Constrained control of bistable reaction-diffusion equations: Gene-flow and spatially heterogeneous models

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Abstract

In this article, we study gene-flow models and the influence of spatial heterogeneity on the dynamics of bistable reaction-diffusion equations from the control point of view. We establish controllability results under geometric assumptions on the domain where the system evolves and regularity assumptions on the spatial heterogeneity. The non-linearity is assumed to be of bistable type and to have three spatially homogeneous equilibria. We investigate whether or not it is possible, starting from any initial datum, to drive the population to one of these equilibria through a boundary control u, under the natural constraints $0 \leqslant u \leqslant 1$. In the case of the gene-flow model, the situation is similar to [35, 33] and the results only depend on the geometry of the domain. In the case of a spatially heterogeneous environment, we distinguish between slowly varying environments and of rapidly varying ones. We develop a new method to prove that controllability to 0, θ or 1 still holds in the slowly varying case, and give examples of rapidly varying environments where controllability to 0 or 1 no longer holds. This lack of controllability is established by studying the existence of non-trivial solutions which act as barriers for controlling the dynamics, which are of independent interest. Our article is completed by several numerical exepriments that confirm our analysis.

Keywords: Control, reaction-diffusion equations, bistable equations, spatial heterogeneity, gene-flow models, staircase method .

AMS classification: 49J20, 34F10, 35K57 .

Contents

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1 Introduction

1.1 Setting and main results

Motivations Reaction-diffusion equations have drawn a lot of attention from the mathematical community over the last decades, but most usually in spatially homogeneous setting, while the literature devoted to spatially heterogeneous domains only started developing recently. This growing interest led to many interesting questions regarding the possible effects of spatial heterogeneity on, for instance, the dynamics of the equation, or on optimization and control problems: how do these heterogeneities impact the dynamics or the criteria under consideration? Can the results obtained in the homogeneous case be obtained in the heterogeneous one, which is more relevant for applications? In this article, we study some of these questions and the influence of spatial heterogeneity from the angle of control theory. Some of our proofs and results are, however, of independent interest for reaction-diffusion equations.

We investigate a boundary control problem arising naturally from population dynamics models and which has several interpretations. For instance, one might consider the following situation: given a population of mosquitoes, a proportion of which is carrying a disease, is it possible, acting only on the proportion of sick mosquitoes on the boundary, to drive this population to a state where only sane mosquitoes remain? Such questions have drawn the attention of the mathematical community in the past years, see for instance [1] Another example might be that of linguistic dynamics: considering a population of individuals, a part of which is monolingual (speaking only the dominant language), the other part of which is bilingual (speaking the dominant and a minority language), is it possible, acting only on the proportion of bilingual speakers on the boundary of the domain, to drive the population to a state where there remains a non-zero proportion of bilingual speakers, thus ensuring the survival of the minority language? Such models are proposed, for instance, in [39]. In both cases, the influence of the spatial heterogeneity has still not been investigated, and the aim of this work is to provide some informations on such matters.

We give, in Section 1.2, more bibliographical references related to modelling issues and the mathematical analysis of the equations studied here.

The equation and the control system We now present the main equations that will be studied here. We refer to Section 1.2 for more informations on modelling.

In this article, we consider a boundary control problem for bistable reaction-diffusion equations. Such bistable equations are well-suited to describe the evolution of a proportion of a population and are characterized by the so-called *Allee effect*: there exists a threshold for the proportion of the population under scrutiny such that, in the absence of spatial diffusion, above this threshold, this subgroup will invade the whole domain (and drive the other subgroup to extinction) while, under this threshold, this subgroup of the population will go extinct. This Allee effect is, on a mathematical level, taken into account via a *bistable non-linearity*, that is, a function $f : \mathbb{R} \to \mathbb{R}$ such that

1. f is $\mathscr{C}^\infty$ on $[0, 1]$,
2. There exists $\theta \in (0; 1)$ such that $0\,, \theta$ and 1 are the only three roots of f in $[0, 1]$, This parameter θ accounts for the Allee effect mentioned above.
3. $f'(0)\,, f'(1) < 0$ and $f'(\theta) > 0$,
4. Without loss of generality, we assume that $\int_0^1 f > 0$.

We give an example of such a bistable non-linearity in Figure 1 below:

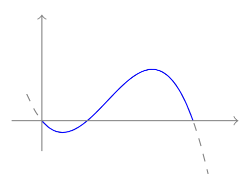

Figure 1: Graph of a typical bistable non-linearity.

The typical example of such a non-linearity is

$$f(\xi) = \xi(\xi - \theta)(1 - \xi),$$

and in this case requiring that $\int_0^1 f > 0$ is equivalent to asking that θ satisfies $\theta < \frac{1}{2}$.

Models with spatial diffusion were studied from the angle of control theory in [35, 33], see Section 1.2. Here, we want to study more precise version of this equation and take into account two phenomenons of great relevance for applications, see Section 1.2: *gene-flow models* and *spatially heterogeneous models*. To write these models in a synthetic way, we will consider, in general, a function $N = N(x, p)$. As will be explained later, gene-flow models correspond to $N = N(p)$ and spatially heterogeneous models correspond to $N = N(x)$.

With a bistable non-linearity f and such a function N, in a domain $\Omega \subset \mathbb{R}^d$, the equation we consider writes, in its most general form

$$\frac{\partial p}{\partial t} - \Delta p - 2\langle \nabla \left(\ln(N(x, p))\right), \nabla p \rangle = f(p). \tag{1}$$

Here, once again, *p stands for a proportion of the total population* (for instance, the proportion of infected mosquitoes or of monolingual speakers). Of particular relevance are the spatially homogeneous steady-states of this equation: $p \equiv 0$, $p \equiv \theta$ and $p \equiv 1$. *Our objective in this article is to investigate whether or not it is possible to control any initial datum to these spatially heterogeneous steady-states.*

Let us formalize this control problem. Given an initial datum $p_0 \in L^2(\Omega)$ such that

$$0 \leqslant p_0 \leqslant 1$$

we consider the control system

$$\begin{cases} \dfrac{\partial p}{\partial t} - \Delta p - 2\langle \nabla \ln(N), \nabla p\rangle = f(p) & \text{in } (0,T) \times \Omega\,, \\ \\ p = u(t,x) & \text{on } (0,T) \times \partial\Omega, \\ \\ p(t=0,\cdot) = 0 \leqslant p_0 \leqslant 1, \end{cases} \tag{2}$$

where, for every $t \geqslant 0\,, x \in \partial\Omega$,

$$u(t,x) \in [0,1] \tag{3}$$

is the control function. Our goal is the following:

Given any initial datum $0 \leqslant p_0 \leqslant 1$,
is it possible to drive p_0 to 0, θ, or 1 in
(in)finite time with a control u satisfying (3)?

In other words, can we drive any initial datum to one of the spatially homogeneous steady-states of the equation? If one thinks about infected mosquitoes, driving any initial population to 0 is relevant for controlling the disease while, if one thinks about mono or bilingual speakers, driving the initial datum to the intermediate steady-state θ ensures the survival of the minority language.

Let us denote the steady-states as follows

$$\forall a \in \{0, \theta, 1\}\,, z_a \equiv a.$$

By controllability, we mean the following: let $a \in \{0, \theta, 1\}$, then

- Controllability in finite time: we say that p_0 *is controllable to a in finite time* if there exists a finite time $T < \infty$ such that there exists a control u satisfying the constraints (3) and such that the solution $p = p(t,x)$ of (2) satisfies

$$p(T,\cdot) = z_a \text{ in } \Omega.$$

- Controllability in infinite time: we say that p_0 *is controllable to z_a in infinite time* if there exists a control u satisfying the constraints (3) such that the solution $p = p(t,x)$ of (2) satisfies

$$p(t,\cdot) \underset{t\to\infty}{\xrightarrow{\mathscr{C}^0(\Omega)}} z_a.$$

Remark 1. Note that, in the definition of controllability in finite time, we do not ask that the controllability time be small; it might actually be large because of the constraint $0 \leqslant u \leqslant 1$, and the question of the minimal controllability time for this problem is, as far as the authors know, still open.

Definition 1. *We say that (2) is controllable to z_a in (in)finite time if is is controllable to z_a in (in)finite time for any initial datum $0 \leqslant p_0 \leqslant 1$.*

Here, for modelling reasons (which we present in the next paragraph), we only consider two cases for the flux $N = N(x,p)$:

- <u>The gene-flow model:</u> In this case, the function $N = N(x,p)$ assumes the form

$$N(x,p) = N(p). \tag{H_1}$$

This model is referred to as the gene-flow models and appears in many situations (we refer to Section 1.2 and mention that this corresponds to a limit case of a system of coupled reaction-diffusion equations). In this case, the environment is spatially homogeneous, and we prove that the controllability results established in [35, 33] still hold under the same assumptions.

- <u>The spatially heterogeneous model:</u> In this case, $N = N(x,p)$ is of the form

$$N = N(x). \tag{H_2}$$

This case corresponds to a spatially heterogeneous environment: when p is a proportion of a total population, this term accounts for the spatial variations of the total population, see [29] and Section 1.2. We mention that this corresponds to another limit in a system of coupled reaction-diffusion equations. Regarding this spatially heterogeneous model, we will focus on two situations: a slowly varying environment, in a sense made precise in the statement of Theorem 2, and a rapidly varying environment, see Theorem 4. In the first case, we prove that controllability still holds while, in the second case, we give an exemple of N that proves that it is in general hopeless to try and control the equation in a rapidly varying environment under the constraints (3).

1.2 Motivations and known results

1.2.1 Modelling considerations

In this paragraph, we lay out the biological motivations for our work.

