{E}) + 0\cdot\deg(\mathcal{E})$$

because $\mathcal{E}$ is locally free of rank $\mathrm{rk}(\mathcal{E})$.

In general, we proceed by induction on $\mathrm{rk}(\mathcal{M})$. We have just proven the case $\mathrm{rk}(\mathcal{M}) = 0$. Suppose that the result is true for any coherent sheaf on X of rank less than $\mathrm{rk}(\mathcal{M})$. Choose any invertible subsheaf $\mathcal{L}$ of $\mathcal{M}$ (it exists as a subsheaf of the locally free part of $\mathcal{M}$, by observation 7.3.1). We have the exact sequence

$$0\to\mathcal{L}\to\mathcal{M}\to\mathcal{N}\to 0$$

with $\mathrm{rk}(\mathcal{N}) = \mathrm{rk}(\mathcal{M}) - 1 < \mathrm{rk}(\mathcal{M})$. Tensoring with $\mathcal{E}$, we obtain the exact sequence

$$0\to\mathcal{L}\otimes\mathcal{E}\to\mathcal{M}\otimes\mathcal{E}\to\mathcal{N}\otimes\mathcal{E}\to 0$$

and thus, by induction hypothesis we have that

$$\begin{aligned}\deg(\mathcal{M}\otimes\mathcal{E}) &= \deg(\mathcal{L}\otimes\mathcal{E}) + \deg(\mathcal{N}\otimes\mathcal{E}) = \\ &= \deg(\mathcal{L})\mathrm{rk}(\mathcal{E}) + \deg(\mathcal{E}) + \deg(\mathcal{N})\mathrm{rk}(\mathcal{E}) + \mathrm{rk}(\mathcal{N})\deg(\mathcal{E}) = \\ &= \deg(\mathcal{M}) + \mathrm{rk}(\mathcal{M})\deg(\mathcal{E})\end{aligned}$$

□

Observation 7.4.2. *Given* $\mathcal{M},\mathcal{N}\in\mathrm{Coh}(X)$, *we can't give a concise formula for* $\deg(\mathcal{M}\otimes\mathcal{N})$, *because it would involve computing terms of the form* $\deg(\mathcal{T}\otimes\mathcal{T}')$, *where* $\mathcal{T},\mathcal{T}'$ *are torsion sheaves on* X. *We would have*

$$\deg(\mathcal{T}\otimes\mathcal{T}') = h^0(\mathcal{T}\otimes\mathcal{T}') = \sum_{x\in X}\dim_k(\mathcal{T}_x\otimes\mathcal{T}'_x)$$

Lemma 7.4.3. *Let* $\mathcal{E}$ *be a locally free sheaf on* X *and* $\mathcal{E}^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E},\mathcal{O}_X)$ *the dual sheaf. Then*

$$\deg(\mathcal{E}^\vee) = -\deg(\mathcal{E})$$

Proof. We proceed by induction on $\mathrm{rk}(\mathcal{E})$. This is a well known result for $\mathrm{rk}(\mathcal{E}) = 1$. Suppose that the result holds for every locally free sheaf of rank less than $\mathrm{rk}(\mathcal{E})$.

Let $\mathcal{L}$ be an invertible subsheaf of $\mathcal{E}$ of maximal degree and consider the associated exact sequence

$$0\to\mathcal{L}\to\mathcal{E}\to\mathcal{F}\to 0$$

$\mathcal{F}$ is a locally free sheaf by corollary 7.4.2. Besides, $\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{E}) - 1 < \mathrm{rk}(\mathcal{E})$ and thus $\deg(\mathcal{F}^\vee) = -\deg(\mathcal{F})$ by induction hypothesis. The sequence

$$0\to\mathcal{F}^\vee\to\mathcal{E}^\vee\to\mathcal{L}^\vee\to 0$$

is exact because $\mathcal{F}$ is a locally free sheaf, so

$$\deg(\mathcal{E}^\vee) = \deg(\mathcal{L}^\vee) + \deg(\mathcal{F}^\vee) = -\deg(\mathcal{L}) - \deg(\mathcal{F}) = -\deg(\mathcal{E})$$

□

Proposition 7.4.3. *Let* $\mathcal{O}_X(1)$ *a very ample invertible sheaf on* X *and* $\mathcal{M} \in \mathrm{Coh}(X)$*. The Hilbert polynomial of* $\mathcal{M}$ *with respect to* $\mathcal{O}_X(1)$*,* $P(\mathcal{M}, m) := \chi(\mathcal{M}(m))$*, is given by*

$$P(\mathcal{M}, m) = \chi(\mathcal{M}) + \mathrm{rk}(\mathcal{M})\mathrm{deg}(X)m = \mathrm{rk}(\mathcal{M})(1-g) + \mathrm{deg}(\mathcal{M}) + \mathrm{rk}(\mathcal{M})\mathrm{deg}(X)m$$

Proof. In virtue of propositions 7.3.3 and 7.4.2 we have

$$\begin{aligned}\chi(\mathcal{M}(m)) &= \mathrm{rk}(\mathcal{M}(m))\chi(\mathcal{O}_X) + \mathrm{deg}(\mathcal{M}(m)) = \\ &= \mathrm{rk}(\mathcal{M})\chi(\mathcal{O}_X) + \mathrm{deg}(\mathcal{M}) + \mathrm{rk}(\mathcal{M})\mathrm{deg}(\mathcal{O}_X(1))m = \\ &= \chi(\mathcal{M}) + \mathrm{rk}(\mathcal{M})\mathrm{deg}(X)m\end{aligned}$$

□

Chapter 8

Stable and semistable sheaves

In this chapter, we will introduce the notion of semistable and stable sheaves on a smooth projective algebraic curve. We will study their main properties, introduce the Jordan-Hölder and Harder-Narasimhan filtrations, and prove that, essentially, a family of locally free sheaves on a smooth projective algebraic curve is bounded if and only if it is semistable.

Some references for this chapter are [HL10, Chapters 1-2], [Sch08, Chapter 2, § 2] and [LP97, Chapter 5]

8.1 Basic definitions

Let X be a smooth projective algebraic variety and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X

Definition 8.1.1. *Let $\mathcal{M}$ be a coherent sheaf on X. The dimension of $\mathcal{M}$ is*

$$\dim(\mathcal{M}) := \dim(\mathrm{Supp}(\mathcal{M}))$$

Definition 8.1.2. *A coherent sheaf $\mathcal{M}$ on X is pure of dimension d if every coherent subsheaf $0 \neq \mathcal{N} \subseteq \mathcal{M}$ is of dimension d*

Lemma 8.1.1 ([HL10], § 1.2)**.** *Let $\mathcal{M}$ be a coherent sheaf on X. The Hilbert polynomial of $\mathcal{M}$ can be written as*

$$P(\mathcal{M}, m) = \chi(\mathcal{M} \otimes \mathcal{O}_X(m)) = \sum_{k=0}^{\dim(\mathcal{M})} \frac{a_k(\mathcal{M})}{k!} m^k$$

where $a_k(\mathcal{M}) \in \mathbb{Z}$ for every $k = 0, \ldots, \dim(\mathcal{M})$

Definition 8.1.3. *The reduced Hilbert polynomial of $\mathcal{M}$ is*

$$p(\mathcal{M}, m) := \frac{P(\mathcal{M}, m)}{a_{\dim(\mathcal{M})}}$$

Definition 8.1.4. *A coherent sheaf $\mathcal{M}$ on X is Gieseker-semistable (resp. Gieseker-stable) if it is pure and for every coherent subsheaf $0 \neq \mathcal{N} \subsetneq \mathcal{M}$ we have that*

$$p(\mathcal{N}, m) \leqslant p(\mathcal{M}, m) \textit{ for } m \gg 0 \quad \textit{(resp. } p(\mathcal{N}, n) < p(\mathcal{M}, m) \textit{ for } m \gg 0\textit{)}$$

8.2 Stability on curves

Let X be a smooth projective algebraic curve of genus g. Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X and denote by $d_X = \deg(\mathcal{O}_X(1))$ the degree of X.

Let $\mathcal{M}$ be a coherent sheaf on X of rank r and degree d. In proposition 7.4.3 we gave an explicit computation of the Hilbert polynomial of $\mathcal{M}$ with respect to $\mathcal{O}_X(1)$

$$P(\mathcal{M}, m) = \chi(\mathcal{M} \otimes \mathcal{O}_X(n)) = r(1-g) + d + r d_X m$$

Lemma 8.2.1. *The reduced Hilbert polynomial of $\mathcal{M}$ is*

(a) If $r > 0$

$$p(\mathcal{M}, m) = \frac{1-g}{d_X} + \frac{1}{d_X}\frac{d}{r} + m$$

(b) If $r = 0$

$$p(\mathcal{M}, m) = 1$$

Definition 8.2.1. *If $r > 0$, $\mu(\mathcal{M}) := \frac{d}{r}$ is called the slope of $\mathcal{M}$*

Definition 8.2.2. *Let $\mathcal{M}$ be a coherent sheaf on X of positive rank. $\mathcal{M}$ is slope-semistable (resp. slope-stable) if for every coherent subsheaf $0 \neq \mathcal{N} \subsetneq \mathcal{M}$ we have that*

$$\mu(\mathcal{N}) \leqslant \mu(\mathcal{M}) \quad \textit{(resp. } \mu(\mathcal{N}) < \mu(\mathcal{M})\textit{)}$$

Lemma 8.2.2. *Let $\mathcal{M}$ be a coherent sheaf on X*

(a) $\mathcal{M}$ is pure of dimension 1 if and only if $\mathcal{M}$ is locally free

(b) $\mathcal{M}$ is pure of dimension 0 if and only if $\mathcal{M}$ is a torsion sheaf

Proof. Every locally free sheaf on X is pure of dimension 1 by lemma 7.3.1.

On the other hand, $\mathcal{M}$ is pure on dimension 1 if $\text{Supp}(\mathcal{M}) = X$ and every proper coherent subsheaf of $\mathcal{M}$ has dimension 1; in particular this means that $\mathcal{M}$ can't contain torsion subsheaves and thus it must be locally free in virtue of lemma 7.3.1.

Finally, we have that

$$\mathcal{M} \text{ is pure of dimension } 0 \Leftrightarrow \dim(\mathcal{M}) = 0 \Leftrightarrow \mathcal{M} \text{ is a torsion sheaf}$$

□

In a smooth projective algebraic curve, Gieseker-stability and slope-stability turn out to be equivalent concepts

Lemma 8.2.3. *Let $\mathcal{M}$ be a coherent sheaf on X*

(a) If $\mathcal{M}$ is pure of dimension 1, then $\mathcal{M}$ is Gieseker-semistable (resp. Gieseker-stable) if and only if it is slope-semistable (resp. slope-stable)

(b) *If $\mathcal{M}$ is pure of dimension 0, then $\mathcal{M}$ is Gieseker-semistable*

Proof. If $\mathcal{M}$ is a pure sheaf of dimension 1 we have that, by lemma 8.2.1

$$p(\mathcal{M}, m) - p(\mathcal{N}, m) = \frac{1}{d_X}(\mu(\mathcal{M}) - \mu(\mathcal{N}))$$

so the result follows.

Suppose now that $\mathcal{M}$ is pure of dimension 0. Then $\mathcal{M}$ is a torsion sheaf on X by lemma 8.2.2, and thus $\mathcal{M}$ is semistable because its reduced Hilbert polynomial is equal to 1 □

Observation 8.2.1. *In general, Gieseker-stability is better suited for constructions of geometric invariant theory. On the other hand, slope-stability behaves well under pullbacks, products and coproducts. The equivalence of these two notions in the case of curves simplifies their study.*

For a general smooth projective algebraic variety X we can also define the slope of a coherent sheaf, and for a pure coherent sheaf $\mathcal{M}$ of dimension $d = \dim(X)$ we have that (see [HL10, Lemma 1.2.13])

$\mathcal{M}$ is slope-stable $\Rightarrow$ $\mathcal{M}$ is Gieseker-stable $\Rightarrow$ $\mathcal{M}$ is Gieseker-semistable $\Rightarrow$ $\mathcal{M}$ is slope-semistable

Observation 8.2.2. *In view of lemma 8.2.3, we will say that a pure coherent sheaf of dimension 1 on X is semistable (resp. stable) to mean that it's Gieseker-semistable or slope-semistable (resp. Gieseker-stable of slope-stable)*

Lemma 8.2.4. *Let $\mathcal{M}$ be a coherent sheaf on X with $\mathrm{rk}(\mathcal{M}) > 0$ and consider an exact sequence of coherent sheaves*

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{P} \to 0$$

Then

$$\mu(\mathcal{M}) = \frac{\mathrm{rk}(\mathcal{N})\mu(\mathcal{N}) + \mathrm{rk}(\mathcal{P})\mu(\mathcal{P})}{\mathrm{rk}(\mathcal{M})}$$

Proof. The proof follows easily from the additivity of the degree of coherent sheaves and the definition of slope

$$\mu(\mathcal{M}) = \frac{\deg(\mathcal{M})}{\mathrm{rk}(\mathcal{M})} = \frac{\deg(\mathcal{N}) + \deg(\mathcal{P})}{\mathrm{rk}(\mathcal{M})} = \frac{\mathrm{rk}(\mathcal{N}) \cdot \mu(\mathcal{N}) + \mathrm{rk}(\mathcal{P}) \cdot \mu(\mathcal{P})}{\mathrm{rk}(\mathcal{M})}$$

□

We will now focus on pure sheaves of dimension 1 on X, i.e., locally free sheaves on X

Lemma 8.2.5. *Let $\mathcal{E}$ be a locally free sheaf on X. The following statements are equivalent*

(a) *$\mathcal{E}$ is semistable (resp. stable)*

(b) *For every sub-bundle (recall definition 7.4.1)* $0 \neq \mathcal{F} \subsetneq \mathcal{E}$, *then* $\mu(\mathcal{F}) \leqslant \mu(\mathcal{E})$ *(resp.* $\mu(\mathcal{F}) < \mu(\mathcal{E})$*)*

(c) *For every proper (i.e. distinct from zero or* $\mathcal{E}$*) coherent quotient sheaf* $\mathcal{E} \to \mathcal{M} \to 0$, *then* $\mu(\mathcal{E}) \leqslant \mu(\mathcal{M})$ *(resp.* $\mu(\mathcal{E}) < \mu(\mathcal{M})$*)*

Proof. Clearly $(a) \Rightarrow (b)$. By corollary 7.4.3, every coherent subsheaf of $\mathcal{E}$ is contained in some sub-bundle of $\mathcal{E}$, so $(b) \Rightarrow (a)$ by lemma 7.4.1.