Reaction-diffusion equations such as (2) have been used since the seminal works [14, 20] to give mathematical models of population dynamics. The bistable non-linearity accounting for the Allee affect is omnipresent in mathematical biology and we refer, for instance, to [2, 3, 4] for some of its uses in population dynamics. We also point to [11, 28, 31] for modelling issues, or to [18, 39], where a game theory approach is undertaken. Here, p stands for the frequency of some trait, or of the proportion of a type of a population, and the drift term accounts for either the spatial heterogeneity of the environment, or for the gene-flow phenomenon.

We note that gene-flow models have been used in the modelling of evolutionary processes of differentiation, see [15, 25]. We point, for further references regarding the adaptative point of view on gene-flow, to [7], as well as [12]. A mathematical study of the impact of gene-flow models on adaptative dynamics is carried out in [27], while a traveling-wave point of view is studied in [29].

In [29, Section 6], a possible derivation of the equations under study in our article is carried out. We can briefly sketch their arguments as follows: let us consider a population with size $N = n_1 + n_2$ where, for $i = 1, 2$, n_i is the number of individuals with trait i (e.g infected or sane mosquitoes). Let us define the proportion $p = \frac{n_1}{N}$. We assume the population evolves in a spatially heterogeneous environment Ω, and that the heterogeneity is modelled by a resources distribution $m : \Omega \to \mathbb{R}$. We introduce the death rates associated to each group $d_1 > d_2$, the fertility rates $F_1 < F_2$. The following system is proposed in [37] to model this situation:

$$\begin{cases} \frac{\partial n_1}{\partial t} - \Delta n_1 = F_1 n_1 (1 - \frac{N}{m}) - \delta d_2 n_1 \,, \\ \\ \frac{\partial n_2}{\partial t} - \Delta n_2 = F_2 n_2 (1-p)(1 - \frac{N}{m}) - d_2 n_2 \,, \end{cases} \tag{4}$$

for some $\delta > 0$.
One then shows that p solves

$$\frac{\partial p}{\partial t} - \mu \Delta p - 2\langle \nabla \ln(N)\,, \nabla p\rangle = p(1-p)\left(F_1(1-N)(p-1) + d_1(1-\delta)\right).$$

The authors of [29] distinguish two limits

- Homogeneous environment and large birth rate:

  Assuming that $F_2 >> 1$ and that m is constant, it is possible to show that there exists $h = h(p)$ such that, when $F_2 \to \infty$, p solves

  $$\frac{\partial p}{\partial t} - \mu \Delta p + 2|\nabla p|^2 \frac{h'(p)}{h(p)} = p(1-p)(p-\theta)$$

  for some $\theta \in (0;1)$ which is the gene-flow model studied in this article.

- Heterogeneous environment and large birth rate:

  We apply the same reasoning, with $F_2 >> 1$ and the additional assumption that $\left|\frac{\Delta m}{m}\right| << 1$. We then obtain

  $$\frac{\partial p}{\partial t} - \mu \Delta p + 2\left\langle \frac{\nabla m}{m}, \nabla p \right\rangle = p(1-p)(p-\theta),$$

  for some $\theta \in (0;1)$, which is the spatially heterogeneous model under consideration here.

As was explained earlier in this Section, one can think of the unknown p as the infection frequency in a population of mosquitoes, as is the case in [1, 29] or as the proportion of mono or bilingual speakers as proposed in [39]. This last interpretation was one of the motivations of [33, 38]. Thus, we may think of wanting to drive p_0 to θ as wanting to reach an equilibrium regarding the languages spoken inside a community, for instance to preserve the existence of this minority language.

The main contribution of this article is understanding how spatial heterogeneity might affect this controllability. We insist upon the fact that such questions pertain to a growing field, see [5, 21, 26, 36] for an optimization approach to spatial heterogeneity for monostable case. In the case of bistable equations, a possible reference from the mathematical point of view is [29]. We refer to [36] for a more biology oriented presentation of such topics in mathematical biology.

Namely, we will prove that, provided the environment is not rapidly varying, the controllability properties still hold, while giving examples where sharp changes keep us from controlling the equation. Intuitively, this result makes sense: if there is a sharp transition in the environment in the center of the domain, it is hopeless to control what is happening inside the domain only using the boundary. To give an example of such quick transitions, we will investigate the case where the spatial heterogeneity is a gaussian, and give a qualitative analysis of the controllability properties when the variance is either small or large.

1.2.2 Known results regarding the constrained controllability of bistable equations

The influence of spatial heterogeneity on population dynamics and its interplay with optimization problems has drawn a lot of attention in the past years. Regarding the controllability properties of these equations, the available literature is scarce.

In [33], the controllability to 0, θ or 1 of the equation

$$\frac{\partial p}{\partial t} - \Delta p = f(p)$$

with a constraints on the boundary control is carried out using a phase portrait analysis. In their case, the domain is $\Omega = [-L, L]$. Namely, they prove, using comparison principles that, regardless of L, the static strategy $u = 1$ allows you to control to $z_1 \equiv 1$ in infinite time. They prove that there exists a threshold L^* such that control to 0 is possible of and only if $L < L^*$, in which case the static strategy $u \equiv 0$ works. This threshold is established by proving that there exists L^* such that, for any $L \geqslant L^*$ there exists a non trivial solution to the equation with homogeneous Dirichlet boundary conditions; this solution acts as a barrier and prevents controllability. Finally, they prove, using a precise analysis of the phase portrait of the equation and the staircase method of [13] that the equation is controllable to $z_\theta \equiv \theta$ in finite time if and only if $L < L^*$.

These results were extended, for the same equation, to the multi-dimensional case in [35].

In [38], the equation

$$\frac{\partial p}{\partial t} - \Delta p = p(p - \theta(t))(1 - p)$$

is considered, but this time, it is the Allee parameter $\theta = \theta(t)$ that is the control parameter and the target is a travelling wave solution.

In [1], an optimal control problem for the equation without diffusion

$$\frac{\partial p}{\partial t} = f(p) + u(t),$$

and with an interior control u (rather than a boundary one) is considered. We underline that, in their study, u only depends on the time, and not on the space variable.

Finally, we mention [29], in which the existence of traveling-waves for the gene-flow model (H_1) is established and (non)-existence and properties of traveling-waves solutions for the (H_2) model are studied. The authors prove that, under certain assumptions on the heterogeneity N (for instance, a high exponential growth on a large enough interval of $\mathbb{R}$), the invasion of the front is blocked. This result seems loosely to the lack of controllability in a rapidly varying environment, see Theorem 4.

1.3 Statement of the main controllability results

We recall that we work with Equation (2)

$$\begin{cases} \dfrac{\partial p}{\partial t} - \Delta p - 2\langle \nabla \ln(N), \nabla p \rangle = f(p) & \text{in } \mathbb{R}_+ \times \Omega\,, \\ p = u(t, x) & \text{on } (0, T) \times \partial\Omega, \\ p(t = 0, \cdot) = p_0 & \text{in } \Omega, \end{cases}$$

where $u \in [0, 1]$ and that we want to control any initial datum to $0, \theta$ or 1.

1.3.1 A brief remark on the statement of the Theorems

We are going to present controllability and non-controllability results for the gene-flow models and the spatially heterogeneous ones. Regarding obstructions to controllability, the main obstacles are the existence of non-trivial steady-states, namely solutions to

$$-\Delta\varphi - 2\left\langle \frac{\nabla N}{N}, \nabla\varphi \right\rangle = f(\varphi) \text{ in } \Omega$$

associated with the boundary conditions $\varphi = 0$ or $\varphi = 1$. However, given that the existence of non-trivial solutions for the Dirichlet boundary conditions $\varphi = 0$ is obtained through a sub and super solution methods, the natural quantity appearing is the inradius of the domain, i.e

$$\rho_\Omega = \sup\{r > 0\,, \exists x \in \Omega\,, \mathbb{B}(x, r) \subset \Omega\},$$

while non-existence of non-trivial solutions is usually done through the study of the first Laplace-Dirichlet eigenvalue

$$\lambda_1^D(\Omega) := \inf_{u \in W_0^{1,2}(\Omega)\,, u \neq 0} \frac{\displaystyle\int_\Omega |\nabla u|^2}{\displaystyle\int_\Omega u^2},$$

which explains why both quantities ρ_Ω and $\lambda_1^D(\Omega)$ appear in the statements. Using Hayman-type inequalities, see [8], we could rewrite $\lambda_1^D(\Omega)$ in terms of the inradius when the set Ω is convex. Indeed, it is proved in [8, Proposition 7.75] that, when Ω is a convex set with $\rho_\Omega < \infty$ then

$$\frac{1}{c\rho_\Omega^2} \leqslant \lambda_1^D(\Omega) \leqslant \frac{C}{\rho_\Omega^2},$$

so that the theorems can be recast in terms of inradius only in the case of convex domains.