Let's see that $(a) \Rightarrow (c)$. Suppose that $\mathcal{E}$ is semistable and consider an exact sequence in $\mathrm{Coh}(X)$

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{M} \to 0$$

By lemma 8.2.4, we have that

$$\mu(\mathcal{E}) = \frac{\mathrm{rk}(\mathcal{F})\mu(\mathcal{F}) + \mathrm{rk}(\mathcal{M})\mu(\mathcal{M})}{\mathrm{rk}(\mathcal{E})} \leqslant \frac{\mathrm{rk}(\mathcal{F})\mu(\mathcal{E}) + \mathrm{rk}(\mathcal{M})\mu(\mathcal{M})}{\mathrm{rk}(\mathcal{E})}$$

and thus

$$\mathrm{rk}(\mathcal{E})\mu(\mathcal{E}) - \mathrm{rk}(\mathcal{F})\mu(\mathcal{E}) \leqslant \mathrm{rk}(\mathcal{M})\mu(\mathcal{M})$$

but $\mathrm{rk}(\mathcal{E})\mu(\mathcal{E}) - \mathrm{rk}(\mathcal{F})\mu(\mathcal{E}) = \mathrm{rk}(\mathcal{M})\mu(\mathcal{E})$, so $(a) \Rightarrow (c)$. The implication $(c) \Rightarrow (a)$ can be proven similarly □

Example 8.2.1. *Every invertible sheaf* $\mathcal{L}$ *on* X *is stable. Indeed, for every invertible subsheaf* $\mathcal{L}' \subsetneq \mathcal{L}$, *by lemma 7.3.5 we have that* $\deg(\mathcal{L}') < \deg(\mathcal{L})$ *and thus* $\mu(\mathcal{L}') < \mu(\mathcal{L})$

Proposition 8.2.1. *Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be semistable locally free sheaves on* X. *Then*

- $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \neq 0 \Rightarrow \mu(\mathcal{E}) \leqslant \mu(\mathcal{F})$
- *Suppose that* $\mu(\mathcal{E}) = \mu(\mathcal{F})$ *and let* $\varphi : \mathcal{E} \to \mathcal{F}$ *be a non-zero homomorphism. Then*
  - *If* $\mathcal{E}$ *is stable, then* $\varphi : \mathcal{E} \to \mathcal{F}$ *is injective*
  - *If* $\mathcal{F}$ *is stable, then* $\varphi : \mathcal{E} \to \mathcal{F}$ *is surjective*

Proof. Suppose that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \neq 0$. Then, there is a non-zero homomorphism $\varphi : \mathcal{E} \to \mathcal{F}$. Clearly, $\mathrm{Im}(\varphi)$ is a coherent sheaf on X and, by lemma 8.2.5, we have that

$$\mu(\mathcal{E}) \leqslant \mu(\mathrm{Im}(\varphi)) \leqslant \mu(\mathcal{F}) \tag{8.1}$$

so the result follows.

Suppose now that $\mu(\mathcal{E}) = \mu(\mathcal{F})$. Then, all the inequalities in equation 8.1 are equalities, and the result follows from the definition of slope-stability □

Corollary 8.2.1. *Let* $\mathcal{E}$ *be a stable locally free sheaf on* X. *Then,* $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$ *is isomorphic to* k *as a* k*-algebra*

Proof. Clearly, $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E}) = H^0(X, \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}))$ is a finite dimensional k-algebra. Besides, $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$ is a division algebra in virtue of proposition 8.2.1.

Every finite dimensional division algebra A over an algebraically closed field k is trivial, because every element $a \in A$ not in k would span a commutative algebraic field extension $k \hookrightarrow k[a]$ □

8.3 Jordan-Hölder and Harder-Narasimhan filtrations

Let $\mu \in \mathbb{Q}$ and denote by $\mathcal{S}(\mu)$ the category of semistable locally free sheaves of slope μ on X

Lemma 8.3.1. *Let*

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

be an exact sequence of locally free sheaves on X *such that* $\mathcal{F}, \mathcal{G} \in \mathcal{S}(\mu)$*. Then,* $\mathcal{E} \in \mathcal{S}(\mu)$

Proof. By lemmas 8.2.4 and 7.1.3, we have that

$$\mu(\mathcal{E}) = \frac{\mathrm{rk}(\mathcal{F})\mu + \mathrm{rk}(\mathcal{G})\mu}{\mathrm{rk}(\mathcal{E})} = \mu$$

Let $\mathcal{F}'$ be a coherent subsheaf of $\mathcal{E}$ and denote $\mathcal{F} \cap \mathcal{F}' := \mathrm{Ker}(\mathcal{F}' \to \mathcal{G})$. We have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} \cap \mathcal{F}' & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F}'/(\mathcal{F} \cap \mathcal{F}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{G} & \longrightarrow & 0
\end{array}$$

whose rows and columns are exact.

Applying again lemma 8.2.4 and using that, by hypothesis, $\mu(\mathcal{F} \cap \mathcal{F}') \leqslant \mu(\mathcal{F}) = \mu$ and $\mu(\mathcal{F}'/(\mathcal{F} \cap \mathcal{F})) \leqslant \mu(\mathcal{G}) = \mu$, we obtain that $\mu(\mathcal{F}') \leqslant \mu(\mathcal{F}) = \mu$, so we conclude □

Observation 8.3.1. *Note that, if* $\mathcal{F}, \mathcal{E} \in \mathcal{S}(\mu)$ *and* $\mathcal{F} \subseteq \mathcal{E}$ *with* $\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{E})$*, then it must be* $\deg(\mathcal{F}) = \deg(\mathcal{E})$ *and thus* $\mathcal{E} = \mathcal{F}$ *by lemma 7.4.1*

Proposition 8.3.1. $\mathcal{S}(\mu)$ *is an abelian category. Furthermore, given any* $\mathcal{E} \in \mathcal{S}(\mu)$*, then every descending chain of elements of* $\mathcal{S}(\mu)$ *contained in* $\mathcal{E}$ *stabilizes, i.e.,* $\mathcal{S}(\mu)$ *is an Artinian category*

Proof. $\mathcal{S}(\mu)$ is clearly a (linearly) additive category, because its hom-sets are finite dimensional vector spaces over $k = H^0(X, \mathcal{O}_X)$. Finite coproducts exist in $\mathcal{S}(\mu)$ by lemma 8.3.1.

Let $\mathcal{E}, \mathcal{F} \in \mathcal{S}(\mu)$ and let $f : \mathcal{E} \to \mathcal{F}$ be a homomorphism. We have to prove that $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are objects of $\mathcal{S}(\mu)$. Clearly, we have that $\mathrm{Im}(f)$ is a coherent subsheaf of $\mathcal{F}$ and thus it's locally free by lemma 7.3.2. Besides, by the semistability of $\mathcal{E}$ and $\mathcal{F}$ we have that

$$\mu = \mu(\mathcal{E}) \leqslant \mu(\mathrm{Im}(f)) \leqslant \mu(\mathcal{F}) = \mu$$

so $\mu(\mathrm{Im}(f)) = \mu$. This proves that $\mathrm{Im}(f)$ must be semistable, because if it were not the case it would contain coherent subsheaves of slope greater than μ, contradicting the semistability of $\mathcal{F}$. A similar argument shows that $\text{Ker}(f) \in \mathcal{S}(\mu)$.

Finally, let's see that $\mathcal{S}(\mu)$ is an Artinian category. If $\mathcal{F}, \mathcal{E} \in \mathcal{S}(\mu)$ and $\mathcal{F} \subsetneq \mathcal{E}$ is a coherent subsheaf of $\mathcal{E}$ then, in virtue of observation 8.3.1, we have that $\text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})$. This proves that the rank decreases in subobjects, and thus every chain of subobjects must stabilize because the rank is always non-negative □

Definition 8.3.1. *Let $\mathcal{E} \in \mathcal{S}(\mu)$. A Jordan-Hölder filtration of $\mathcal{E}$ is a filtration*

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_r = \mathcal{E}$$

such that

- $\mathcal{E}_i \in \mathcal{S}(\mu)$ *for every* $i = 1, \ldots, r$
- $\text{gr}_i(\mathcal{E}) := \mathcal{E}_i/\mathcal{E}_{i-1}$ *is stable for every* $i = 1, \ldots, r$

r *is called the length of the Jordan-Hölder filtration and* $\text{gr}(\mathcal{E}) := \oplus_{i=1}^{r} \text{gr}_i(\mathcal{E})$ *is called the associated graded object*

Proposition 8.3.2. *Let $\mathcal{E} \in \mathcal{S}(\mu)$. Jordan-Hölder filtrations of $\mathcal{E}$ always exist. Furthermore, all Jordan-Hölder filtrations of $\mathcal{E}$ have the same length and their associated graded objects are isomorphic*

Proof. Let's see that Jordan-Hölder filtrations of $\mathcal{E}$ always exist. There is some stable subsheaf $\mathcal{E}_1 \subseteq \mathcal{E}$ of $\mathcal{E}$ of slope μ, because if it were not the case then $\mathcal{E}$ would have non stabilizing decreasing sequences of subsheaves in $\mathcal{S}(\mu)$, and that's impossible because $\mathcal{S}(\mu)$ is an Artinian category.

By proposition 8.3.1, $\mathcal{E}/\mathcal{E}_1$ an object of $\mathcal{S}(\mu)$, and by the same argument as before it contains some stable subsheaf of slope μ. We can continue the argument until we obtain a Jordan-Hölder filtration of $\mathcal{E}$. This process is finite, because the rank of different elements of the filtration is different and the rank of every element of the filtration is bounded between 0 and $\text{rk}(\mathcal{E})$.

Let

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \ldots \subsetneq \mathcal{E}_r = \mathcal{E}$$
$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_s = \mathcal{E}$$

be Jordan-Hölder filtrations of $\mathcal{E}$. Let's prove that $r = s$ and that $\oplus_{i=1}^{r} \mathcal{E}_i/\mathcal{E}_{i-1} \simeq \oplus_{i=1}^{s} \mathcal{F}_i/\mathcal{F}_{i-1}$. We proceed by induction on $\text{rk}(\mathcal{E})$.

If $\text{rk}(\mathcal{E}) = 1$, then $\mathcal{E}$ is an invertible sheaf and it's stable by example 8.2.1, so $\mathcal{E}$ has only one Jordan-Hölder filtration $0 \subsetneq \mathcal{E}$ and the result follows.

Suppose that the result is true for every locally free sheaf of rank less than $\text{rk}(\mathcal{E})$. Let

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \ldots \subsetneq \mathcal{E}_r = \mathcal{E}$$
$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_s = \mathcal{E}$$

be Jordan-Hölder filtrations of $\mathcal{E}$. There is some $j = 1, \dots, s$ such that $\mathcal{E}_1 \subseteq \mathcal{F}_j$ and $\mathcal{E}_1 \nsubseteq \mathcal{F}_{j-1}$, so there is a non zero homomorphism $\mathcal{E}_1 \to \mathcal{F}_j/\mathcal{F}_{j-1}$ of stable sheaves with the same slope, which must be an isomorphism by proposition 8.2.1.

Consider the induced filtrations of $\mathcal{E}/\mathcal{E}_1$

$$0 \subsetneq \mathcal{E}_2/\mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}/\mathcal{E}_1$$
$$0 \subsetneq \mathcal{F}_1/(\mathcal{E} \cap \mathcal{F}_1) \subsetneq \dots \subsetneq \mathcal{E}_1/\mathcal{E}_1$$

of lengths $r-1$ and $s-1$ respectively; we are denoting $\mathcal{E}_1 \cap \mathcal{F}_i := \mathrm{Ker}(\mathcal{F}_i \to \mathcal{E}/\mathcal{E}_1)$. These are Jordan-Hölder filtrations because

$$(\mathcal{E}_i/\mathcal{E}_1)/(\mathcal{E}_{i-1}/\mathcal{E}_1) \simeq \mathcal{E}_1/\mathcal{E}_{i-1} \text{ and } ((\mathcal{F}_i/\mathcal{E}_1 \cap \mathcal{F}_i))/(\mathcal{F}_{i-1}/(\mathcal{E}_1 \cap \mathcal{F}_{i-1})) \simeq \mathcal{F}_i/\mathcal{F}_{i-1}$$

The second isomorphism exists because the snake lemma provides a non zero surjective homomorphism

$$((\mathcal{F}_i/\mathcal{E}_1 \cap \mathcal{F}_i))/(\mathcal{F}_{i-1}/(\mathcal{E}_1 \cap \mathcal{F}_{i-1})) \to \mathcal{F}_i/\mathcal{F}_{i-1} \to 0$$

which is an isomorphism by proposition 8.2.1 because $\mathcal{F}_i/\mathcal{F}_{i-1}$ is stable.