1.3.2 Gene-flow models

For the gene flow model (H_1), i.e when N assumes the form

$$N = N(u),$$

the main equation of (2) reads

$$\frac{\partial p}{\partial t} - \Delta p - 2\frac{N'}{N}(p)|\nabla p|^2 = f(p).$$

Then the controllability properties of the equation are the same as in [35]:

Theorem 1. *Let, for any $\Omega \subset \mathbb{R}^d$, ρ_Ω be its inradius:*

$$\rho_\Omega = \sup\{r > 0\,, \exists x \in \Omega\,, \mathbb{B}(x,r) \subset \Omega\}. \tag{5}$$

When N satisfies (H_1), there exists $\rho^* = \rho_*(f)$ such that, for any smooth bounded domain Ω,

1. *Lack of controllability for large inradii: If $\rho_\Omega > \rho^*$, then (2) is not controllable to 0 in (in)finite time in the sense of Definition 1: there exist initial data $0 \leqslant p_0 \leqslant 1$ such that, for any control u satisfying the constraints (3), the solution p of (2) does not converge to 0 as $t \to \infty$.*

2. *Controllability for large Dirichlet eigenvalue If $\lambda_1^D(\Omega) > ||f||_{L^\infty}$, then (2) is controllable to 0, 1 in infinite time for any initial datum $0 \leqslant p_0 \leqslant 1$, and to θ in finite time for any initial datum $0 \leqslant p_0 \leqslant 1$.*

Hence the situation is exactly the same as in [35, 33], and we give, in Figure 2 and 3 schematic representations of domains where controllability might hold or not.

Remark 2. The result makes sense: even if the domain has a large measure, if it is also very thin, it makes sense that a boundary control should work while if it has a big bulge, it is intuitive that a lack of boundary controllability should occur:

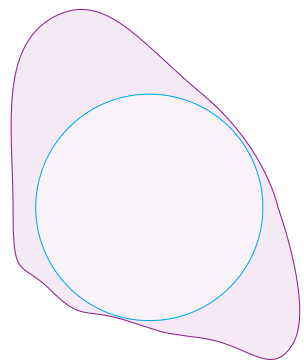

Figure 2: A domain with a large inradius, for which constrained boundary control does not enable us to control the population to an intermediate trait.

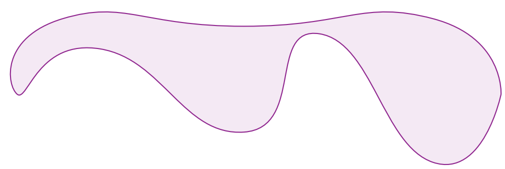

Figure 3: A domain with a large eigenvalue, for which constrained boundary control enables us to control the population to an intermediate trait.

1.3.3 Spatially heterogeneous models

In this case, we work under assumption H_2, i.e with $N = N(x)$ in Ω.
As explained in the introduction, we need to distinguish between two cases: that of a slowly varying environment and that of sharp changes in the environment.

Slowly varying environment In the first part of this paragraph, we consider the case of a slowly varying total population size: we consider, for a homogeneous steady state $z_a \equiv a$, $a \in \{0, \theta, 1\}$, a function $n \in \mathscr{C}^1(\mathbb{R}^d; \mathbb{R})$ and a parameter $\varepsilon > 0$ the control system

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p - \varepsilon \langle \nabla n, \nabla p \rangle = f(p) & \text{in } \mathbb{R}_+ \times \Omega\,, \\ p = u(t,x) & \text{on } \partial\Omega, \\ 0 \leqslant u \leqslant 1, \\ p(t=0,\cdot) = p_0\,, 0 \leqslant p_0 \leqslant 1\,, \end{cases} \tag{6}$$

which models an environment with small spatial changes in the total population size; this amounts to requiring that

$$\left| \frac{\nabla N}{N} \right| << 1,$$

where N satisfies (H_2). Indeed, we can then formally write

$$N \approx N_0 + \frac{\varepsilon}{2} n(x),$$

where N_0 is a constant[1].

Remark 3. For simplicity, we assume that n is defined on $\mathbb{R}^d$ rather than on Ω. Since we already assumed that N was $\mathscr{C}^1$, this amounts to requiring that n can be extended in a $\mathscr{C}^1$ function outside of Ω, which once again would follow from regularity assumptions on Ω.

Theorem 2. *Let, for any $\Omega \subset \mathbb{R}^d$, ρ_Ω be its inradius (defined in Equation (5)).*
Let $n \in \mathscr{C}^1(\mathbb{R}^d)$.

1. *Lack of controllability for large inradii: There exists $\rho^* = \rho_*(n, f) > 0$ such that if $\rho_\Omega > \rho^*$, then (6) is not controllable to 0 in (in)finite time in the sense of Definition 1: there exist initial data p_0 such that, for any control u satisfying the constraints (3), the solution p of (6) does not converge to 0 as $t \to \infty$.*

2. *Controllability for large Dirichlet eigenvalue and small spatial variations: If $\lambda_1^D(\Omega) > ||f||_{L^\infty}$, there exists $\varepsilon_* = \varepsilon_*(n, f, \Omega)$ such that, when $\varepsilon \leqslant \varepsilon_*$, the Equation (6) is controllable to 0 and 1 in infinite time f and to θ in finite time in the sense of Definition 1.*

To prove this theorem, we have to introduce perturbative arguments to the staircase method of [13], which we believe sheds a new light on this method as well as on the influence of spatial heterogeneity on reaction-diffusion equations.

The case of radial drifts The previous result, however general, is proved using a very implicit method that does not enable us to give explicit bounds on the perturbation ε. In the case where the total population size $n : \Omega \to \mathbb{R}_+^*$ can be extended into a radial function $n : \mathbb{R}^d \to \mathbb{R}_+^*$ we can give en explicit bound on the decay rate of N to ensure the controllability of

$$\begin{cases} \dfrac{\partial p}{\partial t} - \Delta p - \langle \dfrac{\nabla N}{N}(x), \nabla p\rangle = f(p) \text{ in } \Omega \times (0,T)\,, \\ p = u(t,x) \text{ on } \partial\Omega \times (0,T), \\ 0 \leqslant p\,, u(t,x) \leqslant 1, \\ p(t=0,\cdot) = \varphi_0\,, 0 \leqslant \varphi_0 \leqslant 1\,, \end{cases} \tag{7}$$

In other words, when the total population size is the restriction to the domain Ω of a radial function, we can obtain controllability results.

Theorem 3. *Let Ω be a bounded smooth domain in $\mathbb{R}^d$. Let $N \in \mathscr{C}^1(\mathbb{R}^d; \mathbb{R}_+^*)\,, \inf N > 0$ and N be radially symmetric. Let*

$$\lambda_1^D(\Omega, N) := \inf_{u \in W_0^{1,2}(\Omega)} \frac{\displaystyle\int_\Omega N^2 |\nabla u|^2}{\displaystyle\int_\Omega N^2 u^2}$$

be the weighted eigenvalue associated with N.
If

$$||f'||_{L^\infty} \leqslant \lambda_1^D(\Omega, N) \tag{8}$$

and if

$$N'(r) \geqslant -\frac{d-1}{2r} N(r), \tag{A_1}$$

then the equation (7) is controllable to z_0 in infinite time and to θ in finite time, for any initial datum $0 \leqslant p_0 \leqslant 1$.

This Theorem is proved using energy methods and adapting the proofs of [35].

[1] We can assume, without loss of generality, that $N_0 = 1$. Indeed, the equation (2) is invariant under the scaling $N \mapsto \lambda N$ where $\lambda \in \mathbb{R}_+^*$.

Lack of controllability for rapidly varying total population size: blocking phenomenons
As mentioned, the lack of controllability occurs when barriers appear. For instance, if a non-trivial solution to

$$\begin{cases} -\Delta\varphi - 2\langle \frac{\nabla N}{N}, \nabla\varphi\rangle = f(\varphi) & \text{in } \Omega\,, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

exists, then it must reach its maximum above θ and thus, from the maximum principle, it is not possible to drive an initial datum $p_0 \geqslant \varphi_0$ to 0 with constrained controls. This kind of counter-examples appear when the drift is absent, see [35, 33]. They are usually constructed by means of sub and super solutions of the equation. What is more surprising however is that adding a drift actually leads to the existence of non-trivial solutions to

$$\begin{cases} -\Delta\varphi_1 - 2\langle \frac{\nabla N}{N}, \nabla\varphi_1\rangle = f(\varphi_1) & \text{in } \Omega\,, \\ \varphi = 1 & \text{on } \partial\Omega, \end{cases}$$

which never happens when no drift is present, meaning that driving the population from an initial datum $p_0 \leqslant \varphi_1$ to z_1 is impossible. Here, we need to carry out a precise analysis of the equation: Equation (11) has a variational formulation but since $z_1 \equiv 1$ is always a global minimizer of the natural energy associated with Equation (11), using an energy argument is not possible.
In this paragraph we give an explicit example of some N in the one-dimensional case such that the equation is not controllable to either 0, θ or 1. Let, for any $\sigma > 0$, the gaussian of variance $\sqrt{\sigma}$ be defined as

$$N_\sigma(x) := \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}},$$

so that the control problem (2) becomes, in the one dimensional case

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} - \dfrac{\partial\varphi}{\partial x^2} + \dfrac{2x}{\sigma}\dfrac{\partial\varphi}{\partial x} = f(\varphi) \text{ in } \Omega\,, \\ \varphi(-L) = u(t,-L)\,, \varphi(L) = u(t,L), \\ 0 \leqslant \varphi\,, u(t,x) \leqslant 1, \\ \varphi(t=0,\cdot) = \varphi_0\,, 0 \leqslant \varphi_0 \leqslant 1\,, \\ \varphi(t=T,\cdot) = z_a\,, T \in]0;+\infty], \end{cases} \tag{9}$$

Introduce the following barrier equations (i.e, if there exists a non-trivial solution to these equations, controllability might fail) on some interval $[-L, L]$:

$$\begin{cases} -\frac{\partial^2\varphi}{\partial x^2} + \frac{2x}{\sigma}\frac{\partial\varphi}{\partial x} = f(\varphi) \text{ in } [-L, L]\,, \\ \varphi(\pm L) = 0, \\ 0 \leqslant \varphi \leqslant 1, \end{cases} \tag{10}$$

and

$$\begin{cases} -\frac{\partial^2\varphi}{\partial x^2} + \frac{2x}{\sigma}\frac{\partial\varphi}{\partial x} = f(\varphi) \text{ in } [-L, L]\,, \\ \varphi(\pm L) = 1, \\ 0 \leqslant \varphi \leqslant 1, \end{cases} \tag{11}$$

Theorem 4. 1. *Existence of critical lengths L_σ: For any $\sigma > 0$, there exists $L_\sigma(0) > 0$ (resp. $L_\sigma(1)$) such that the Equation (10) (resp. Equation (11)) has a non-trivial solution in $\Omega = [-L_\sigma(0); L_\sigma(0)]$ (resp. $[-L_\sigma(1); L_\sigma(1)]$). As a consequence, for $a = 0, 1$, Equation (9) is not controllable in infinite time to a or θ on $[-L_\sigma(a); L_\sigma(a)]$.*
For any $L \geqslant L_\sigma^*(1)$, Equation (11) has a non-trivial solution on $[-L, L]$.