We have that $\mathrm{rk}(\mathcal{E}/\mathcal{E}_1) < \mathrm{rk}(\mathcal{E})$ and thus, by induction hypothesis,

$$r = s \text{ and } \oplus_{i=2}^{r} \mathcal{E}_i/\mathcal{E}_{i-1} \simeq \oplus_{i \neq j} \mathcal{F}_i/\mathcal{F}_{j-1}$$

On the other hand, we already saw that $\mathcal{E}_1 \simeq \mathcal{F}_j/\mathcal{F}_{j-1}$, so the result follows ☐

Definition 8.3.2. *A semistable sheaf on X is called polystable if it's the direct sum of stable sheaves. Two semistable sheaves on X are S-equivalent if their associated graded objects are isomorphic*

We will now introduce the Harder-Narasimhan filtration of a locally free sheaf on X. It can be seen as a measure of how far is a sheaf from being semistable

Lemma 8.3.2. *Let $\mathcal{E}$ be a locally free sheaf on X. The set of slopes of coherent subsheaves of $\mathcal{E}$ is bounded above*

Proof. Let $\mathcal{F} \subseteq \mathcal{E}$ be a coherent subsheaf of $\mathcal{E}$. Since $\mathrm{rk}(\mathcal{F}) > 0$, we have that

$$\mu(\mathcal{F}) = \frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} \leqslant \deg(\mathcal{F})$$

so we conclude by proposition 7.4.1 ☐

Proposition 8.3.3. *Let $\mathcal{E}$ be a locally free sheaf on X. There is a filtration of $\mathcal{E}$ by sub-bundles (recall definition 7.4.1)*

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_r = \mathcal{E}$$

such that

- $\mathcal{E}_i/\mathcal{E}_{i-1}$ *is semistable for every* $i = 1, \dots, r$

- $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) > \mu(\mathcal{E}_{i-1}/\mathcal{E}_{i-2})$ *for every* $i = 2, \dots, r$
- *The filtration is unique*

This filtration is called the Harder-Narasimhan filtration of $\mathcal{E}$. We denote $\mu_i(\mathcal{E}) := \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ and

$$\mu_{max}(\mathcal{E}) := \mu_1(\mathcal{E}) \quad \mu_{min}(\mathcal{E}) := \mu_r(\mathcal{E})$$

The increasing sequence

$$\mu_{max}(\mathcal{E}) > \dots > \mu_{min}(\mathcal{E})$$

is called the Harder-Narasimhan sequence of $\mathcal{E}$. The sub-bundle $\mathcal{E}_1 \subseteq \mathcal{E}$ is called the maximal destabilizing sub-bundle of $\mathcal{E}$

Proof. (Of proposition 8.3.3) Let $\mathcal{E}_1 \subseteq \mathcal{E}$ be a coherent subsheaf of $\mathcal{E}$ satisfying the following conditions

- $\mu(\mathcal{E}_1)$ is maximal among the slopes of coherent subsheaves of $\mathcal{E}$ (this is possible by lemma 8.3.2)
- $\text{rk}(\mathcal{E}_1)$ is maximal among the ranks of coherent subsheaves of $\mathcal{E}$ of maximal slope

Because of these conditions, $\mathcal{E}_1$ is a coherent subsheaf of $\mathcal{E}$ of maximal degree among coherent subsheaves of $\mathcal{E}$ with rank $\text{rk}(\mathcal{E}_1)$. By a very similar argument to the proof of corollary 7.4.3, $\mathcal{E}_1$ is a sub-bundle of $\mathcal{E}$. $\mathcal{E}_1$ is called a maximal destabilizing sub-bundle of $\mathcal{E}$ (we will see later that $\mathcal{E}_1$ is unique satisfying these two properties).

We have the inclusions

$$0 \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}$$

Clearly, $\mathcal{E}_1$ is semistable because it has maximal slope. Let $\overline{\mathcal{E}}_2$ be a maximal destabilizing sub-bundle of $\mathcal{E}/\mathcal{E}_1$. Then, $\overline{\mathcal{E}}_2 = \mathcal{E}_2/\mathcal{E}_1$ for a coherent sheaf $\mathcal{E}_2$ of $\mathcal{E}$ containing $\mathcal{E}_1$. We have that $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2)$ because, if it were $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$, then we would also have $\text{rk}(\mathcal{E}_1) = \text{rk}(\mathcal{E}_2)$ by the properties of $\mathcal{E}_1$ and thus we would have $\mathcal{E}_1 \simeq \mathcal{E}_2$ by lemma 7.4.1. From this, using lemma 8.2.4 we have that

$$\text{rk}(\mathcal{E}_2)\mu(\mathcal{E}_1) > \text{rk}(\mathcal{E}_2) = \text{rk}(\mathcal{E}_1)\mu(\mathcal{E}_1) + \text{rk}(\mathcal{E}_2/\mathcal{E}_1)\mu(\mathcal{E}_1/\mathcal{E}_2)$$

and thus $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_1/\mathcal{E}_2)$. We can continue the process to obtain a filtration of $\mathcal{E}$ satisfying the desired conditions.

Let

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_r = \mathcal{E}$$

be a filtration of $\mathcal{E}$ constructed in this way, and let

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \dots \subsetneq \mathcal{F}_s = \mathcal{E}$$

be another filtration of $\mathcal{E}$ satifying the same properties. Let's see that each term is equal. There is some $j = 1, \dots, s$ such that $\mathcal{E}_1 \subseteq \mathcal{F}_j$ and $\mathcal{E}_1 \nsubseteq \mathcal{F}_{j-1}$; this induces a non zero homomorphism $\mathcal{E}_1 \to \mathcal{F}_j/\mathcal{F}_{j-1}$ between semistable sheaves. By the properties of $\mathcal{E}_1$, proposition 8.2.1 and the properties of the filtration we have the chain of inequalities

$$\mu(\mathcal{F}_1) \leqslant \mu(\mathcal{E}_1) \leqslant \mu(\mathcal{F}_j/\mathcal{F}_{j-1}) \leqslant \mu(\mathcal{F}_1)$$

so all these inequalities are equalities and thus

- $\mathcal{F}_j = \mathcal{F}_1$
- $\mu(\mathcal{F}_1) = \mu(\mathcal{E}_1)$ and $\mathcal{E}_1 \subseteq \mathcal{F}_1 \Rightarrow \mathrm{rk}(\mathcal{E}_1) = \mathrm{rk}(\mathcal{F}_1)$

so $\mathcal{E}_1 \simeq \mathcal{F}_1$. We can use the same argument to conclude that $\mathcal{E}_2/\mathcal{E}_1 \simeq \mathcal{F}_2/\mathcal{F}_1$ and thus $\mathcal{E}_2 \simeq \mathcal{F}_2$. Uniqueness follows □

8.4 Boundedness

Let X be a smooth projective algebraic curve

Definition 8.4.1. *Let $\mathcal{A}$ be a family of isomorphism classes of locally free sheaves on X. $\mathcal{A}$ is bounded if there is a finite type scheme S and a coherent sheaf $\mathcal{U}$ on $X \times S$, flat over S (with respect to the natural projection on S), such that for every $\mathcal{E} \in \mathcal{A}$ there is some $s \in S^\bullet(k)$ with $\mathcal{U}_s \simeq \mathcal{E}$, where $\mathcal{U}_s$ is the pullback of $\mathcal{U}$ by the inclusion $\{s\} \hookrightarrow S$*

In this section we will give necessary and sufficient conditions for a family of isomorphism classes of locally free sheaves on X to be bounded. We will prove that these conditions are fundamentally related to semistability

Observation 8.4.1. *Let S be a finite type scheme and $\mathcal{M}$ a coherent sheaf on $X \times S$, flat over S. By the semicontinuity theorem (see for example [Mum70, Chapter II, § 5]), the map $s \mapsto \chi(\mathcal{M}_s \otimes \mathcal{O}_X(n))$ is constant on each connected component of S, so the possible values for the rank and degree of the sheaves in the family $\{\mathcal{M}_s\}_{s \in S^\bullet(k)}$ are finite (because S is a finite type scheme). In particular, any bounded family $\mathcal{A}$ of locally free sheaves on X can take only a finite number of values of rank and degree*

Lemma 8.4.1. *Let $\mathcal{M}$ be a flat coherent $\mathcal{O}_X$-module such that $H^1(X, \mathcal{M}) = 0$. Then, the following properties are equivalent*

- *$\mathcal{M}$ is generated by its global sections*
- *$H^1(X, \mathcal{M} \otimes \mathcal{O}_X(-x)) = 0$ for every $x \in X^\bullet(k)$*

Proof. By Nakayama's lemma, $\mathcal{M}$ is generated by its global sections if and only if, for every $x \in X^\bullet(k)$, the natural homomorphism $H^0(X, \mathcal{M}) \to \mathcal{M}_x/\mathfrak{m}_x\mathcal{M}_x$ is surjective.

Let $x \in X^\bullet(k)$. There is an exact sequence

$$0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to k(x) \to 0$$

Since $\mathcal{M}$ is a flat $\mathcal{O}_X$-module, we obtain the exact sequence

$$0 \to \mathcal{M} \otimes \mathcal{O}_X(-x) \to \mathcal{M} \to \mathcal{M}_x/\mathfrak{m}_x\mathcal{M}_x \to 0$$

Taking cohomology, we obtain the long exact sequence

$$\cdots \to H^0(X, \mathcal{M}) \to \mathcal{M}_x/\mathfrak{m}_x\mathcal{M}_x \to H^1(X, \mathcal{M} \otimes \mathcal{O}_X(-x)) \to 0$$

so we conclude □

Theorem 8.4.1 ([Scho8], Proposition 2.2.3.2)**.** *Let $\mathcal{A}$ be a family of isomorphism classes of locally free sheaves on X. Then, $\mathcal{A}$ is a bounded family if and only if the following conditions are satisfied*

(a) The set $\{(\deg(\mathcal{E}), \mathrm{rk}(\mathcal{E})) : \mathcal{E} \in \mathcal{A}\}$ *is finite*

(b) There is a natural number $n_0 > 0$ such that, for every $n \geqslant n_0$ and every $\mathcal{E} \in \mathcal{A}$

- $H^1(X, \mathcal{E}(n)) = 0$
- $\mathcal{E}(n)$ *is generated by its global sections*

Proof. Suppose that $\mathcal{A}$ is a bounded family. Then, there is a finite type scheme S over k and a coherent sheaf $\mathcal{U}$ on $X \times S$, flat over S via the projection $\pi_S : X \times S \to S$, such that the elements of $\mathcal{A}$ appear as elements of the family $\{\mathcal{U}_s\}_{s \in S^\bullet(k)}$ of coherent sheaves on X.

Observation 8.4.1 proves that condition (a) holds. Fix $s_0 \in S^\bullet(k)$. By Serre's theorems, there is a natural number $n_0 > 0$ such that, for every $n \geqslant n_0$, $H^1(X, \mathcal{U}_{s_0}(n)) = 0$ and $\mathcal{U}_{s_0}(n)$ is generated by its global sections. In virtue of lemma 8.4.1, we can express this fact as a cohomological condition: $H^1(X, \mathcal{U}_{s_0}(n) \otimes \mathcal{O}_X(-x)) = 0$ for every $x \in X^\bullet(k)$.

Let $\pi_X : X \times S \to X$ be the natural projection on X and consider, for every $x \in X^\bullet(k)$, the coherent sheaves on $X \times S$

$$\mathcal{U} \otimes \pi_X^* \mathcal{O}_X(n) \text{ and } \mathcal{U} \otimes \pi_X^* \mathcal{O}_X(n - x)$$

These are coherent flat sheaves over S such that

$$(\mathcal{U} \otimes \pi_X^* \mathcal{O}_X(n))_s \simeq \mathcal{U}_s(n) \text{ and } (\mathcal{U} \otimes \pi_X^* \mathcal{O}_X(n - x))_s \simeq \mathcal{U}_s(n) \otimes \mathcal{O}_X(-x)$$

for every $s \in S^\bullet(k)$. By construction, we have that

- $H^1(X, (\mathcal{U} \otimes \pi_X^* \mathcal{O}_X(n))_{s_0}) = H^1(X, \mathcal{U}_{s_0}(n)) = 0$
- $H^1(X, (\mathcal{U} \otimes \pi_X^* \mathcal{O}_X(n - x))_{s_0}) = H^1(X, \mathcal{U}_{s_0}(n) \otimes \mathcal{O}_X(-x)) = 0$

and by the semicontinuity theorem

- $H^1(X, \mathcal{U}_s(n)) = 0$ (8.2)
- $H^1(X, \mathcal{U}_s(n) \otimes \mathcal{O}_X(-x)) = 0 \;\; \forall x \in X^\bullet(k)$ (8.3)

for every closed point s in the connected component of S containing s_0 and every $n \geqslant n_0$.

Since S is a finite type scheme, we can take n_0 sufficiently large so that the previous conditions hold for every $s \in S^\bullet(k)$ and every $n \geqslant n_0$. In virtue of lemma 8.4.1, we have proven that for every $n \geqslant n_0$ and every $s \in S^\bullet(k)$

- $H^1(X, \mathcal{U}_s(n)) = 0$
- $\mathcal{U}_s(n)$ is generated by its global sections

so the same follows for the elements of $\mathcal{A}$.

Suppose now that $\mathcal{A}$ satisfies the conditions of the theorem. We can suppose, without any loss of generality, that the elements of $\mathcal{A}$ have a fixed rank r and degree d. By hypothesis, there is a natural number $N \geqslant 0$ such that, for every $\mathcal{E} \in \mathcal{A}$,

- $H^1(X, \mathcal{E}(N)) = 0$
- $\mathcal{E}$ is generated by its global sections, i.e., there is a surjective homomorphism $H^0(X, \mathcal{E}(N)) \otimes_k \mathcal{O}_X \to \mathcal{E}(N) \to 0$

In particular, every element $\mathcal{E} \in \mathcal{A}$ can be expressed as a quotient

$$H^0(X, \mathcal{E}(N)) \otimes_k \mathcal{O}_X(-N) \to \mathcal{E} \to 0$$

Furthermore, by proposition 7.3.3 and using that $H^1(X, \mathcal{E}(N)) = 0$

$$h^0(\mathcal{E}(N)) = \chi(\mathcal{E}(N)) = r(1-g) + d + \deg(X)rN$$

Fix a k-vector space H of dimension $r(1-g) + d + \deg(X)rN$ and a linear isomorphism $H \simeq H^0(X, \mathcal{E}(N)$ for every $\mathcal{E} \in \mathcal{A}$. Then, every $\mathcal{E} \in \mathcal{A}$ can be expressed as a quotient

$$H \otimes_k \mathcal{O}_X(-N) \to \mathcal{E} \to 0$$

such that the induced homomorphism

$$H \to H^0(X, \mathcal{E}(N))$$

is an isomorphism.

In other words, every $\mathcal{E} \in \mathcal{A}$ is a closed point of the Quot scheme $\mathrm{Quot}_X^{P(m)}(H \otimes_k \mathcal{O}_X(-N))$ where $P(m) = P(\mathcal{E}, m)$ is the Hilbert polynomial of any $\mathcal{E} \in \mathcal{A}$. So every element of $\mathcal{A}$ arises as a pullback of the universal quotient sheaf $\mathcal{U}$ on $X \times \mathrm{Quot}_X^{P(m)}(H \otimes_k \mathcal{O}_X(-N))$, and thus $\mathcal{A}$ is a bounded family □

Example 8.4.1. *Denote by $\mathcal{A}(r, d)$ the family of isomorphism classes of locally free sheaves on X of rank r and degree d. Then, $\mathcal{A}(r, d)$ is not bounded if $r \geqslant 2$.*

Indeed, consider the subfamily

$$\{\mathcal{E}_n = \mathcal{O}_X(-n) \oplus \mathcal{O}_X(d+n) \oplus \mathcal{O}_X^{\oplus(r-2)}\}_{n \in \mathbb{Z}} \subset \mathcal{A}(r, d)$$

For every $m \geqslant 0$, we have that

$$h^1(\mathcal{E}_n(m)) \geqslant h^1(\mathcal{O}_X(m-n)) \geqslant r(g-1) + (m-n)$$

so there is not any $m \geqslant 0$ such that $h^1(\mathcal{E}_n(m)) = 0$ for every $n \in \mathbb{Z}$, and thus $\mathcal{A}(r, d)$ is not a bounded family by theorem 8.4.1

We will now relate the conditions of theorem 8.4.1 to slope-semistability

Proposition 8.4.1 ([Scho8], Proposition 2.2.3.7). *Let $\mathcal{A}$ be a family of isomorphism classes of locally free sheaves on X. Then, $\mathcal{A}$ is bounded if and only if the following conditions are satisfied*

- *The set* $\{(\deg(\mathcal{E}), \mathrm{rk}(\mathcal{E})) : \mathcal{E} \in \mathcal{A}\}$ *is finite*
- *There is a constant* $C \in \mathbb{R}$ *such that, for every* $\mathcal{E} \in \mathcal{A}$,

$$\mu_{\max}(\mathcal{E}) \leqslant \mu(\mathcal{E}) + C$$

(recall the definition of $\mu_{\max}(\mathcal{E})$ *in proposition 8.3.3)*

Proof. First, note that the second condition of the proposition implies any of the following two equivalent assertions

(a) There is a constant $C \in \mathbb{R}$ such that, for every $\mathcal{E} \in \mathcal{A}$ and every coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$, we have that $\mu(\mathcal{F}) \leqslant \mu(\mathcal{E}) + C$

(b) There is a constant $C \in \mathbb{R}$ such that, for every $\mathcal{E} \in \mathcal{A}$ and every sub-bundle $\mathcal{F} \subseteq \mathcal{E}$, we have that $C + \mu(\mathcal{E}) \leqslant \mu(\mathcal{E}/\mathcal{F})$

(a) follows directly from the definition of $\mu_{\max}(\mathcal{E})$, and the equivalence of the two is a direct consequence of lemma 8.2.4.

Suppose that $\mathcal{A}$ is a bounded family. By theorem 8.4.1, we can find a sufficiently large integer N such that, for every $\mathcal{E} \in \mathcal{A}$, we have that $H^1(X, \mathcal{E}(N)) = 0$. Suppose that the condition of the proposition is not true. Then, for every $C \in \mathbb{R}$ we could find some $\mathcal{E} \in \mathcal{A}$ such that there is a sub-bundle $\mathcal{F} \subseteq \mathcal{E}$ with $C + \mu(\mathcal{E}) > \mu(\mathcal{Q})$, where $\mathcal{Q} := \mathcal{E}/\mathcal{F}$.

Since the set $\{\mu(\mathcal{M}) : \mathcal{M} \in \mathcal{A}\}$ is finite by hypothesis, we can choose $C \in \mathbb{R}$ such that $\mu(\mathcal{Q}) < g - 1 - N\deg(X)$. Besides, since $\mathcal{F}$ is a sub-bundle of $\mathcal{E}$, then $\mathcal{Q}^\vee$ is a subsheaf of $\mathcal{E}^\vee$. Let ω_X be the canonical sheaf on X. Using Serre duality, the fact that $\mathcal{Q}^\vee$ is a subsheaf of $\mathcal{E}^\vee$ and the Riemann-Roch formula, we have that

$$\begin{aligned}
0 = \frac{h^1(\mathcal{E}(N))}{\mathrm{rk}(\mathcal{Q})} &= \\
&= \frac{h^0(\mathcal{E}^\vee(-N) \otimes \omega_X)}{\mathrm{rk}(\mathcal{Q})} \geqslant \\
&\geqslant \frac{h^0(\mathcal{Q}^\vee(-N) \otimes \omega_X)}{\mathrm{rk}(\mathcal{Q})} = \\
&= \frac{h^1(\mathcal{Q}(N))}{\mathrm{rk}(\mathcal{Q})} = \\
&= \frac{h^0(\mathcal{Q}(N)) + \mathrm{rk}(\mathcal{Q})(g-1) - \deg(\mathcal{Q}) - \deg(X)\mathrm{rk}(\mathcal{Q})N}{\mathrm{rk}(\mathcal{Q})} \geqslant \\
&\geqslant g - 1 - \mu(\mathcal{Q}) - \deg(X)N > 0
\end{aligned}$$

which is a contradiction.

Suppose now that the conditions of the proposition hold. We will prove that $\mathcal{A}$ is bounded using the criteria of theorem 8.4.1.

Let N be an integer such that there is some $\mathcal{E} \in \mathcal{A}$ with $H^1(X, \mathcal{E}(N)) \neq 0$. By Serre duality, we have that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}(N), \omega_X) \simeq H^1(X, \mathcal{E}(N))^* \neq 0$, so there is a non-zero homomorphism $f : \mathcal{E}(N) \to \omega_X$. Consider the associated exact sequence

$$0 \to \mathrm{Ker}(f) \to \mathcal{E}(N) \to \mathrm{Im}(f) \to 0$$

By lemma 8.2.4, we have that

$$\mathrm{rk}(\mathcal{E})\mu(\mathcal{E}(N)) = \mathrm{rk}(\mathrm{Ker}(f))\mu(\mathrm{Ker}(f)) + \mathrm{rk}(\mathrm{Im}(f))\mu(\mathrm{Im}(f)) \tag{8.4}$$

Besides, $\mathrm{Ker}(f)(-N) \subseteq \mathcal{E}$, so by hypothesis there exists some $C \in \mathbb{R}$ (we can suppose that $C \geqslant 0$) such that

$$\mu(\mathrm{Ker}(f)) \leqslant \mu(\mathcal{E}) + C + N\deg(X)$$

On the other hand, we have that $\mu(\mathcal{E}(N)) = \mu(\mathcal{E}) + N\deg(X)$. Substituting this on equation 8.4, we have that

$$\mathrm{rk}(\mathcal{E})(\mu(\mathcal{E}) + N\deg(X)) \leqslant \mathrm{rk}(\mathrm{Ker}(f))(\mu(\mathcal{E}) + N\deg(X) + C) + \mathrm{rk}(\mathrm{Im}(f))\mu(\mathrm{Im}(f)) \tag{8.5}$$

$\mathrm{Im}(f)$ is an invertible subsheaf of ω_X, so $\mu(\mathrm{Im}(f)) \leqslant \mu(\omega_X) = 2g - 2$. Putting this on equation 8.5 and using lemma 7.1.3, we arrive at

$$\mu(\mathcal{E}) + 2 - 2g - \frac{\mathrm{rk}(\mathrm{Ker}(f))}{\mathrm{rk}(\mathrm{Im}(f))}C < -N\deg(X)$$

But $\frac{\mathrm{rk}(\mathrm{Ker}(f))}{\mathrm{rk}(\mathrm{Im}(f))} = \frac{\mathrm{rk}(\mathcal{E})}{\mathrm{rk}(\mathrm{Im}(f))} - 1 \leqslant \mathrm{rk}(\mathcal{E}) - 1$, so we have that

$$(\mathrm{rk}(\mathcal{E}) - 1)C + 2g - 2 - \mu(\mathcal{E}) \geqslant N\deg(X)$$

By hypothesis, the set $\{(\deg(\mathcal{E}), \mathrm{rk}(\mathcal{E})) : \mathcal{E} \in \mathcal{A}\}$ is finite, so we can find bounds for $\mathrm{rk}(\mathcal{E}) - 1$ and $-\mu(\mathcal{E})$, and we deduce that there is some $M \in \mathbb{R}$ such that $M \geqslant N$.

In other words, for every integer $N \geqslant M$ and every $\mathcal{E} \in \mathcal{A}$, we have that $H^1(X, \mathcal{E}(N)) = 0$. Using the same ideas, we can consider a larger M so that, for every $x \in X^\bullet(k)$ and every $\mathcal{E} \in \mathcal{A}$, we have that $H^1(X, \mathcal{E}(N) \otimes \mathcal{O}_X(-x)) = 0$. By lemma 8.4.1 and theorem 8.4.1, we conclude that $\mathcal{A}$ is a bounded family □

Chapter 9

The moduli space of semistable sheaves

In this chapter we will define the moduli problem of semistable sheaves on a smooth projective algebraic curve. We will prove the existence of moduli spaces using the techniques of geometric invariant theory developed in part I. We will suppose that $\mathrm{char}(k) = 0$.

Some references for this chapter are [Hos15, Chapter 8], [LP97, Chapters 7-8] and [HL10, Chapter 4]

9.1 The moduli functor

Let X be a smooth projective algebraic curve. Let $r > 0$ and $d \in \mathbb{Z}$

Definition 9.1.1. *Let S be a scheme. A flat family of semistable (resp. stable) sheaves on X parameterized by S is a coherent sheaf $\mathcal{M}$ on $X \times S$, flat over S, such that $\mathcal{M}_s$ is a semistable (resp. stable) locally free sheaf on X for every $s \in S^{\bullet}(k)$*

For every scheme S, denote by $\mathcal{M}_X^{ss}(r,d)(S)$ the set of isomorphism classes of flat families of semistable sheaves of rank r and degree d on X parameterized by S

Lemma 9.1.1. *The correspondence*

$$\begin{aligned} \mathcal{M}_X^{ss}(r,d) : \mathrm{Sch} &\to \mathrm{Sets} \\ S &\mapsto \mathcal{M}_X^{ss}(r,d)(S) \end{aligned}$$

is a contravariant functor.

If $\mathcal{M}_X^{s}(r,d)(S)$ denotes the set of flat families of stable locally free sheaves of rank r and degree d on X parameterized by S, the correspondence $\mathcal{M}_X^{s}(r,d) : \mathrm{Sch} \to \mathrm{Sets}$ is also functorial

Proof. Let $f : S \to T$ be a morphism of schemes. We have the cartesian square

$$\begin{array}{ccc} X \times S & \xrightarrow{\mathrm{id}_X \times f} & X \times T \\ {\scriptstyle \pi_S}\downarrow & & \downarrow{\scriptstyle \pi_T} \\ S & \xrightarrow[f]{} & T \end{array}$$

Consider

$$\mathcal{M}_X^{ss}(r,d)(f) : \mathcal{M}_X^{ss}(r,d)(T) \to \mathcal{M}_X^{ss}(r,d)(S)$$
$$\mathcal{M} \mapsto \mathcal{M}_X^{ss}(r,d)(f)(\mathcal{M}) := (id_X \times f)^*\mathcal{M}$$

Clearly, $(id_X \times f)^*(\mathcal{M})$ is a coherent sheaf on $X \times S$, flat over S. Besides, for every $s \in S^\bullet(k)$ then, if $i_s := (Id_X, s)$, we have that

$$\begin{aligned}[(id_X \times f)^*\mathcal{M}]_s &= i_s^*(id_X \times f)^*\mathcal{M} = \\ &= [(id_X \times f) \circ i_s]^*\mathcal{M} = \\ &= i_{f(s)}^*\mathcal{M} = \\ &= \mathcal{M}_{f(s)}\end{aligned}$$

so $(id_X \times f)^*\mathcal{M} \in \mathcal{M}_X^{ss}(r,d)(S)$, and thus $\mathcal{M}_X^{ss}(r,d)(f)$ is well defined. This proves that $\mathcal{M}_X^{ss}(r,d) : \mathrm{Sch}_k \to \mathrm{Sets}$ is a contravariant functor.

The analogous result for $\mathcal{M}_X^{s}(r,d)$ can be proven in a very similar way □

Let S be a scheme, $\pi_S : X \times S \to S$ the projection on S and $\mathcal{M} \in \mathcal{M}_X^{ss}(r,d)(S)$. Then, for every $\mathcal{L} \in \mathrm{Pic}(S)$, it's easy to see that $\mathcal{M} \otimes \pi_S^*\mathcal{L} \in \mathcal{M}_X^{ss}(r,d)(S)$.

Consider the following equivalence relation on $\mathcal{M}_X^{ss}(r,d)(S)$

$$\mathcal{M} \sim_S \mathcal{N} \text{ in } \mathcal{M}_X^{ss}(r,d)(S) \Leftrightarrow \mathcal{M} \simeq \mathcal{N} \otimes \pi_S^*\mathcal{L} \text{ for some } \mathcal{L} \in \mathrm{Pic}(S)$$

Denote by $\widetilde{\mathcal{M}}_X^{ss}(r,d)$ the functor defined by $\widetilde{\mathcal{M}}_X^{ss}(r,d)(S) := \mathcal{M}_X^{ss}(r,d)(S)/\sim_S$ for every scheme S (we define $\widetilde{\mathcal{M}}_X^{s}(r,d)$ similarly)

Proposition 9.1.1 ([HL10], Proposition 2.3.1)**.** *Let $\mathcal{M}$ be a flat family of coherent sheaves on X parameterized by S. The sets $\{s \in S^\bullet(k) : \mathcal{M}_s$ is semistable$\}$ and $\{s \in S^\bullet(k) : \mathcal{M}_s$ is stable$\}$ are the sets of closed points of uniquely determined open subsets of S. In other words, being stable or semistable are open conditions in flat families*

Observation 9.1.1. *Note that, by proposition 9.1.1, $\mathcal{M}_X^{s}(r,d)$ and $\widetilde{\mathcal{M}}_X^{s}(r,d)$ are, respectively, open subfunctors of $\mathcal{M}_X^{ss}(r,d)$ and $\widetilde{\mathcal{M}}_X^{ss}(r,d)$*

Lemma 9.1.2. *A scheme corepresents $\mathcal{M}_X^{ss}(r,d)$ (resp. $\mathcal{M}_X^{s}(r,d)$) if and only if it corepresents $\widetilde{\mathcal{M}}_X^{ss}(r,d)$ (resp. $\widetilde{\mathcal{M}}_X^{s}(r,d)$)*

Proof. There is a natural morphism of functors $\eta : \mathcal{M}_X^{ss}(r,d) \to \widetilde{\mathcal{M}}_X^{ss}(r,d)$. Let T be a scheme and $\xi : \mathcal{M}_X^{ss}(r,d) \to T^\bullet$ a morphism of functors. If we prove that there is a unique morphism of functors $\widetilde{\xi} : \widetilde{\mathcal{M}}_X^{ss}(r,d) \to T^\bullet$ such that $\xi = \widetilde{\xi} \circ \eta$, then we would conclude.

Let S be a scheme and $\xi_S : \mathcal{M}_X^{ss}(r,d)(S) \to T^\bullet(S)$. If we prove that for any $\mathcal{L} \in \mathrm{Pic}(S)$ we have $\xi_S(\pi_S^*\mathcal{L}) = \xi_S(\pi_S^*\mathcal{O}_S) = \xi_S(\mathcal{O}_{T\times S})$, then clearly ξ_S would factor via η_S and we would conclude.