2. *Asymptotic analysis of L_σ: Define $L_\sigma(a)^*$ as the minimal length of the interval such that controllability fails on $[-L_\sigma(a)^*; L_\sigma(a)^*]$, then*

$$L_\sigma(1)^* \underset{\sigma\to\infty}{\rightarrow} +\infty \,, L_\sigma(1)^* \underset{\sigma\to 0}{\rightarrow} 0 \,, L_\sigma^*(0) \underset{\sigma\to 0}{\rightarrow} 0.$$

In other words, the sharper the transition, the smaller the interval where lack of controllability occurs.

3. *Double-blocking phenomenon: There exists L_σ^{**} such that both Equations (10) and (11) have a non-trivial solution on $[-L_\sigma^{**}, L_\sigma^{**}]$. Equation (9) is not controllable to either 0, θ or 1 on $[-L_\sigma^{**}, L_\sigma^{**}]$.*

We illustrate the existence of non-trivial solutions in Figures 4, 5, 6 and 7.
To prove this Theorem, we will study the energy

$$\mathscr{E} : (u, v) \mapsto \frac{1}{2}v^2 + F(u).$$

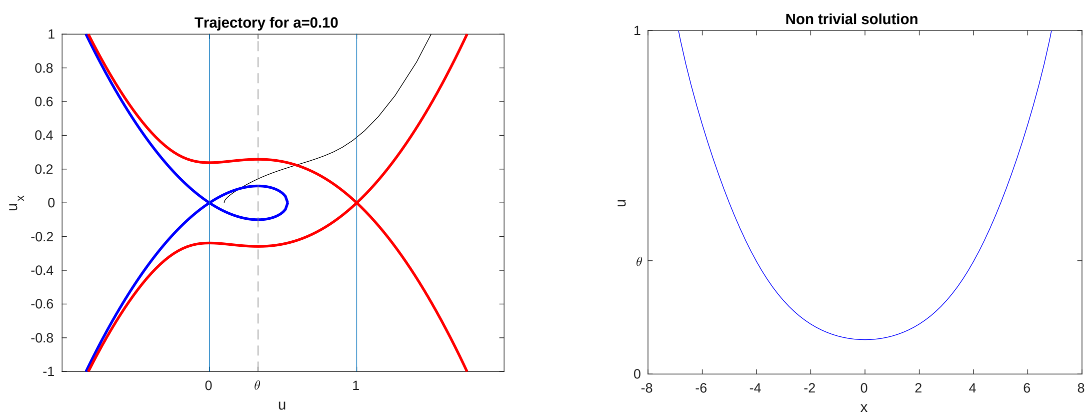

Figure 4: $\sigma = 40$ and $f(s) = s(1-s)(s-\theta)$, $\theta = 0.33$. Phase portrait (Left): the trajectory corresponding to the nontrivial solution is in black, the energy set $\{\mathscr{E} = F(1)\}$ in red, the energy set $\{\mathscr{E} = F(0)\}$ in blue. Nontrivial solution of (11) (Right).

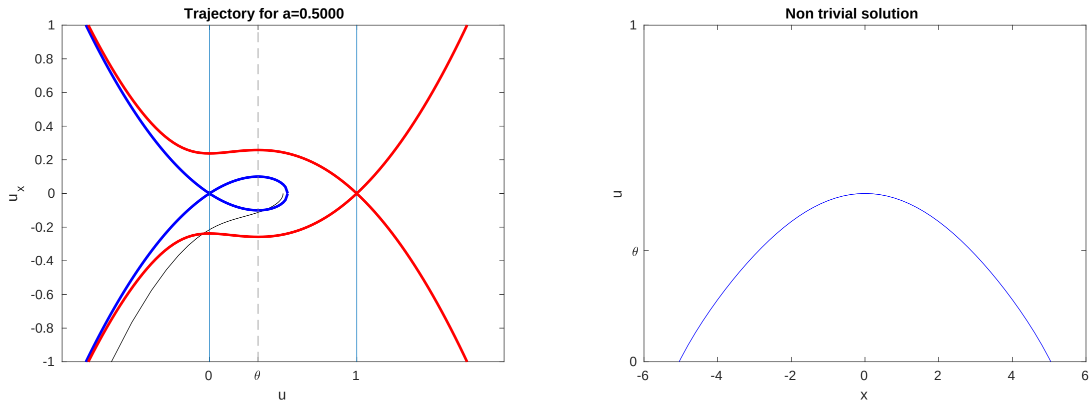

Figure 5: Same class of parameters σ, θ, f. Phase portrait (Left): the trajectory corresponding to the nontrivial solution is in black, the energy set $\{\mathscr{E} = F(1)\}$ in red, the energy set $\{\mathscr{E} = F(0)\}$ in blue. Nontrivial solution of (10) (Right).

We also observe this "double-blocking" phenomenon (i.e the existence of non-trivial solutions to (11) and (10) in the same interval) numerically, when trying to control an initial datum to θ:

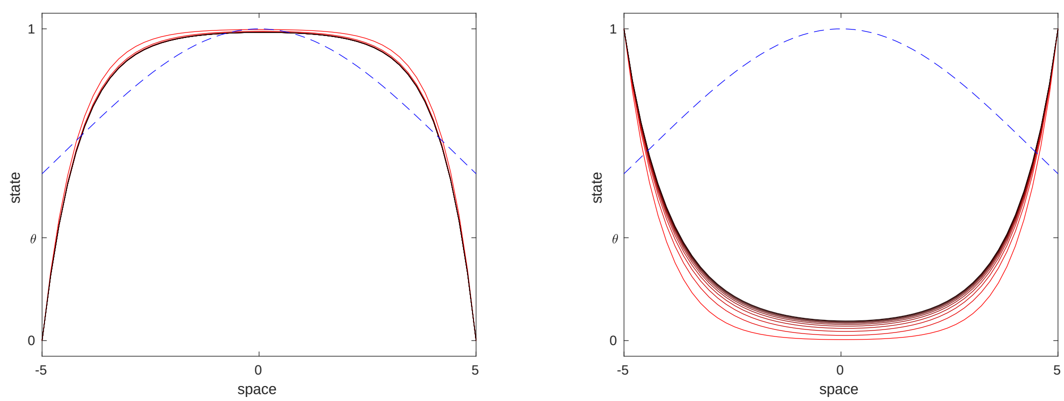

Figure 6: $N(x) = e^{\frac{-x^2}{\sigma}}$, $\sigma = 40$, $L = 5$, (Left) initial datum $u_0 = 1$, (Right) intial datum $u_0 = 0$.

There can also be controllability from 0 to θ, but not from 1 to θ, as shown, numerically, below:

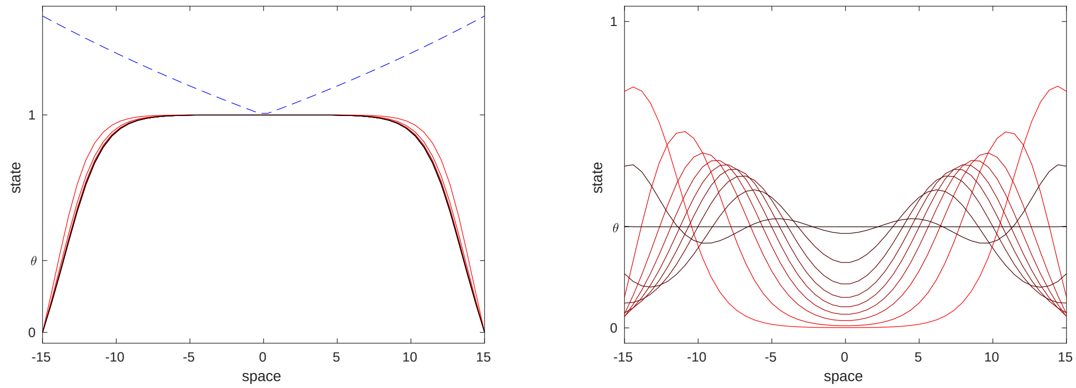

Figure 7: $N(x) = e^{\frac{|x|}{\sigma}}$, $\sigma = 40$, $T = 150$, $L = 15$. (Left) initial datum $u_0 = 1$, (Right) intial datum $u_0 = 0$.

Remark 4. As noted, these sharp changes in the total population size have been known, since [29], to provoke blocking phenomenons for the traveling-waves solutions of the bistable equation, and our results seems to lead to the same kind of interpretation: when a sudden change occurs in N, it is hopeless for a population coming from the boundary to settle everywhere in the domain. We prove this result using a careful analysis of the phase portrait for the non-autonomous system to establish existence of non-trivial solutions to the steady-state equations with homogeneous Dirichlet boundary conditions equal to either 0 or 1.