By definition, $\xi_S(\pi_S^*\mathcal{L}) \in T^\bullet(S) = \mathrm{Hom}(S, T)$. If we prove that for every closed point $p : \operatorname{Spec} k \to S$ we have $\xi_S(\pi_S^*\mathcal{L}) \circ p = \xi_S(\mathcal{O}_{X\times S}) \circ p = \xi_{\operatorname{Spec} k}(\mathcal{O}_X)$, then we conclude. This is immediate since

$$\begin{aligned}\xi_S(\pi_S^*\mathcal{L}) \circ p &= \xi_{\operatorname{Spec} k}((\mathrm{Id}_X \times p)^*\pi_S^*\mathcal{L}) = \\ &= \xi_{\operatorname{Spec} k}(\mathcal{L}_p/\mathfrak{m}_p\mathcal{L}_p \otimes_k \mathcal{O}_X) = \\ &= \xi_{\operatorname{Spec} k}(\mathcal{O}_X)\end{aligned}$$

□

9.2 Construction of the moduli space

Let $r > 0$ and $d \in \mathbb{Z}$

Definition 9.2.1. *A scheme corepresenting the functor* $\mathcal{M}_X^{ss}(r, d) : \mathrm{Sch} \to \mathrm{Sets}$ *is called a moduli space of semistable sheaves of rank* r *and degree* d

In this section, we will prove that there are always moduli spaces of semistable sheaves, and that there exists a coarse moduli space for $\mathcal{M}_X^s(r, d)$ (see [Hos15, Chapter 2, § 5] for the definition of a coarse moduli space).

We outline here the main steps that we will follow in the construction

(a) We will start by reducing the moduli problem to semistable sheaves with sufficiently high degree. Using proposition 8.2.1, this condition on the degree implies the vanishing of the first cohomology group of the sheaves and the property of being globally generated

(b) We will then prove some technical results due to Le Potier, that express the condition of being semistable in terms of a growing condition on the Hilbert polynomial

(c) We will transform the moduli problem into a problem of geometric invariant theory, so that the moduli space of semistable sheaves is isomorphic to the categorical quotient of an open subset Ω of a certain Quot scheme by an action of $SL(P(N))$, for some $P(N)$

(d) Finally, we will use the previous results of Le Potier to identify Ω as the set of semistable points associated to a linearization of an action of $SL(P(N))$ on the Quot scheme. This would prove that the categorical quotient $\Omega//SL(P(N))$ exists by theorem 4.2.1, and we would conclude by (c)

9.2.1 Reduction to sheaves of large degree

Lemma 9.2.1. *Let* $\mathcal{E}$ *be a semistable locally free sheaf on* X*. If* $\deg(\mathcal{E}) > \mathrm{rk}(\mathcal{E}) \cdot (2g - 1)$*, then* $H^1(X, \mathcal{E}) = 0$ *and* $\mathcal{E}$ *it's generated by its global sections*

Proof. Denote by ω_X the canonical sheaf on X. By Serre duality, we have that

$$H^1(X,\mathcal{E})^* \simeq H^0(X,\mathcal{E}^\vee \otimes \omega_X) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\omega_X)$$

Since $\mathcal{E}$ and ω_X are semistable sheaves and $\mu(\omega_X) = \deg(\omega_X) = 2g-2$, by proposition 8.2.1 we have that $H^1(X,\mathcal{E}) = 0$.

On the other hand, by lemma 8.4.1 we have that $\mathcal{E}$ is generated by its global sections if and only if $H^1(X,\mathcal{E}\otimes\mathcal{O}_X(-x)) = 0$ for every $x \in X^\bullet(k)$.

Again, by Serre duality this is equivalent to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}\otimes\mathcal{O}_X(-x),\omega_X) = 0$ for every $x \in X^\bullet(k)$, and the result follows from an easy computation and proposition 8.2.1, because $\mu(\mathcal{E}\otimes\mathcal{O}_X(-x)) = \frac{\deg(\mathcal{E})-\mathrm{rk}(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$ □

Lemma 9.2.2. *Let $\mathcal{E}$ be a semistable (resp. stable) locally free sheaf on X and $\mathcal{L}$ an invertible sheaf. Then, $\mathcal{E}\otimes\mathcal{L}$ is semistable (resp. stable)*

Proof. Let $\mathcal{F}$ be a coherent subsheaf of $\mathcal{E}\otimes\mathcal{L}$. Then, $\mathcal{F}\otimes\mathcal{L}^{-1}$ is a coherent subsheaf of $\mathcal{E}$, and the result follows from the semistability (resp. stability) of $\mathcal{E}$ and the fact that $\mu(\mathcal{E}\otimes\mathcal{L}) = \mu(\mathcal{E}) + \deg(\mathcal{L})$ and $\mu(\mathcal{F}\otimes\mathcal{L}^{-1}) = \mu(\mathcal{F}) - \deg(\mathcal{L})$ □

Observation 9.2.1. *The proof of a more general statement than lemma 9.2.2 can be found in [Zha]*

Proposition 9.2.1. *Let S be a scheme, and denote by $\pi_X : X \times S \to X$ the natural projection on X. For every $n \in \mathbb{Z}$, there is a bijection*

$$\mathcal{M}_X^{ss}(r,d)(S) \to \mathcal{M}_X^{ss}(r,d+r\deg(X)\cdot n)(S)$$
$$\mathcal{M} \mapsto \mathcal{M}\otimes\pi_X^*\mathcal{O}_X(n)$$

Furthermore, this bijection extends to an isomorphism of functors $\mathcal{M}_X^{ss}(r,d) \simeq \mathcal{M}_X^{ss}(r,d+r\deg(X)\cdot n)$ for every $n \in \mathbb{Z}$. The same result holds for the functor of flat families of stable sheaves

Proof. Let $\mathcal{M} \in \mathcal{M}_X^{ss}(r,d)(S)$. Then, $\mathcal{M}\otimes\pi_X^*\mathcal{O}_X(n)$ is a coherent sheaf on $X\times S$, flat over S.

For every $s \in S^\bullet(k)$, let $i_s = (\mathrm{Id}_X, s)$. We have that

$$i_s^*(\mathcal{M}\otimes\pi_X^*\mathcal{O}_X(n)) \simeq \mathcal{M}_s\otimes(\pi_X\circ i_s)^*\mathcal{O}_X(n) \simeq \mathcal{M}_s\otimes\mathcal{O}_X(n)$$

By lemma 9.2.2, $\mathcal{M}_s\otimes\mathcal{O}_X(n)$ is a semistable sheaf, and clearly it has rank r and degree $d + r\deg(X)\cdot n$.

This proves that the map is well defined. It's easy to see that its inverse is given by tensoring with $\pi_X^*\mathcal{O}_X(-n)$, and functoriality follows immediately. The same ideas prove the analogous result for the functor of flat families of stable sheaves □

Proposition 9.2.1 shows that we can reduce the problem to moduli functors $\mathcal{M}_X^{ss}(r,d)$ with $d > r(2g-1)$. This has the advantage that, by lemma 9.2.1, every semistable locally free sheaf $\mathcal{E}$ of rank r and degree d has $H^1(X,\mathcal{E}) = 0$ and it's globally generated

9.2.2 Some results of Le Potier

In this subsection, we will prove some technical results that are useful for the construction of the moduli space of semistable sheaves. They will allow us to identify semistable locally free sheaves as those that satisfy a certain growth condition on their Hilbert polynomials. The results presented here can be found in [LP97, Chapter 7, § 1]

Lemma 9.2.3. *Let $\mathcal{E}$ be a semistable locally free sheaf on X. Then*

$$\frac{h^0(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} \leqslant [\mu(\mathcal{E})+1]_+$$

Where $[a]_+ := \sup(a, 0)$

Proof. We proceed by induction on $\deg(\mathcal{E})$. If $\deg(\mathcal{E}) < 0$ then, by proposition 8.2.1,

$$H^0(X, \mathcal{E}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{E}) = 0$$

so the inequality follows.

Suppose that the result is true for every semistable locally free sheaf on X of degree less than $\deg(\mathcal{E})$. Let $x \in X^\bullet(k)$ and consider the exact sequence

$$0 \to \mathcal{E}(-x) \to \mathcal{E} \to \mathcal{E}_x/\mathfrak{m}_x\mathcal{E}_x \to 0$$

By lemma 9.2.2, $\mathcal{E}(-x)$ is a semistable sheaf of degree

$$\deg(\mathcal{E}(-x)) = \deg(\mathcal{E}) - \mathrm{rk}(\mathcal{E}) < \deg(\mathcal{E})$$

and thus, by induction hypothesis and noting that $\mu(\mathcal{E}) > 0$, we obtain

$$\begin{aligned} h^0(\mathcal{E}) &= h^0(\mathcal{E}(-x)) + h^0(\mathcal{E}_x/\mathfrak{m}_x\mathcal{E}_x) = \\ &= h^0(\mathcal{E}(-x)) + \mathrm{rk}(\mathcal{E}) \leqslant \\ &\leqslant \mathrm{rk}(\mathcal{E})[\mu(\mathcal{E}(-x))+1]_+ + \mathrm{rk}(\mathcal{E}) = \\ &= \mathrm{rk}(\mathcal{E})\mu(\mathcal{E}) + \mathrm{rk}(\mathcal{E}) = \\ &= \mathrm{rk}(\mathcal{E})[\mu(\mathcal{E})+1]_+ \end{aligned}$$

so we conclude ☐

Note that, if $\mathcal{E}$ is a locally free sheaf and

$$0 \subsetneq \mathcal{E}_1 \subsetneq \ldots \subsetneq \mathcal{E}_r = \mathcal{E}$$

is the Harder-Narasimhan filtration of $\mathcal{E}$ (recall proposition 8.3.3) then, by lemma 9.2.3, we have that

$$\frac{h^0(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} \leqslant \sum_{i=1}^{r} \frac{\mathrm{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})}{\mathrm{rk}(\mathcal{E})}[\mu(\mathcal{E}_i/\mathcal{E}_{i-1})+1]_+ \tag{9.1}$$

Proposition 9.2.2. *Denote by* $\mathrm{Coh}_X(r,d)$ *the family of isomorphism classes of coherent sheaves of rank* $r > 0$ *and degree* $d \in \mathbb{Z}$ *on* X. *There is an integer* $N(r,d)$ *such that, for every* $n \geqslant N(r,d)$ *and every* $\mathcal{M} \in \mathrm{Coh}_X(r,d)$, *the following assertions are equivalent*

(a) $\mathcal{M}$ *is a semistable locally free sheaf*

(b) For every coherent subsheaf $\mathcal{N} \subseteq \mathcal{M}$, *we have that*

$$h^0(\mathcal{N}(n)) \leqslant \frac{h^0(\mathcal{M}(n))}{\mathrm{rk}(\mathcal{M})} \cdot \mathrm{rk}(\mathcal{N})$$

Proof. Let's see that (a) implies (b). Let $\mathcal{E}$ be a semistable locally free sheaf on X and let a be an integer such that $a < \mu - g \cdot r$ (where $\mu = \mu(\mathcal{E}) = d/r$). Let $\mathcal{F} \subseteq \mathcal{E}$ a coherent subsheaf (which is locally free by lemma 7.3.2) such that $\mu_{min}(\mathcal{F}) \leqslant a$. Using lemma 9.2.3 and equation 9.1 above it's easy to show that, for every $n \geqslant -(a+1)$,

$$\frac{h^0(\mathcal{F}(n))}{\mathrm{rk}(\mathcal{F})} < \mu + n\deg(X) + 1 - g$$

We can find a sufficiently large integer $N(r,d) \geqslant -(a+1)$ such that, for every semistable locally free sheaf $\mathcal{E} \in \mathcal{S}(r,d)$ and every $n \geqslant N(r,d)$, then $h^1(\mathcal{E}(n)) = 0$. By the Riemann-Roch formula, we have that

$$\frac{h^0(\mathcal{E}(n))}{r} = \frac{\chi(\mathcal{E}(n))}{r} = \frac{r(1-g) + d + rn\deg(X)}{r} = 1 - g + \mu + n\deg(X)$$

So for every coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$ such that $\mu_{min}(\mathcal{F}) \leqslant a$ and every $n \geqslant N(r,d)$

$$\frac{h^0(\mathcal{F}(n))}{\mathrm{rk}(\mathcal{F})} \leqslant \frac{h^0(\mathcal{E}(n))}{\mathrm{rk}(\mathcal{E})}$$

We also have to consider coherent subsheaves $\mathcal{F} \subset \mathcal{E}$ such that $\mu_{min}(\mathcal{F}) > a$. Consider the family

$$\mathcal{A} := \{\mathcal{F} \subseteq \mathcal{E} \text{ coherent subsheaf} : \mu_{min}(\mathcal{F}) > a\}$$

For every $\mathcal{F} \in \mathcal{A}$, we have the associated Harder-Narasimhan filtration

$$0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_s = \mathcal{F}$$

It's easy to see that

$$\mu(\mathcal{F}) = \mu_{max}(\mathcal{F}) + \mu(\mathcal{F}_2/\mathcal{F}_1) + \ldots + \mu(\mathcal{F}/\mathcal{F}_{s-1}) > \mu_{max}(\mathcal{F}) + (s-1)a$$

But the length of Harder-Narasimhan filtrations of coherent subsheaves of $\mathcal{E}$ is bounded above by r, so there exists a constant $-(r-1)a \in \mathbb{R}$ such that, for every $\mathcal{F} \in \mathcal{A}$, we have that

$$\mu_{max}(\mathcal{F}) < \mu(\mathcal{F}) - (r-1)a$$

and thus the family $\mathcal{A}$ is bounded by proposition 8.4.1.

If necessary, we can make $N(r,d)$ larger such that, for every $n \geqslant N(r,d)$ and every $\mathcal{F} \in \mathcal{A}$, then $h^1(\mathcal{F}(n)) = 0$ and, by the semistability of $\mathcal{E}$, we have that

$$\frac{h^0(\mathcal{F}(n))}{\mathrm{rk}(\mathcal{F})} = \frac{\chi(\mathcal{F}(n))}{\mathrm{rk}(\mathcal{F})} = 1 - g + \mu(\mathcal{F}) + n\deg(X) \leqslant 1 - g + \mu + n\deg(X) = \frac{h^0(\mathcal{E}(n))}{r}$$

obtaining the desired inequality.

Let's see now that (b) implies (a). We can assume that $\mu + N(r,d) \geqslant g$. Let $\mathcal{M}$ be a coherent sheaf on X satisfying (b). The inequality says that every torsion subsheaf of $\mathcal{M}$ has no global sections and thus must be zero, so $\mathcal{M}$ is torsion-free and thus locally free by lemma 7.3.1.

We are going to prove that $\mathcal{M}$ is semistable using semistable quotient sheaves. Let $\mathcal{M} \to \mathcal{E} \to 0$ be a semistable quotient sheaf such that $\mu(\mathcal{E}) \leqslant \mu$. Then we have that, using the hypothesis and lemma 9.2.3

$$\begin{aligned} 1 - g + \mu + n\deg(X) = \frac{\chi(\mathcal{M}(n))}{r} &\leqslant \\ \leqslant \frac{h^0(\mathcal{M}(n))}{r} &\leqslant \\ \leqslant \frac{h^0(\mathcal{E}(n))}{\mathrm{rk}(\mathcal{E})} &\leqslant \\ \leqslant [\mu(\mathcal{E}) + n + 1]_+ & \end{aligned}$$

from which we conclude that $\mu - g \leqslant \mu(\mathcal{E})$.

It easily follows from a similar argument as before that the family $\mathcal{B}$ of coherent sheaves satisfying the hypothesis of the proposition for some integer n is bounded, and the same is true for the family of quotients $\mathcal{E}$ we are considering, since they are semistable and their slope is bounded above by μ. We can take $N(r,d)$ large enough so that for every $n \geqslant N(r,d)$ we have $\chi(\mathcal{E}(n)) = h^0(\mathcal{E}(n))$ and $\chi(\mathcal{M}(n)) = h^0(\mathcal{M}(n))$. From the inequality in the hypothesis, we arrive at

$$1 - g + \mu + n\deg(X) \leqslant 1 - g + \mu(\mathcal{E}) + n\deg(X)$$

concluding that $\mu = \mu(\mathcal{E})$ and thus $\mathcal{M}$ is semistable. □

Lemma 9.2.4. *There is an integer* $N(r,d)$ *satisfying the conditions of proposition 9.2.2 such that for every semistable locally free sheaf* $\mathcal{E} \in \mathcal{S}(r,d)$, *every coherent subsheaf* $\mathcal{F} \subseteq \mathcal{E}$ *and every* $n \geqslant N(r,d)$ *the following are equivalent*

(a) $\mu(\mathcal{F}) = \mu(\mathcal{E})$

(b)

$$\frac{h^0(\mathcal{F}(n))}{\mathrm{rk}(\mathcal{F})} = \frac{h^0(\mathcal{E}(n))}{\mathrm{rk}(\mathcal{E})}$$

Proof. Consider the family $\mathcal{A} := \{\mathcal{F} \subseteq \mathcal{E} \text{ coherent} : \mathcal{E} \in \mathcal{S}(r,d) \text{ and } \mu(\mathcal{F}) = \mu(\mathcal{E}) = d/r\}$. Every coherent subsheaf $\mathcal{F} \in \mathcal{A}$ is necessarily semistable because each $\mathcal{E}$ is semistable. Besides, there is only a finite number of possible values of rank and degree of coherent subsheaves of any $\mathcal{E} \in \mathcal{S}(r,d)$, so the family $\mathcal{A}$ is bounded by proposition 8.4.1.

We can then choose $N(r,d)$ sufficiently large so that $h^1(\mathcal{F}(n)) = 0$ for every $\mathcal{F} \in \mathcal{A}$ and every $n \geqslant N(r,d)$ (in particular, $h^1(\mathcal{E}(n)) = 0$). In this case, we have that

$$\begin{aligned}\frac{h^0(\mathcal{F}(n))}{\mathrm{rk}(\mathcal{F})} = \frac{\chi(\mathcal{F}(n))}{\mathrm{rk}(\mathcal{F})} = 1 - g + \mu(\mathcal{F}) + n\deg(X) = \\ = 1 - g + \mu(\mathcal{E}) + n\deg(X) = \frac{\chi(\mathcal{E}(n))}{\mathrm{rk}(\mathcal{E})} = \frac{h^0(\mathcal{E}(n))}{\mathrm{rk}(\mathcal{E})}\end{aligned}$$

And thus (b) follows.

Suppose now that (b) holds for a coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$. From the proof of proposition 9.2.2, we can see that $\mu_{min}(\mathcal{F}) > a$, and thus the family of such coherent subsheaves is bounded. The same argument as above shows that $\mu(\mathcal{F}) = \mu(\mathcal{E})$ □

9.2.3 GIT set up for the construction of the moduli space

Let $r > 0$ and $d > r(2g-1)$. Let $N(r,d)$ be as in proposition 9.2.2 and $N \geqslant N(r,d)$.

For every $\mathcal{E} \in \mathcal{S}(r,d)$, we have that $\mathcal{E}(N)$ is globally generated and $H^1(X, \mathcal{E}(N)) = 0$. There is a surjective homomorphism

$$H^0(X, \mathcal{E}(N)) \otimes_k \mathcal{O}_X \to \mathcal{E}(N) \to 0$$

and, by the Riemann-Roch formula, we have that

$$h^0(\mathcal{E}(N)) = r(1-g) + d + r\deg(X) \cdot N$$

Let H be a vector space of dimension $r(1-g) + d + r\deg(X) \cdot N$. For every $\mathcal{E} \in \mathcal{S}(r,d)$, there is a linear isomorphism $H \simeq H^0(X, \mathcal{E}(N))$, and thus every $\mathcal{E} \in \mathcal{S}(r,d)$ can be presented as a quotient

$$H \otimes_k \mathcal{O}_X(-N) \to \mathcal{E} \to 0$$

Denote $\mathcal{H} := H \otimes_k \mathcal{O}_X(-N)$. Every $\mathcal{E} \in \mathcal{S}(r,d)$ defines a closed point of the Quot scheme $\mathfrak{Q} := \mathrm{Quot}_X^{P(m)}(\mathcal{H})$, with $P(m) = r(1-g) + d + r\deg(X) \cdot m$.

Consider the subset of $\mathfrak{Q}^\bullet(k)$ given by quotients $\rho : \mathcal{H} \to \mathcal{E} \to 0$ such that

(a) $\mathcal{E}$ is semistable

(b) The natural homomorphism $H^0(\rho(N)) : H \to H^0(X, \mathcal{E}(N))$ is an isomorphism

These conditions are open by proposition 9.1.1 and the semicontinuity theorem. Denote by Ω the open subset of $\mathfrak{Q}$ that they define. The subset $\Omega^s \subseteq \Omega$ given by stable sheaves is also open by proposition 9.1.1.

A natural candidate for a moduli space of semistable (resp. stable) sheaves can be found at Ω (resp. Ω^s), but there is a crucial ambiguity: there are many possible linear isomorphisms $H \to H^0(X, \mathcal{E}(N))$ for each $\mathcal{E} \in \mathcal{S}(r, d)$.

We will identify these possible isomorphisms by considering a right action of $GL(P(N))$ on $\mathfrak{Q}$ defined by

$$\mathfrak{Q}^\bullet(S) \times GL(P(N))^\bullet(S) \to \mathfrak{Q}^\bullet(S)$$
$$(\rho, g) \mapsto \rho \circ (g \otimes \mathrm{Id}_{\mathcal{O}_{X\times S}(-N)})$$

for points with values in some scheme S.

Clearly, Ω and Ω^s are $GL(P(N))$-invariant open subsets of $\mathfrak{Q}$ with respect to this action.

Let $\alpha : \mathcal{H} \to \mathcal{E}$ and $\beta : \mathcal{H} \to \mathcal{F}$ be closed points of $\mathfrak{Q}$. Recall that $\alpha = \beta$ if and only if there is a sheaf homomorphism $\phi : \mathcal{E} \to \mathcal{F}$ such that $\beta = \phi \circ \alpha$

Lemma 9.2.5. *Let $\rho : \mathcal{H} \to \mathcal{E}$ be a closed point in $\mathfrak{Q}$ such that $\mathcal{E}(N)$ is generated by its global sections and the natural homomorphism $H^0(\rho(N)) : H \to H^0(C, \mathcal{E}(N))$ is an isomorphism. There is an injective homomorphism $\mathrm{Aut}_{\mathcal{O}_X}(\mathcal{E}) \to GL(P(N))^\bullet(k)$ whose image is the stabilizer subgroup $GL(P(N))_\rho$*

Proof. Consider

$$\Lambda : \mathrm{Aut}_{\mathcal{O}_X}(\mathcal{E}) \to GL(P(N))^\bullet(k)$$
$$\varphi \mapsto \Lambda(\varphi) := H^0(\rho(N))^{-1} \circ H^0(\varphi(N)) \circ H^0(\rho(N))$$

Λ is clearly a group homomorphism, and it takes values in $GL(P(N))_\rho$, because a direct computation shows that $\varphi^{-1} \circ (\Lambda(\varphi) \cdot \rho) = \rho$ for every $\varphi \in \mathrm{Aut}_{\mathcal{O}_X}(\mathcal{E})$.

Besides, Λ is injective, because if $\Lambda(\varphi) = \mathrm{Id}$, then necessarily $H^0(\varphi(N)) = \mathrm{Id}$ and thus $\varphi(N) \circ \rho(N) = \rho(N)$, but $\rho(N)$ is a sheaf epimorphism because $\mathcal{E}(N)$ is generated by its global sections and thus $\varphi(N) = \mathrm{Id}$, so $\varphi = \mathrm{Id}$ and we conclude □

From lemma 9.2.5, we deduce that

$$\Omega^\bullet(k)/GL(P(N))^\bullet(k) = \{\text{Semistable sheaves of rank } r \text{ and degree } d \text{ on } X\} \Big/ \simeq$$

We will now reduce the construction of the moduli space of semistable sheaves to a problem in geometric invariant theory.

First, we will recall some basic definitions

Definition 9.2.2. *Let* S *be a scheme and* $\mathcal{E}$ *a locally free sheaf of rank* r *on* S*, and let* $\mathbb{H}\mathrm{om}_X(\mathcal{O}_X^{\oplus r}, \mathcal{E}) \to X$ *be the vector bundle associated to the locally free sheaf* $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus r}, \mathcal{E})$*. The frame bundle of* $\mathcal{E}$ *is the open sub-bundle* $R(\mathcal{E}) = \mathbb{I}\mathrm{som}_X(\mathcal{O}_X^{\oplus r}, \mathcal{E})$ *of* $\mathbb{H}\mathrm{om}_X(\mathcal{O}_X^{\oplus r}, \mathcal{E}) \to X$ *given by isomorphisms* $\mathcal{O}_X^{\oplus r} \xrightarrow{\simeq} \mathcal{E}$

We state without proof the following well-known result

Lemma 9.2.6. *Let* S *be a scheme and* $\mathcal{E}$ *a locally free sheaf of rank* r *on* S*.* $\mathrm{GL}(r)$ *acts on* $R(\mathcal{E})$ *and, with respect to this action,* $R(\mathcal{E}) \to S$ *is a principal* $\mathrm{GL}(r)$*-bundle*

Proposition 9.2.3 ([HL10], Lemma 4.3.1)**.** *With the previous notations, let* M *be a scheme. Then,* $\Omega \to M$ *(resp.* $\Omega^s \to M$*) is a categorical quotient for the action of* $\mathrm{GL}(P(N))$ *on* Ω *(resp.* Ω^s*) if and only if* M *corepresents* $\mathcal{M}_X^{ss}(r,d)$ *(resp.* $\mathcal{M}_X^{s}(r,d)$*)*

Proof. Let S be a scheme and denote by $\pi_X : X \times S \to X$ and $\pi_S : X \times S \to S$ the natural projections. Let $\mathcal{M} \in \mathcal{M}_X^{ss}(r,d)(S)$. By theorem 8.4.1 and the theorem of Grauert-Grothendieck (see for example [Mum70, Chapter 2, § 5, Corollary 2]), we have that $V_{\mathcal{M}} := (\pi_S)_*(\mathcal{M} \otimes \pi_X^* \mathcal{O}_X(N))$ is a locally free sheaf on S of rank $P(N)$. Using the adjunction of π_S^* and $(\pi_S)_*$, we obtain a canonical surjective homomorphism $\varphi_{\mathcal{M}} : \pi_S^* V_{\mathcal{M}} \otimes \pi_X^* \mathcal{O}_X(-N) \to \mathcal{M}$.

Let $\pi : R(V_{\mathcal{M}}) \to S$ be the frame bundle associated to $V_{\mathcal{M}}$ and denote by $q_{\mathcal{M}} : \mathcal{O}_{R(V_{\mathcal{M}})}^{\oplus P(N)} \simeq H \otimes_k \mathcal{O}_{R(V_{\mathcal{M}})} \to \pi^* V_{\mathcal{M}}$ the canonical isomorphism. Composing $\varphi_{\mathcal{M}}$ and $q_{\mathcal{M}}$, we obtain a surjective homomorphism $H \otimes \mathcal{O}_{R(V_{\mathcal{M}})}(-N) \to (\mathrm{Id}_X \times \pi)^* \mathcal{M}$ that defines a point of $\mathfrak{Q}$ with values in $R(V_{\mathcal{M}})$. Equivalently, we have a morphism $\Phi_{\mathcal{M}} : R(V_{\mathcal{M}}) \to \mathfrak{Q}$.

Since $\mathcal{M}$ is a family of semistable sheaves, clearly $\Phi_{\mathcal{M}}(R(V_{\mathcal{M}})) \subseteq \Omega$, and $\Phi_{\mathcal{M}} : R(V_{\mathcal{M}}) \to \Omega$ is $\mathrm{GL}(P(N))$-equivariant. From this, we have that $\Phi_{\mathcal{M}}$ induces a morphism of functors $\widetilde{\Phi}_{\mathcal{M}} : \underline{R(V_{\mathcal{M}})/\mathrm{GL}(P(N))} \to \underline{\Omega/\mathrm{GL}(P(N))}$. Besides, $\pi : R(V_{\mathcal{M}}) \to S$ is a principal $\mathrm{GL}(r)$-bundle, so by lemma 2.1.1 we have that S corepresents $\underline{R(V_{\mathcal{M}})/\mathrm{GL}(P(N))}$ and thus induces a morphism $S^\bullet \to \underline{\Omega/\mathrm{GL}(P(N))}$ or, equivalently, an element of $\underline{\Omega/\mathrm{GL}(P(N))}(S)$ by Yoneda's lemma.