We note that our proofs could be extended to the multi-dimensional case, when considering a multi-dimensional gaussian distribution.

2 Proof of Theorem 1: gene-flow models

Proof of Theorem 1. The proof consists in a simple transformation of the equation, already used in [29, Proof of Theorem 1], which will turn the equation into the classical bistable reaction diffusion equation already considered in [35]. We consider the equation

$$\frac{\partial p}{\partial t} - \Delta p - 2\frac{N'}{N}(p)|\nabla p|^2 = f(p), \tag{12}$$

and introduce the anti-derivative of N as

$$\mathcal{N} : x \mapsto \int_0^x N^2(\xi)d\xi.$$

We first note that multiplying N by any factor λ leaves the equation (12) invariant. We thus fix

$$\int_0^1 N^2(\xi)d\xi = 1.$$

Multiplying (12) by N^2 we get

$$N^2(p)\frac{\partial p}{\partial t} - N^2(p)\Delta p - 2N'(p)|\nabla p|^2 = (\mathcal{N}(p))_t - \nabla \cdot \left(N^2(p)\nabla p\right) = (\mathcal{N}(p))_t - \Delta(\mathcal{N}(p)).$$

Hence, as $\mathscr{N}$ is a diffeomorphism the function $\tilde{p} := \mathscr{N}(p)$ satisfies

$$\frac{\partial \tilde{p}}{\partial t} - \Delta \tilde{p} = \tilde{f}\left(\mathscr{N}^{-1}(\tilde{p})\right) N^2 \left(\mathscr{N}^{-1}(\tilde{p})\right) =: \tilde{f}(\tilde{p}).$$

However, it is easy to see that, f being bistable, so is $\tilde{f}$. Furthermore, $\mathscr{N}$ is a $\mathscr{C}^1$ diffeomorphism of $[0,1]$, and it is easy to see that p is controllable to $0, \theta$ or 1 if and only if $\tilde{p}$ is controllable to $0, \theta$ or 1, and we are thus reduced to the statement of [35, Theorem 1.2], from which the conclusion follows. □

3 Proof of Theorem 2: slowly varying total population size

3.1 Lack of controllability to 0 for large inradius

We prove here the first point of Theorem 2. Recall that we want to prove that, if the inradius ρ_Ω is bigger than a threshold ρ^* depending only on f, then equation (6) is not controllable to 0 in (in)finite time.

Following [33], we claim that this lack of controllability occurs when the equation

$$\begin{cases} -\Delta \eta - \varepsilon \langle \nabla n \,, \nabla \eta \rangle = f(\eta) & \text{in } \Omega\,, \\ \eta = 0 & \text{on } \partial\Omega\,, \\ 0 \leqslant \eta \leqslant 1 \end{cases} \tag{13}$$

has a non-trivial solution, i.e a solution such that $\eta \neq 0$. Indeed, we have the following Claim:

Claim 1. *If there exists a non-trivial solution $\eta \neq 0$ to* (13)*, then* (6) *is not controllable to 0 in infinite time.*

Proof of Claim 1. This is an easy consequence of the maximum principle. Indeed, let η be a non-trivial solution of (13) and let p_0 be any initial datum satisfying

$$\eta \leqslant p_0 \leqslant 1.$$

Let $u : \mathbb{R}_+ \times \partial\Omega \to [0,1]$ be a boundary control. Let p^u be the solution of

$$\begin{cases} \frac{\partial p^u}{\partial t} - \Delta p^u - \varepsilon \langle \nabla n \,, \nabla p^u \rangle = f(p^u) \text{ in } \Omega\,, \\ p^u = u \text{ on } \mathbb{R}_+^* \times \partial\Omega\,, \\ p^u(t=0\,, \cdot) = p_0, \end{cases} \tag{14}$$

From the parabolic maximum principle [34, Theorem 12], we have for every $t \in \mathbb{R}_+$,

$$\eta(t, \cdot) \leqslant p^u(t, \cdot),$$

so that p^u cannot converge to 0 as $t \to \infty$. This concludes the proof. □

It thus remains to establish the following Lemma:

Lemma 1. *There exists $\rho^* = \rho^*(n, f)$ such that, for any Ω satisfying*

$$\rho_\Omega > \rho^*$$

there exists a non-trivial solution $\eta \neq 0$ to equation (13).

Since the proof of this Lemma is a straightforward adaptation of [35, Proposition 3.1], we postpone it to Appendix A.

3.2 Controllability to 0 and 1

We now prove the second part of Theorem 2, which we rewrite as the following claim:

Claim 2. *1. Controllability to 0: There exists $\rho_* = \rho_*(n, f)$ such that, for any Ω, if $\rho_\Omega \leqslant \rho_*$, Equation (2) is controllable to 0 in infinite time.*

2. Controllability to 1: There exists $\overline{\varepsilon} > 0$ such that, for any $\varepsilon \leqslant \overline{\varepsilon}$, Equation (2) is controllable to 1 in infinite time.

Proof of Claim 2. 1. Controllability to 0:

The key part is the following thing:

There exists $\rho_* > 0$ such that, if $\rho_\Omega < \rho_*$, then $y \equiv 0$ is the only solution to

$$\begin{cases} -\Delta y - \varepsilon \langle \nabla n \,, \nabla y \rangle = f(y)\,, & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Indeed, assuming that the uniqueness result (15) holds, consider the static control $u \equiv 0$ and the solution of

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p - \varepsilon \langle \nabla n \,, \nabla p \rangle = f(p)\,, & \text{in } \mathbb{R}_+ \times \Omega\,, \\ p = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ p(t=0,\cdot) = p^0 & \text{in } \Omega. \end{cases}$$

From standard parabolic regularity and the Arzela-Ascoli theorem, p converges uniformly in Ω to a solution $\overline{p}$ of

$$\begin{cases} -\Delta \overline{p} - \varepsilon \langle \nabla n \,, \nabla \overline{p} \rangle = f(\overline{p})\,, & \text{in } \mathbb{R}_+ \times \Omega \\ \overline{p} = 0 & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

However, by the uniqueness result (15), we have $\overline{p} = 0$, whence

$$p(t,\cdot) \underset{t\to\infty}{\overset{\mathscr{C}^0(\overline{\Omega})}{\to}} 0,$$

which means that the static strategy drives p_0 to 0.
Finally, we claim that (15) follows from spectral arguments: first of all, uniqueness holds for

$$\begin{cases} -\Delta y - \varepsilon \langle \nabla n \,, \nabla y \rangle = f(y)\,, & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases}$$

if the first eigenvalue $\lambda(\varepsilon, n, \Omega)$ of the operator

$$\mathcal{L}_{\varepsilon,n} = -\nabla \cdot (e^{\varepsilon n} \nabla u)$$

with Dirichlet boundary conditions satisfies

$$\lambda_1(\varepsilon, n, \Omega) > ||f'||_{L^\infty} e^{\varepsilon ||n||_{L^\infty}}\,,$$

as is standard from classical theory for non-linear elliptic PDE, see [6].
We now notice that, n being positive, the Rayleigh quotient formulation for the eigenvalue

$$\lambda(\varepsilon, n, \Omega) = \inf_{u \in W_0^{1,2}(\Omega)} \frac{\int_\Omega e^{\varepsilon n} |\nabla u|^2}{\int_\Omega u^2}$$

yields that

$$\lambda(\varepsilon, n, \Omega) \geqslant \lambda_1^D(\Omega)$$

where $\lambda_1^D(\Omega)$ is the first eigenvalue of the Laplace operator with Dirichlet boundary conditions. Thus we are reduced to checking that

$$\lambda_1^D(\Omega) > ||f'||_{L^\infty} e^{\varepsilon ||n||_{L^\infty}},$$

as claimed. If the condition $\lambda_1^D(\Omega) > ||f'||_{L^\infty}$, taking the limit as $\varepsilon \to 0$ yields the desired result.

2. Controllability to 1 Using the same arguments, we claim that controllability to 1 can be achieved through the static control $u \equiv 1$ provided the only solution to

$$\begin{cases} -\Delta \overline{p} - \varepsilon \langle \nabla n \,, \nabla \overline{p} \rangle = f(\overline{p})\,, & \text{in } \Omega\,, \\ \overline{p} = 1 & \text{on } \partial\Omega, \\ 0 \leqslant \overline{p} \leqslant 1 \end{cases} \tag{16}$$

is $\overline{p} \equiv 1$.
We already know (see [35, 33]) that uniqueness holds for $\varepsilon = 0$. Now this implies that uniqueness holds for ε small enough. Indeed, argue by contradiction and assume that, for every $\varepsilon > 0$ there exists a non-trivial solution $\overline{p}_\varepsilon$ to (16). Since $\overline{p}_\varepsilon \neq 1$, p reaches a minimum at some $\overline{x}_\varepsilon \in \Omega$, and so

$$f(\overline{p}_\varepsilon(\overline{x}_\varepsilon)) < 0$$

which means that

$$\overline{p}_\varepsilon(\overline{x}_\varepsilon) < \theta.$$

Standard elliptic estimates entail that, as $\varepsilon \to 0$, p_ε converges in $W^{1,2}(\Omega)$ and in $\mathscr{C}^0(\overline{\Omega})$ to $\overline{p}$ satisfying

$$\begin{cases} -\Delta \overline{p} = f(\overline{p}) & \text{in } \Omega\,, \\ \overline{p} = 1 & \text{on } \partial\Omega, \\ 0 \leqslant \overline{p} \leqslant 1 \end{cases} \tag{17}$$

and such that there exists a point $\overline{x}$ satisfying

$$\overline{p}(\overline{x}) < \theta$$

which is a contradiction since we now uniqueness holds for (16). This concludes the proof.