We have thus defined, for every $\mathcal{M} \in \mathcal{M}_X^{ss}(r,d)(S)$, an element of $\underline{\Omega/\mathrm{GL}(P(N))}(S)$. It's easy to prove that this defines a morphism of functors $\mathcal{M}_X^{ss}(r,d) \to \underline{\Omega/\mathrm{GL}(P(N))}$.

On the other hand, let $\mathcal{U}$ be the restriction of the universal quotient sheaf on $X \times \mathfrak{Q}$ to $X \times \Omega$. $\mathcal{U}$ defines an inverse morphism of functors $\underline{\Omega/\mathrm{GL}(P(N))} \to \mathcal{M}_X^{ss}(r,d)$, and thus corepresenting $\mathcal{M}_X^{ss}(r,d)$ is the same as corepresenting $\underline{\Omega/\mathrm{GL}(P(N))}$ □

9.2.4 Semistable points of the $\mathrm{SL}(P(N))$-action

In virtue of lemma 9.2.5, the action of $\mathrm{GL}(P(N))$ on $\mathfrak{Q}$ induces an action of $\mathrm{SL}(P(N))$. We will work with respect to the action of $\mathrm{SL}(P(N))$.

In this subsection, we will find a linearization of the action of $\mathrm{SL}(P(N))$ on $\mathfrak{Q}$ such that its set of semistable (resp. stable) points is Ω (resp. Ω^s). In virtue of proposition

9.2.3 and theorem 4.2.1, this will end the construction of the moduli space.

The idea will be to embed $\mathfrak{Q}$ inside a suitable grassmannian scheme and use the stability criterion of proposition 5.3.1 to find the semistable (resp. stable) points with respect to the linearization associated to the embedding.

Let S be a scheme and denote by $\pi_X : X \times S \to X$ and $\pi_S : X \times S \to S$ the natural projections. By Mumford-Castelnuovo regularity (see [Nit05, § 2]), there is a sufficiently large integer $q \geqslant N$ such that, for every S-valued point $\pi_X^* \mathcal{H} \to \mathcal{E}$ of $\mathfrak{Q}$, then

- $(\pi_S)_* \mathcal{E}(q)$ is a locally free sheaf on S of rank $P(q) = r(1-g) + d + r\deg(X) \cdot q$
- $(\pi_S)_* \pi_X^* \mathcal{H}(q) \simeq \mathcal{O}_S \otimes_k H^0(X, \mathcal{H}(q)) \simeq \mathcal{O}_S \otimes_k (H \otimes H^0(X, \mathcal{O}_X(q-N)))$

In particular, $(\pi_S)_* \pi_X^* \mathcal{H}(q) \to (\pi_S)_* \mathcal{E}(q)$ is a S-valued point of the grassmannian scheme $\mathfrak{G} := \mathrm{Grass}_k^{P(q)}(H \otimes H^0(X, \mathcal{O}_X(q-N)))$.

This correspondence defines a morphism of functors $\mathfrak{Q}^\bullet \to \mathfrak{G}^\bullet$ that is a closed $SL(P(N))$-equivariant embedding (see [Nit05, § 5] for the details), where $SL(P(N))$ acts on $\mathfrak{G}$ on the H-component as in section 5.3 of chapter 5.

Furthermore, we can embed $\mathfrak{G}$ in $\mathbb{P}(\Lambda^{max}(H \otimes H^0(X, \mathcal{O}_X(q-N))))$ via the Plücker embedding. Denote by $\mathcal{L}_q$ the induced $SL(P(N))$-linearized sheaf on $\mathfrak{Q}$

Definition 9.2.3. *Let $H' \subseteq H$ be a vector subspace and let $\rho : \mathcal{H} \to \mathcal{M}$ be a closed point in $\mathfrak{Q}$. The coherent subsheaf of $\mathcal{M}$ generated by H' is*

$$\mathcal{N} := \mathrm{Coker}(\mathrm{Ker}(H' \otimes_k \mathcal{O}_X(-N) \to \mathcal{M}) \to H' \otimes_k \mathcal{O}_X(-N))$$

Lemma 9.2.7. *Let $\rho : \mathcal{H} \to \mathcal{M}$ be a closed point in $\mathfrak{Q}$. The following properties are equivalent*

(a) $\rho \in \mathfrak{Q}^{ss}(\mathcal{L}_r)$ *(resp.* $\rho \in \mathfrak{Q}^{s}(\mathcal{L}_r)$*)*

(b) For each vector subspace $0 \neq H' \subsetneq H$, denote by $\mathcal{N}$ be the coherent subsheaf of $\mathcal{M}$ generated by H'. Then

$$\frac{h^0(\mathcal{N}(q))}{\dim(H')} \geqslant \frac{h^0(\mathcal{M}(q))}{\dim(H)} \left(\textit{resp. } \frac{h^0(\mathcal{N}(q))}{\dim(H')} > \frac{h^0(\mathcal{M}(q))}{\dim(H)} \right)$$

(c) For each vector subspace $0 \neq H' \subsetneq H$, denote by $\mathcal{N}$ be the coherent subsheaf of $\mathcal{M}$ generated by H'. Then

$$\frac{\chi(\mathcal{N}(m))}{\dim(H')} \geqslant \frac{\chi(\mathcal{M}(m))}{\dim(H')} \textit{ for every } m \gg 0 \left(\textit{resp. } \frac{\chi(\mathcal{N}(m))}{\dim(H')} > \frac{\chi(\mathcal{M}(m))}{\dim(H')} \textit{ for every } m \gg 0 \right)$$

Proof. This in an immediate consequence of proposition 5.3.1, because $\mathfrak{G}$ is a grassmannian scheme of quotients of a vector space of the form $H \otimes_k V$. Proposition 5.3.1 was stated for grassmannian schemes of vector subspaces, but the same holds for grassmannian schemes of quotients by just reversing the inequalities □

Proposition 9.2.4. *With the previous notations, we have that*

- $\mathfrak{Q}^{ss}(\mathcal{L}_r) = \Omega$
- $\mathfrak{Q}^{s}(\mathcal{L}_r) = \Omega^s$

Proof. Let $\rho : \mathcal{H} \to \mathcal{E}$ be a closed point in Ω. For every coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$ generated by a vector subspace $H' \subseteq H$, we have the commuting square

$$\begin{array}{ccc} H & \xrightarrow{\simeq} & H^0(C, \mathcal{E}(N)) \\ \uparrow & & \uparrow \\ H' & \xrightarrow[\simeq]{} & H^0(C, \mathcal{F}(N)) \end{array}$$

where the vertical arrows are inclusions. From the diagram and proposition 9.2.2, we have that

$$\frac{\dim(H')}{\mathrm{rk}(\mathcal{F})} = \frac{h^0(\mathcal{F}(N))}{\mathrm{rk}(\mathcal{F})} \leqslant \frac{h^0(\mathcal{E}(N))}{\mathrm{rk}(\mathcal{E})} = \frac{\dim(H)}{\mathrm{rk}(\mathcal{E})}$$

And thus

$$\frac{\mathrm{rk}(\mathcal{E})}{\dim(H)} \leqslant \frac{\mathrm{rk}(\mathcal{F})}{\dim(H')}$$

But these are the leading coefficients of the rational polynomials $\frac{\chi(\mathcal{E}(m))}{\dim(H)}$ and $\frac{\chi(\mathcal{F}(m))}{\dim(H')}$, so we conclude that $\rho \in \mathfrak{Q}^{ss}(\mathcal{L}_q)$ by lemma 9.2.7.

Furthermore, from lemma 9.2.4 we deduce that

$$\mu(\mathcal{F}) = \mu(\mathcal{E}) \Leftrightarrow \frac{\mathrm{rk}(\mathcal{E})}{\dim(H)} = \frac{\mathrm{rk}(\mathcal{F})}{\dim(H')}$$

So $\Omega^s \subseteq \mathfrak{Q}^s(\mathcal{L}_q)$.

Suppose now that $\rho : \mathcal{H} \to \mathcal{M}$ is a closed point of $\mathfrak{Q}$, semistable with respect to the action of $SL(P(N))$.

Let's see that $H^1(X, \mathcal{M}(N)) = 0$. By Serre duality, we have that

$$H^1(X, \mathcal{M}(N))^* \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}(N), \omega_X)$$

Suppose that $H^1(X, \mathcal{M}(N)) \neq 0$. Then, there is a non-zero homomorphism $\mathcal{M}(N) \to \omega_X$. Denote by $\mathcal{L}$ the image of this homomorphism; $\mathcal{L}$ is an invertible subsheaf of ω_X.

From the fact that $\mathcal{M}$ is $SL(P(N))$-semistable, we have that

$$\frac{\chi(\mathcal{M}(m))}{\dim(H)} \geqslant \frac{\chi(\mathcal{L}(m))}{h^0(\mathcal{L})}$$

and thus, using the Riemann-Roch formula and comparing the dominant terms, we obtain the inequality

$$\frac{r}{\dim(H)} \geqslant \frac{1}{h^0(\mathcal{L})}$$

Finally, we have that

$$1 - g + \mu + N \cdot \deg(X) = \frac{\chi(\mathcal{M}(N))}{r} = \frac{\dim(H)}{r} \leqslant h^0(\mathcal{L}) \leqslant h^0(\omega_X) = g$$

On the other hand, we have that by hypothesis $d > r(2g-1)$ and hence $\mu > 2g-1$, so

$$g \geqslant 1 - g + \mu + N\deg(X) > g + N\deg(X)$$

and thus $N\deg(X) < 0$, which is impossible because $N, \deg(X) \geqslant 0$, so it must be $H^1(X, \mathcal{M}(N)) = 0$.

In particular, this means that $h^0(\mathcal{M}(N)) = \chi(\mathcal{M}(N)) = \dim(H)$. Besides, if the natural homomorphism $H^0(\rho(N)) : H \to H^0(C, \mathcal{M}(N))$ had a kernel, then by the $SL(P(N))$-semistability of $\mathcal{M}$ we would have that

$$0 \leqslant \frac{rk(\mathcal{M})}{\dim(H)} \leqslant 0$$

which is impossible, so the natural homomorphism $H^0(\rho(N)) : H \to H^0(X, \mathcal{M}(N))$ is an isomorphism.

Let's now see that $\mathcal{M}$ is a semistable sheaf. Let $\mathcal{N} \subseteq \mathcal{M}$ be a coherent subsheaf and denote by H' the vector subspace of H that it generates. $\mathcal{M}$ is $SL(P(N))$-semistable, so we have the inequality

$$\frac{rk(\mathcal{N})}{\dim(H')} \geqslant \frac{rk(\mathcal{M})}{\dim(H)}$$

but $H \simeq H^0(X, \mathcal{M}(N))$ and thus $H' \simeq H^0(X, \mathcal{N}(N))$, so

$$\frac{h^0(\mathcal{M}(N))}{rk(\mathcal{M})} \geqslant \frac{h^0(\mathcal{N}(N))}{rk(\mathcal{N})}$$

Since $N \geqslant N(r, d)$, $\mathcal{M}$ is semistable by proposition 9.2.2. This proves that ρ is a closed point of Ω, so $\Omega = \mathfrak{Q}^{ss}(\mathcal{L}_q)$.

The fact that $\Omega^s = \mathfrak{Q}^s(\mathcal{L}_q)$ can be proved in a completely analogous way. □

We finally obtain the main result of this chapter

Theorem 9.2.1. *Let X be a smooth projective algebraic curve, $r > 0$ and $d \in \mathbb{Z}$. There exists a moduli space of semistable sheaves of rank r and degree d, denoted by $M_X^{ss}(r, d)$. Furthermore, $M_X^{ss}(r, d)$ is a projective variety, and there is an open subscheme $M_X^s(r, d)$ of $M_X^{ss}(r, d)$ that is a coarse moduli space for $\mathcal{M}_X^s(r, d)$.*

Besides, the closed points of $M_X^{ss}(r, d)$ are in a one to one correspondence with polystable sheaves (recall definition 8.3.2) of rank r and degree d on X

Proof. Proposition 9.2.4 and theorem 4.2.1 shows that there is a good quotient $M_X^{ss}(r,d) := \Omega//SL(P(N))$, and proposition 9.2.3 proves that $M_X^{ss}(r,d)$ is the moduli space of semistable sheaves of rank r and degree d on X.

By the same arguments, $M_X^s(r,d) := \Omega^s//SL(P(N))$ is an open subscheme of $M_X^{ss}(r,d)$ that corepresents the functor $\mathcal{M}_X^s(r,d)$.

The facts that the closed points of $M_X^{ss}(r,d)$ are in a one to one correspondence with polystable sheaves of rank r and degree d and that $M_X^s(r,d)$ is a coarse moduli space for $\mathcal{M}_X^s(r,d)$ are a direct consequence of the following lemma □

Lemma 9.2.8. *Let $\mathcal{E}$ and $\mathcal{F}$ be semistable sheaves on X of rank r and degree d. Then, $\mathcal{E}$ and $\mathcal{F}$ define the same closed point of $M_X^{ss}(r,d)$ if and only if $\mathcal{E}$ and $\mathcal{F}$ are S-equivalent (recall definition 8.3.2)*

Proof. Let $\mathcal{E}$ be a semistable sheaf on X of rank r and degree d. We are going to prove that, for any exact sequence of semistable sheaves

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

then $\mathcal{E}$ and $\mathcal{E}' \oplus \mathcal{E}''$ define the same closed point in $M_X^{ss}(r,d)$. Applying this result repeatedly to any Jordan-Hölder filtration of a semistable sheaf, we would prove that S-equivalent sheaves define the same closed point on $M_X^{ss}(r,d)$.

Denote by $\pi_X : \mathbb{A}^1 \times X \to X$ the projection on X and let $i = (0, \mathrm{Id}_X) : X \to \mathbb{A}^1 \times X$ be the inclusion of X in $\mathbb{A}^1 \times X$ via the closed point $0 \in \mathbb{A}^1$. We have the following exact sequence of coherent sheaves on $X \times \mathbb{A}^1$

$$0 \to \mathcal{F} \to \pi_X^* \mathcal{E} \to i_* \mathcal{E}'' \to 0$$

and $\mathcal{F}$ is a flat family of semistable sheaves of rank r and degree d on X parameterized by $\mathbb{A}^1$ such that, by construction

$$\mathcal{F}_0 \simeq \mathcal{E}' \oplus \mathcal{E}'' \qquad \mathcal{F}_t \simeq \mathcal{E} \text{ for any } t \in (\mathbb{A}^1)^\bullet(k) \text{ distinct from } 0$$

so $\mathcal{F}$ defines a morphism $\mathbb{A}^1 \to M_X^{ss}(r,d)$ that is constant on $\mathbb{A}^1 - \{0\}$, and hence constant because $\mathcal{M}_X^{ss}(r,d)$ is a projective variety. This proves that $\mathcal{E}$ and $\mathcal{E}' \oplus \mathcal{E}''$ define the same closed point of $M_X^{ss}(r,d)$.

Let's now prove that, if $\mathcal{E}$ is a polystable sheaf on X of rank r and degree d, then $SL(P(N)) \cdot \mathcal{E}$ is closed. If we proved this, we would conclude, because the closed points of the GIT quotient are in a one-to-one correspondence with closed orbits.

Let C be a curve, $p_0 \in C^\bullet(k)$ and let $\mathcal{M}$ be a flat family of semistable sheaves on X parameterized by C such that $\mathcal{M}_p \simeq \mathcal{E}$ for every $p \in C^\bullet(k)$ with $p \neq p_0$. If we prove that $\mathcal{M}_{p_0} \simeq \mathcal{E}$, we conclude by the valuative criterion of properness.

Let $\mathcal{E} \simeq \oplus_i \mathcal{E}_i$ be the decomposition of $\mathcal{E}$ as a direct sum of stable sheaves. Denote by $\pi_X : X \times C \to X$ the projection on X and consider the flat family $\pi_X^* \mathcal{E}^\vee \otimes \mathcal{M}$ of semistable sheaves on X parameterized by C. Clearly, for every $p \in C^{\bullet}(k)$, we have that

$$H^0(X, (\pi_X^*\mathcal{E}^\vee \otimes \mathcal{M})_p) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M}_p)$$

so, by the semicontinuity theorem, the fact that $\mathcal{M}_p \simeq \mathcal{E}$ for every $p \neq p_0$ and proposition 8.2.1, we have that $\mathcal{M}_{p_0}$ contains $\mathcal{E}_i$ as a direct summand with the same multiplicity as in $\mathcal{E}$ for every i, and thus $\mathcal{M}_{p_0} \simeq \mathcal{E}$, so we conclude □

We will finish this chapter proving that the moduli space of stable sheaves is smooth. First, we recall a smoothness criterion for Quot schemes

Theorem 9.2.2 ([HL10], Propositions 2.2.7-2.2.8)**.** *Let X be a smooth projective scheme and $\mathcal{H}$ a coherent sheaf on X. Let $P(m) \in \mathbb{Q}[m]$ and $Q := \mathrm{Quot}_X^{P(m)}(\mathcal{H})$ the associated Quot scheme. Let $q : \mathcal{H} \to \mathcal{E}$ be a closed point in Q and $\mathcal{K} := \mathrm{Ker}\, q$. We have that*

- $T_q Q \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{E})$
- $$\hom(\mathcal{K}, \mathcal{E}) \geqslant \dim_q Q \geqslant \hom(\mathcal{K}, \mathcal{E}) - \mathrm{ext}^1(\mathcal{K}, \mathcal{E})$$

in particular, if $\mathrm{Ext}^1(\mathcal{K}, \mathcal{E}) = 0$, then Q is smooth in an open neighbourhood of q

Proposition 9.2.5. *The moduli space of stable sheaves $M_X^s(r, d)$ is a smooth quasiprojective algebraic variety of dimension $r^2(g-1)+1$*

Proof. Let $\rho : \mathcal{H} \to \mathcal{E}$ be a closed point of Ω^s and $\mathcal{K} := \mathrm{Ker}\, q$. We have the exact sequence

$$0 \to \mathcal{K} \to \mathcal{H} \to \mathcal{E} \to 0$$

taking $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{E})$, we obtain the long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{E}) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \to \mathrm{Ext}^1(\mathcal{H}, \mathcal{E}) \to \mathrm{Ext}^1(\mathcal{K}, \mathcal{E}) \to 0$$

and we have that

$$\begin{aligned}
\mathrm{Ext}^1(\mathcal{H}, \mathcal{E}) &\simeq H^1(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{E})) \simeq \\
&\simeq H^1(X, \mathcal{H}^\vee \otimes \mathcal{E}) \simeq \\
&\simeq H^1(X, (\mathcal{O}_X^{\oplus P(N)} \otimes \mathcal{O}_X(-N))^\vee \otimes \mathcal{E}) = \\
&= H^1(X, (\mathcal{O}_X^{\oplus P(N)})^\vee \otimes \mathcal{E}(N)) = \\
&= H^1(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus P(N)}, \mathcal{E}(N))) = \\
&= H^1(X, \mathcal{E}(N))^{\oplus P(N)} = \\
&= 0
\end{aligned}$$

this proves that $\mathrm{Ext}^1(\mathcal{K}, \mathcal{E}) = 0$, and thus $\mathfrak{Q}$ is smooth in an open neighbourhood of q by theorem 9.2.2.

On the other hand, $T_q\mathfrak{Q} \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K},\mathcal{E})$. By the same arguments as above, $H^0(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{H},\mathcal{E})) \simeq H^0(X,\mathcal{E}(N))^{P(n)}$ and thus we have the exact sequence

$$0 \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}) \to H^0(X,\mathcal{E}(N))^{\oplus P(N)} \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K},\mathcal{E}) \to \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to 0$$

So

$$\dim_q \mathfrak{Q} = \hom(\mathcal{K},\mathcal{E}) = P(N)\cdot h^0(\mathcal{E}(N)) + \mathrm{ext}^1(\mathcal{E},\mathcal{E}) - \mathrm{end}(\mathcal{E})$$

but

- From the fact that $d > r(2g-1)$, we have $h^0(\mathcal{E}(N)) = P(N)$
- We have the isomorphisms $\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \simeq H^1(C, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \simeq H^1(C,\mathcal{E}^\vee \otimes \mathcal{E})$ so, from the Riemann-Roch formula

$$\begin{aligned}\mathrm{ext}^1(\mathcal{E},\mathcal{E}) &= h^1(\mathcal{E}^\vee\otimes\mathcal{E}) = \\ &= h^0(\mathcal{E}^\vee\otimes\mathcal{E}) - \chi(\mathcal{E}^\vee\otimes\mathcal{E}) = \\ &= \mathrm{end}(\mathcal{E}) - r^2(1-g)\end{aligned}$$

and thus

$$\dim_q \mathfrak{Q} = P(N)^2 + r^2(g-1)$$

This proves that Ω^s is a smooth quasiprojective variety of dimension $P(N)^2 + r^2(g-1)$.

By corollary 6.3.1, the geometric quotient $\pi : \Omega^s \to M_X^s(r,d)$ is a principal $SL(P(N))$-bundle, and thus $M_X^s(r,d)$ is a smooth quasiprojective algebraic variety of dimension

$$\dim M_X^s(r,d) = \dim \Omega^s - \dim SL(P(N)) = \dim \Omega^S - (P(N)^2 - 1) = r^2(g-1) + 1$$

□

Conclusions

In this thesis we have given an introduction to the study of moduli problems, developing the necessary tools from geometric invariant theory in part I, and studying a concrete example in part II: the construction of the moduli space of semistable sheaves on a smooth projective algebraic curve.

Due to a lack of space, we have not given a more complete study of the geometric properties of the moduli spaces of sheaves, apart from the fact that the moduli space of stable sheaves is a smooth quasiprojective algebraic variety. For example, we could have proven that the moduli space of semistable sheaves is irreducible (see [LP97, Chapter 8, § 8.5]) or that the moduli space of stable sheaves of rank r and degree d is a fine moduli space if and only if $\gcd(r, d) = 1$ (see [New78, Chapter 5, § 5] and [Ram73]).

Also, we have not studied the moduli problem of vector bundles from the point of view of complex differential geometry and its relation with the theory of unitary representations (see [NS65]) or Yang-Mills theory (see [AB83]). In particular, the approach of Donaldson (see [Don83] and [Don85]) is the starting point of the so-called Hitchin-Kobayashi correspondence, that relates stable holomorphic bundles over certain complex manifolds with the existence of Einstein-Hermite metrics. For an introduction to these topics, see for example [WGP08, Appendix]

Bibliography

[AB83] Michael Francis Atiyah and Raoul Bott. The yang-mills equations over riemann surfaces. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 308(1505):523–615, 1983.

[An12] Jinpeng An. Rigid geometric structures, isometric actions, and algebraic quotients. *Geometriae Dedicata*, 157(1):153–185, 2012.

[Ati57] Michael Francis Atiyah. Vector bundles over an elliptic curve. *Proceedings of the London Mathematical Society*, 3(1):414–452, 1957.

[Bor91] Armand Borel. *Linear Algebraic Groups, second edition. Graduate texts in mathematics*, volume 126. Springer Science & Business Media, 1991.

[CS86] Gary Cornell and Joseph H Silverman. *Arithmetic Geometry*. Springer-Verlag New York, 1986.

[DH98] Igor V Dolgachev and Yi Hu. Variation of geometric invariant theory quotients. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 87(1):5–51, 1998.

[DK07] Brent Doran and Frances Kirwan. Towards non-reductive geometric invariant theory. *Pure and Applied Mathematics Quarterly*, 3(1):61–105, 2007.

[Dol03] Igor Dolgachev. *Lectures on Invariant Theory. London mathematical society lecture note series.* Number 296. Cambridge University Press, 2003.

[Don83] Simon K Donaldson. A new proof of a theorem of narasimhan and seshadri. *Journal of Differential Geometry*, 18(2):269–277, 1983.

[Don85] Simon K Donaldson. Anti self-dual yang-mills connections over complex algebraic surfaces and stable vector bundles. *Proceedings of the London Mathematical Society*, 3(1):1–26, 1985.

[Dré04] Jean-Marc Drézet. Luna's slice theorem and applications, 2004.

[dS01] C Sancho de Salas. *Grupos Algebraicos y Teoría de Invariantes. Aportaciones matemáticas*, volume 16. Sociedad matemática mexicana, 2001.

[Gro57] Alexander Grothendieck. Sur la classification des fibrés holomorphes sur la sphere de riemann. *American Journal of Mathematics*, 79(1):121–138, 1957.

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry Part I: Schemes. With examples and exercises. Advanced lectures in mathematics*. Springer. Vieweg+Teubner, 2010.

[Hab75] William J Haboush. Reductive groups are geometrically reductive. *Annals of Mathematics*, pages 67–83, 1975.

[Har77] Robin Hartshorne. *Algebraic Geometry. Graduate texts in mathematics*, volume 52. Springer-Verlag New York, 1977.

[HL10] Daniel Huybrechts and Manfred Lehn. *The Geometry of Moduli Spaces of Sheaves, second edition*. Cambridge University Press, 2010.

[Hos15] Victoria Hoskins. Moduli problems and geometric invariant theory. *Lecture notes, Freie Universität, Berlin*, 2015.

[IM65] Nagayoshi Iwahori and Hideya Matsumoto. On some bruhat decomposition and the structure of the hecke rings of p-adic chevalley groups. *Publications Mathématiques de l'IHÉS*, 25:5–48, 1965.

[Kem80] George R Kempf. Some elementary proofs of basic theorems in the cohomology of quasi-coherent sheaves. *The Rocky Mountain Journal of Mathematics*, 10(3):637–645, 1980.

[Liu02] Qing Liu. *Algebraic Geometry and Arithmetic Curves. Oxford graduate texts in mathematics*, volume 6. Oxford Science Publications, 2002.

[LP97] Joseph Le Potier. *Lectures on Vector Bundles. Cambridge studies in advanced mathematics*. Number 54. Cambridge University Press, 1997.

[Lun73] Domingo Luna. Slices étales. *Bull. Soc. Math. France*, 33:81–105, 1973.

[Mat60] Yozô Matsushima. Espaces homogenes de stein des groupes de lie complexes. *Nagoya Mathematical Journal*, 16:205–218, 1960.

[MFK94] David Mumford, John Fogarty, and Frances Kirwan. *Geometric Invariant Theory, third revised edition. A series of modern surveys in mathematics*, volume 34. Springer-Verlag Berlin Heidelberg, 1994.

[Mil17] James S Milne. *Algebraic groups: The Theory of Group Schemes of Finite Type over a Field. Cambdrige studies in advanced mathematics*, volume 170. Cambridge University Press, 2017.

[Muk03] Shigeru Mukai. *An Introduction to Invariants and Moduli. Cambridge studies in advanced mathematics*, volume 81. Cambridge University Press, 2003.

[Mum70] David Mumford. Abelian varieties. 1970.

[Nag59] Masayoshi Nagata. On the 14-th problem of hilbert. *American Journal of Mathematics*, 81(3):766–772, 1959.

[Nag61] Masayoshi Nagata. Complete reducibility of rational representations of a matric group. *Journal of Mathematics of Kyoto University*, 1(1):87–99, 1961.

[Nag63] Masayoshi Nagata. Invariants of group in an affine ring. *Journal of Mathematics of Kyoto University*, 3(3):369–378, 1963.

[New78] PE Newstead. *Introduction to Moduli Problems and Orbit Spaces. Tata Institute of Fundamental Research lectures*, volume 51. 1978.

[Nit05] Nitin Nitsure. Construction of hilbert and quot schemes. *arXiv preprint math/0504590*, 2005.

[NS65] Mudumbai S Narasimhan and Conjeeveram S Seshadri. Stable and unitary vector bundles on a compact riemann surface. *Annals of Mathematics*, pages 540–567, 1965.

[Ram73] Sundararaman Ramanan. The moduli spaces of vector bundles over an algebraic curve. *Mathematische Annalen*, 200(1):69–84, 1973.

[Sch08] Alexander HW Schmitt. *Geometric Invariant Theory and Decorated Principal Bundles. Zurich lectures in advanced mathematics*, volume 11. European Mathematical Society, 2008.

[Sim94] Carlos T Simpson. Moduli of representations of the fundamental group of a smooth projective variety i. *Publications Mathématiques de l'IHÉS*, 79:47–129, 1994.

[SR16] Walter Ferrer Santos and Alvaro Rittatore. *Actions and Invariants of Algebraic Groups, second edition. Monographs and research notes in mathematics*. CRC press, 2016.

[Sta19] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2019.

[WGP08] Raymond O'Neil Wells and Oscar García-Prada. *Differential Analysis on Complex Manifolds, third edition. Graduate texts in mathematics*, volume 65. Springer-Verlag New York, 2008.

[Zha] Yifei Zhao. Tensor product of semistable vector bundles over curves in characteristic zero.

Index