□

3.3 Proof of the controllability to θ for small inradiuses

3.3.1 Structure of the proof: the staircase method

We recall that we want to control the semilinear heat equation

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p - \varepsilon \langle \nabla n \,, \nabla p \rangle = f(p) & \text{in } \Omega\,, \\ p = u(t) & \text{on } \partial\Omega, \\ p(t=0, \cdot) = y_0 \end{cases} \tag{18}$$

to $z_\theta \equiv \theta$.
We give the following local exact controllability result from [33, Lemma 1] or [32, Lemma 2.1], which is the starting point of the method:

Proposition 1. *[Local exact controllability] Let $T > 0$. There exists $\delta_1 > 0$ such that for all steady state y_f of (18), for all $0 \leqslant y_d \leqslant 1$ satisfying*

$$||y_d - y_f||_{\mathscr{C}^0} \leqslant \delta_1$$

then (18) is controllable from y_d to z_a in finite time $T < \infty$ through a control u. Furthermore, letting $\overline{u} = y_f|_{\partial\Omega}$, the control function $u = u(t)$ satisfies

$$||u(t) - \overline{u}||_{\mathscr{C}^0(\partial\Omega)} \leqslant C(T)\delta_1 \tag{19}$$

for some constant $C(T) > 0$.

We now assume that $\rho_\Omega \leqslant \rho^*$, that is, thanks to Claim 2, we assume that we have uniqueness for the equation

$$\begin{cases} -\Delta y - \varepsilon\langle \nabla n \,, \nabla y\rangle = f(y) & \text{in } \Omega\,, \\ y = 0 & \text{on } \partial\Omega \end{cases}$$

The way to prove controllability is to proceed along two different steps:

- Step 1: Starting from any initial conditon $0 \leqslant p_0 \leqslant 1$, we first set the static control

  $$u(t,x) = 0.$$

  Since n is $\mathscr{C}^1$, standard parabolic estimates and the Arzela-Ascoli theorem ensures that the solution p^u of (6) converges uniformly, as $t \to \infty$ to a solution $\overline{\eta}$ of

  $$\begin{cases} -\Delta\overline{\eta} - \varepsilon\langle \nabla n \,, \nabla\overline{\eta}\rangle = f(\overline{\eta}) & \text{in } \Omega\,, \\ \overline{\eta} = 0 & \text{on } \partial\Omega\,, \\ 0 \leqslant \overline{\eta} \leqslant 1 \end{cases} \tag{20}$$

  However, from Claim 2, $\rho_\Omega \leqslant \rho_*(n, f)$ implies that $z_0 \equiv 0$ is the unique solution of this equation. Thus, this static control guarantees that, for every $\delta_1 > 0$, there exists $T_1 > 0$ such that, for any $t \geqslant T_1$

  $$||p^u(t,\cdot)||_{L^\infty} \leqslant \delta.$$

- Step 2: We prove that there exists a steady state p_0 of (18) such that

  $$0 < \inf_{x\in\Omega} p_0(x) \leqslant ||p_0||_{L^\infty} \leqslant \frac{\delta}{2}$$

  and, applying Proposition 1, we drive $p^u(T_1, \cdot)$ to p_0 in finite time.

- Step 3: If we can drive p_0 to θ, then we are done. Thus, we are, in this setting, reduced to the controllability of initial datum in a small neighbourhood of 0 to θ. This is what we are going to prove, using the staircase method.

The staircase method The key idea to do that is the same as in [33], that is, we want to use the staircase method of Coron and Trélat, see [13] for the one-dimensional case (which uses quasi-static deformations) and [32] for a full derivation. We briefly recall the most important features of this method and the way we wish to apply it to our problem.
Assume that there exists a $\mathscr{C}^0$-continuous path of steady-states of (18) $\Gamma = \{p_s\}_{s\in[0,1]}$ such that $p_0 = y_0$ and $p_1 = y_1$.
Then (18) is controllable from y_0 to y_1 in finite time. Indeed, as is usually done, we consider a subdivision

$$0 = s_{i_1} < \cdots < s_{i_N} = 1$$

of $[0,1]$ such that

$$\forall j \in \{0,\dots,N_1\}\,, ||p_{s_i} - p_{s_{i+1}}||_{\mathscr{C}^0(\Omega)} \leqslant \delta_1$$

where δ_1 is the controllability parameter given by the local exact controllability result. We then control each p_{s_i} to $p_{s_{i+1}}$ in finite time using Proposition 1.

This result does not necessarily yield constrained controls, but, thanks to estimate (19) we can enforce these constraints, by choosing a control parameter δ_1 small enough.

Thus, the tricky part seems to be finding a continuous path of steady-states for the perturbed system with slowly varying total population size (6). However, it suffices to have a finite numbers of steady-states that are close enough to each other, starting at y_0 and ending at y_1. We represent the situation in Figure 8 below:

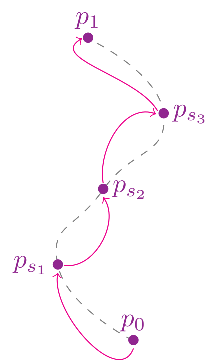

Figure 8: The dashed curve is the path of steady states (for instance in $W^{1,2}(\Omega) \cap \mathscr{C}^0(\overline{\Omega})$), and the points are the close enough steady states. We represent the exact control in finite time T with the pink arrows.

3.3.2 Perturbation of a path of steady-states

We are going to perturb the path of steady-states using the implicit function Theorem in order to get a sequence of close enough steady-states, so that the previous staircase strategy still applies.

Remark 5. Here, if we were to try and prove, for ε small enough, the existence of a continuous path of steady states, the idea would be to start from a path $(p_{s,0})_{s\in[0,1]}$ for $\varepsilon = 0$ (which we know exists from [35, 33]) and to try and perturb it into a path for $\varepsilon > 0$ small enough, thus giving us a path $\{p_{\varepsilon,s}\}_{s\in[0,1]\,,\varepsilon>0}$. However, doing it for the whole path requires some kind of implicit function theorem or, at least, some bifurcation argument. Namely, to construct the path, we would need to ensure that either

$$\mathcal{L}^{s,\varepsilon} := -\nabla \cdot (e^{\varepsilon n}\nabla) - e^{\varepsilon n} f'(p^{0,s})$$

has no zero eigenvalue for $\varepsilon = 0$ or that it has a non-zero crossing number (namely, a non zero number of eigenvalues enter or leave $\mathbb{R}_+^*$ as ε increases from $-\delta$ to δ). In the first case, the implicit function theorem would apply; in the second case, Bifurcation Theory (see [19, Theorem II.7.3]) would ensure the existence of a branch $p_{\varepsilon,s}$ for ε small enough. These conditions seem too hard to check for a general path of continuous of steady states.

Hence, we focus on perturbing a finite number of points close enough on the path since, as we noted, this is enough to ensure exact controllability.

We will strongly rely on the properties of the path of steady-states built in [35, 33].

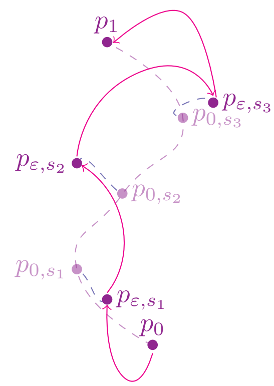

Figure 9: In dark purple, the perturbed steady states, linked to the unperturbed steady states. We do not know whether or not a continuous path of steady states linking these new states exists; however, such points enable us to do exact controllability again and to apply the stair case method.

Henceforth, our goal is the following proposition:

Proposition 2. *Let $\delta > 0$. There exists $N > 0$ and $\overline{\varepsilon} > 0$ such that, for any $\varepsilon \leqslant \overline{\varepsilon}$, there exists a sequence $\{p_{\varepsilon,i}\}_{i=1,\dots N}$ satisfying:*

- *For every $i = 1, \dots, N$, $p_{\varepsilon,i}$ is a steady-state of* (6)*:*

$$-\Delta p_{\varepsilon,i} - \varepsilon \langle \nabla n \,, \nabla p_{\varepsilon,i} \rangle = f(p_{\varepsilon,i}),$$

- $p_{\varepsilon,N} = z_\theta \equiv \theta \,, \quad 0 < \inf p_{\varepsilon,1} \leqslant ||p_{\varepsilon,1}||_{L^\infty} \leqslant \delta$
- *For every $i = 1, \dots, N$,*

$$\frac{\delta}{2} \leqslant p_{\varepsilon,i} \leqslant ||p_{\varepsilon,i}||_{L^\infty} \leqslant 1 - \frac{\delta}{2},$$

- *For every $i = 1, \dots, N-1$,*

$$||p_{\varepsilon,i+1} - p_{\varepsilon,i}||_{L^\infty} \leqslant \delta.$$

As explained, this Proposition gives us the desired conclusion:

Claim 3. *Proposition 2 implies the controllability to θ for any initial datum p_0 in Equation* (2)*.*

Before we prove Proposition 2, we recall how the paths of steady-states are constructed when $\varepsilon = 0$.

Known constructions of a path of steady-states For the multi-dimensional case, it has been shown in [35] that one can construct a path of steady-states linking $z_0 \equiv 0$ to $z_\theta \equiv \theta$ in the following way: let Ω be the domain where the equation is set and let $R_\Omega > 0$ be such that

$$\Omega \subseteq \mathbb{B}(0; R_\Omega).$$

The path of steady state is defined as follows: first of all, if uniqueness holds for

$$\begin{cases}-\Delta y=f(y)\text{ in }\mathbb{B}(0;R_\Omega)\,,\\ y=0.\end{cases}$$

then, for $\eta>0$ small enough, there exists a unique solution to

$$\begin{cases}-\Delta y_\eta=f(y_\eta) & \text{in }\mathbb{B}(0;R_\Omega)\\ y_\eta=\eta & \text{on }\partial\mathbb{B}(0;R_\Omega).\end{cases}$$

Define, for any $s\in[0,1]$, let $p^{0,s}$ be the unique solution to the problem

$$\begin{cases}-\Delta p^{0,s}=f\left(p^{0,s}\right)\text{ in }\mathbb{B}(0;R)\,,\\ p^{0,s}(0)=s\theta+(1-s)y_\eta(0)\,,\\ p^{0,s}\text{ is radial.}\end{cases}\tag{21}$$

Using the polar coordinates, the authors prove that the equation above has a unique solution, and that this solution is admissible, i.e that we even have, for any $0<s_0<1$,

$$0<\inf_{s\in[s_0;1]\,,x\in\mathbb{B}(0;R)}p^{0,s}(x)\leqslant\sup_{s\in[0,1]\,,x\in\mathbb{B}(0;R)}p^{0,s}(x)<1.$$

This is done using energy type methods and gives a path on $\mathbb{B}(0;R_\Omega)$. To construct the path on Ω, it suffices to set

$$\tilde{p}^{0,s}:=p^{0,s}\big|_\Omega\,.$$

Furthermore, by elliptic regularity or by studying the equation in polar coordinates, we see that, for every $s\in[0,1]$,

$$p^{0,s}\in\mathscr{C}^{2,\alpha}(\mathbb{B}(0;R_\Omega))$$

for any $0<\alpha<1$. Instead of perturbing the functions $\tilde{p}^{0,s}\in\mathscr{C}^{2,\alpha}(\Omega)$, we will perturb the functions $p^{0,s}\in\mathscr{C}^2\big(\mathbb{B}(0;R_\Omega)\big)$.

Notation 1. *Henceforth, the parameter $R_\Omega>0$ is fixed and, for any $s\in[0,1]$, $p^{0,s}$ is the unique solution to* (21)*.*

Proof of Proposition 2. Let $\delta>0$. Let $\{s_i\}_{i=1,\dots,N}$ be a sequence of points such that

$$0<p^{0,s_0}\leqslant||p^{0,s_0}||_{L^\infty}\leqslant\frac{\delta}{2},\tag{22}$$

and

$$\forall i\in\{0,\dots,N-1\}\,,\left|\left|p^{0,s_i}-p^{0,s_{i+1}}\right|\right|_{L^\infty}\leqslant\frac{\delta}{4}.\tag{23}$$

We define, for any $i=1,\dots,N$,

$$p_{0,i}=p^{0,s_i}.$$

Fix a parameter $\alpha\in(0;1)$. We define a one-parameter family of mappings as follows: for any $i=1,\dots,N$, let

$$\mathscr{F}_i:\begin{cases}\mathscr{C}^{2,\alpha}\left(\mathbb{B}(0;R_\Omega)\right)\times[-1;1] & \to\mathscr{C}^{0,\alpha}\left(\mathbb{B}(0;R_\Omega)\right)\times\mathscr{C}^0\left(\partial\mathbb{B}(0;R_\Omega)\right),\\ (u,\varepsilon) & \mapsto\left(-\nabla\cdot\left(e^{\varepsilon n}\nabla u\right)-f(u)e^{\varepsilon n}\,,u|_{\partial\mathbb{B}(0;R_\Omega)}-p_0^i|_{\partial\mathbb{B}(0;R_\Omega)}\right).\end{cases}$$

We note that

$$\forall i\in\{0,\dots,N\}\,,\mathscr{F}_i(p_{0,i},0)=0.$$

We wish to apply the implicit function theorem, which is permitted provided the operator

$$\mathscr{L}_i : u \mapsto -\Delta u - f'(p_{0,i})u$$

with Dirichlet boundary conditions is invertible. If this is the case we know that there exists a continuous path $p_{\varepsilon,i}$ starting from $p_{0,i}$ such that

$$\mathscr{F}_i(p^i_\varepsilon, \varepsilon) = 0$$

Denoting, for any differential operator $\mathscr{A}$ its spectrum by $\Sigma(\mathscr{A})$, this invertibility property amounts, thanks to elliptic regularity (see [16]) to requiring that

$$\forall i \in \{1, \dots, N\}, 0 \notin \Sigma(\mathscr{L}_i). \tag{24}$$

If the condition (24) is satisfied, then $p_{0,i}$ perturbs into p^i_ε and we can define

$$\tilde{p}^i_\varepsilon := p^i_\varepsilon\big|_\Omega$$

as a suitable sequence of steady states in Ω. Since we are working with a finite number of points, taking ε small enough guarantees

$$\forall i = 1, \dots, N, ||p_{\varepsilon,i} - p_{0,i}||_{L^\infty} \leqslant \frac{\delta}{4}$$

and we would then have, for any $i = 1, \dots, N_1$,

$$\begin{aligned} ||p_{\varepsilon,i+1} - p_{\varepsilon,i}||_{L^\infty} &\leqslant ||p_{\varepsilon,i+1} - p_{0,i+1}||_{L^\infty} + ||p_{0,i} - p_{\varepsilon,i}||_{L^\infty} + ||p_{0,i+1} - p_{0,i}||_{L^\infty} \\ &\leqslant \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{2} = \delta, \end{aligned}$$

which is what we require of the sequence.
Let us define the set of resonant points (i.e the points where the implicit function Theorem does not apply) as

$$\Gamma := \{j \in \{1, \dots, N\}, 0 \in \Sigma(\mathscr{L}_i)\}.$$

We note that $0 \notin \Gamma$ because the first eigenvalue of

$$\mathscr{L}_0 = -\Delta - f'(0)$$

is positive: indeed, since $f'(0) < 0$, this first eigenvalue is bounded from below by the first Dirichlet eigenvalue of the ball $\mathbb{B}(0; R)$. Hence $0 \notin \Gamma$. We proceed as follows:

1. Whenever $i \notin \Gamma$, we can apply the implicit function Theorem to obtain the existence of a continuous path $p_{\varepsilon,i}$ starting from $p_{0,i}$ such that

   $$p_{\varepsilon,i}|_{\partial\mathbb{B}(0;R_\Omega)} = p_{0,i}|_{\partial\mathbb{B}(0;R_\Omega)}, \mathcal{F}(p_{\varepsilon,i}, \varepsilon) = 0,$$

   so that, taking ε small enough, we can ensure that, for any $i \notin \Gamma$,

   $$||p_{\varepsilon,i} - p_{0,i}||_{L^\infty} \leqslant \frac{\delta}{4}.$$

2. Whenever $i \in \Gamma$, we apply the implicit function theorem on a larger domain $\mathbb{B}(0; R_\Omega + \tilde{\delta})$, $\tilde{\delta} > 0$.

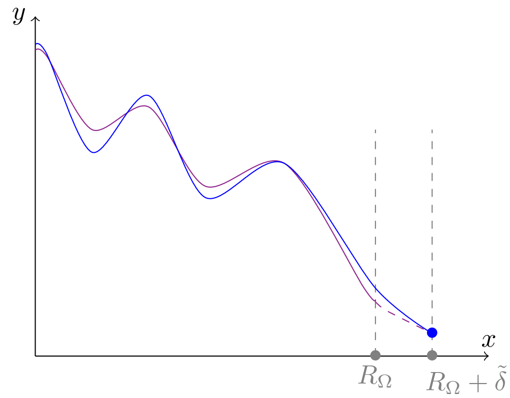

Figure 10: The initial solution $p_{0,i}$ on $\mathbb{B}(0;R_\Omega)$ is continued into a solution on $\mathbb{B}(0;R_\Omega+\tilde\delta)$, and we apply the implicit function theorem on this domain to obtain the blue curve.

Let, for any $i\in\Gamma$, $\lambda_i(k,R_\Omega)$ be the k-th eigenvalue of $\mathscr{L}_i$ with Dirichlet boundary conditions on $\mathbb{B}(0;R_\Omega)$.
Let, for any $i\in\Gamma$,

$$k_i:=\sup\left\{k\,,\lambda_i(k,R)=0\right\}.$$

Obviously, there exists $M>0$ such that $k_i\leqslant M$ uniformly in i, since $\lambda_i(k,R_\Omega)\to\infty$ as $k\to\infty$. We then invoke the monotonicity of the eigenvalues with respect to the domain. Let, for any $\tilde\delta$, $p_{0,i}^{\tilde\delta}$ be the extension of $p_{0,i}$ to $\mathbb{B}(0;R_\Omega+\tilde\delta)$; this is possible given that $p_{0,i}$ is given by the radial equation (21).

Let $\tilde{\mathcal{L}}_i:u\mapsto-\Delta u-f'(p_{0,i}^{\tilde\delta})u$ and $\tilde\lambda_i(\cdot,R_\Omega+\tilde\delta)$ be its eigenvalues. By the min-max principle of Courant (see [17]) we have, for any $k\in\mathbb{N}$ and any $\tilde\delta>0$,

$$\tilde\lambda_i(k,R_\Omega+\tilde\delta)<\lambda_i(k,R_\Omega).$$

Hence, for every $i\in\Gamma$, there exists $\tilde\delta_i>0$ small enough so that, for any $0<\tilde\delta<\tilde\delta_i$,

$$0\notin\Sigma\left(\tilde{\mathcal{L}}_i\right).$$

We then choose $\tilde{\tilde\delta}=\min_{i\in\Gamma}\tilde\delta_i$ and apply the implicit function theorem on $\mathbb{B}\left(0;R_\Omega+\frac{\tilde{\tilde\delta}}{2}\right)$. This gives the existence of $\tilde\varepsilon>0$ such that, for any $\varepsilon<\tilde\varepsilon$ and any $i\in\Gamma$, there exists a solution $p_{\varepsilon,i}^{\tilde{\tilde\delta}}$ of

$$\begin{cases}-\Delta p_{\varepsilon,i}^{\tilde{\tilde\delta}}-\varepsilon\langle\nabla n\,,\nabla p_{\varepsilon,i}^{\tilde{\tilde\delta}}\rangle=f(p_{\varepsilon,i}^{\tilde{\tilde\delta}})\text{ in }\mathbb{B}(0;R_\Omega+\frac{\tilde{\tilde\delta}}{2})\,,\\ p_{\varepsilon,i}^{\tilde{\tilde\delta}}=p_{0,i}\big|_{\mathbb{S}(0;R_\Omega+\frac{\tilde{\tilde\delta}}{2})}\,,\\ p_{\varepsilon,i}\underset{\varepsilon\to0}{\overset{\mathscr{C}^0(\mathbb{B}(0;R_\Omega+\frac{\tilde{\tilde\delta}}{2}))}{\longrightarrow}}p_{0,i}^{\frac{\tilde{\tilde\delta}}{2}}\end{cases}\tag{25}$$

Furthermore,

$$p_{0,i}^{\delta}|_{\mathbb{S}(0;R_\Omega+\delta)}\underset{\delta\to0}{\to}p_{0,i}|_{\partial\mathbb{B}(0;R_\Omega)}.$$

Thus, by choosing $\tilde{\tilde{\delta}}$ small enough, we can guarantee that, by defining

$$\tilde{p}_{\varepsilon,i} := p^{\tilde{\tilde{\delta}}}_{\varepsilon,i}|_{\mathbb{B}(0;R_\Omega)}$$

we have for every ε small enough

$$||\tilde{p}_{\varepsilon,i} - p_{0,i}||_{L^\infty} \leqslant \frac{\delta}{4}.$$

We note that $\tilde{p}_{\varepsilon,i}$ does not satisfy, on $\partial\mathbb{B}(0;R_\Omega)$ the same boundary condition as $p_{0,i}$, but this would be too strong a requirement.

This concludes the proof. □

4 Proof of Theorem 3

Proof of Theorem 3. Proceeding along the same lines as in Theorem 2, we prove that for any drift $N \in \mathscr{C}^\infty(\Omega;\mathbb{R}^d)$ (regardless of whether or not it is the restriction of a radial drift N to the domain Ω), we prove that, when condition (8) then $z_0 \equiv 0$ is the only solution to

$$\begin{cases} -\Delta p - 2\langle \frac{\nabla N}{N}, \nabla p\rangle = f(p) & \text{in } \Omega\,, \\ p = 0 & \text{on } \partial\Omega, \\ 0 \leqslant p \leqslant 1, \end{cases} \tag{26}$$

and note that the main equation is equivalent to

$$-\nabla \cdot \left(N^2 \nabla u\right) = f(p)N^2.$$

Indeed, assuming there exists a non-trivial solution to (26) then from the mean value theorem, we can write

$$f(p) = f'(y)p$$

for some function y and, multiplying the equation by p and integrating by parts gives, using the Rayleigh quotient formulation of $\lambda_1^D(\Omega)$:

$$\lambda_1^D(\Omega)\int_\Omega p^2 \leqslant \int_\Omega |\nabla p|^2 \leqslant \int_\Omega N^2|\nabla p|^2 \leqslant ||f'||_{L^\infty}||N^2||_{L^\infty}\int_\Omega p^2,$$

which is contradiction unless $p = z_0$.

Once we have uniqueness for (26) we follow, for any initial datum p_0, the staircase procedure explained in the proof of Theorem 2: we first set the static control $u = 0$, we drive the solution to a $\mathscr{C}^0$ neighbourhood of z_0, then to a steady-state solution of (7) in this neighboorhood. Thus, we only need to prove the existence of a path of steady states linking z_0 to z_θ. In order to prove that such a path of steady states exists under assumption (A_1), we use an energy method.

Let $R > 0$ be such that $\Omega \subset \mathbb{B}(0;R)$. As in [35], we define, for any $s \in [0,1]$, p_s as the unique solution of

$$\begin{cases} -\Delta p_s - 2\langle \frac{\nabla N}{N}, \nabla p_s\rangle = f(p_s)\,, & \text{in } \mathbb{B}(0;R_\Omega) \\ p_s \text{ is radial in } \mathbb{B}(0;R), \\ p_s(0) = s\theta. \end{cases} \tag{27}$$

We notice that the first equation in (27) rewrites as

$$-\nabla \cdot (N^2 \nabla p_s) = f(p_s)N.$$

Since N is radially symmetric, this amounts to solving, in radial coordinates

$$\begin{cases} -\frac{1}{r^{d-1}}\left(r^{d-1}N^2p_s'\right)' = f(p_s)N^2 \text{ in } [0;R]\,, \\ p_s(0) = s\theta\,, p_s'(0) = 0. \end{cases} \tag{28}$$

We prove the existence and uniqueness of solutions to (28) below but underline that the core difficulty here is ensuring that

$$0 \leqslant p_s \leqslant 1.$$

Claim 4. *For any $s \in [0,1]$, there exists a unique solution to (28).*

Claim 5. *Under Assumption A_1 the path is admissible: we have, for any $s \in [0,1]$,*

$$0 \leqslant p_s \leqslant 1. \tag{29}$$

Furthermore, the path $\{p_s\}_{s\in[0,1]}$ is continuous in the $\mathscr{C}^0$ topology.

Proof of Claim 5.

1. Admissibility of the path under Assumption A_1: We now prove Estimate (29), which proves that the path of steady states is admissible (with respect to the constraints).

   To do so, we introduce the energy functional

   $$\mathscr{E}_1 : x \mapsto \frac{1}{2}(p_s'(x))^2 + F(u(x)).$$

   Differentiating $\mathscr{E}_1$ with respect to x, we get

   $$\begin{aligned} \mathscr{E}_1'(x) &= \left(p_s''(x) + f(p_s)\right) p_s'(x) \\ &= \left(-\frac{d-1}{r} - 2\frac{N'(r)}{N(r)}\right)(p_s'(r))^2 && \text{from Equation (28)} \\ &\leqslant 0 && \text{from Hypothesis } A_1. \end{aligned}$$

   In particular, we have, for any $s \neq 0$, $p_s \neq 0$ in $(0;R)$: arguing by contradiction if, for $\underline{x} \in (0;R)$ we had $p_s(\underline{x}) = 0$ then

   $$\mathscr{E}_1(\underline{x}) = \frac{1}{2}\left(p_s'(\underline{x})\right)^2 \geqslant 0.$$

   However, $\mathscr{E}_1(0) = F(s\theta) < 0$, so that a contradiction follows. For the same reason, $p_s \neq 1$ in $[0;R]$, for otherwise , if $p_s(\overline{x}) = 1$ at some $\overline{x} \in [0,1]$ we would have

   $$\mathscr{E}_1(\overline{x}) \geqslant F(1) > 0,$$

   which is once again a contradiction. It follows that, for any $s \in (0;1]$,

   $$0 \leqslant p_s \leqslant 1,$$

   as claimed. This concludes the proof of the admissibility of the path.

2. Continuity of the path: We want to prove the $\mathscr{C}^0$ continuity of the path. Let $s \in [0,1]$ and let $\{s_k\}_{k\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ be a sequence such that

   $$s_k \underset{k\to\infty}{\rightarrow} s.$$

Let $p_k := p_{s_k}$. Our goal is to show that

$$p_k \underset{k\to\infty}{\overset{\mathscr{C}^0(\mathbb{B}(0;R))}{\to}} p_s. \tag{30}$$

We will use elliptic regularity to ensure that. We first derive a $W^{1,\infty}$ estimate from the one-dimensional equation and use it to derive a $\mathscr{C}^{2,\alpha}$ estimate for the equation set in $\mathbb{B}(0;R)$.
By the admissibility of the path we have, for every $k \in \mathbb{N}$,

$$0 \leqslant p_k \leqslant 1.$$

Passing into radial coordinates and integrating Equation (28) between 0 and x gives

$$-p_k'(x) = \frac{1}{N^2(x)x^{d-1}} \int_0^x f(p_k(t))\, N^2(t) t^{d-1} dt. \tag{31}$$

From Equation (31) we see that $\{p_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $W^{1,\infty}((0;1))$,
We now consider the Equation in $\mathbb{B}(0;R)$, i.e we work with (27). Since $\{p_k\}_{k\in\mathbb{N}}$ is uniformly bounded in any $\mathscr{C}^{0,\alpha}(\mathbb{B}(0;R))$ by the first step and since $N \in \mathscr{C}^\infty(\mathbb{B}(0;R))$, it follows from Hölder elliptic regularity (see [16] that there exists $M \in \mathbb{R}$ such that, for every $k \in \mathbb{N}$,

$$||p_k||_{\mathscr{C}^{2,\alpha}(\mathbb{B}(0;R))} \leqslant M$$

hence $\{p_k\}_{k\in\mathbb{N}}$ converges in $\mathscr{C}^1(\mathbb{B}(0;R))$, up to a subsequence, to p_∞. Passing to the limit in the weak formulation of the equation, we see that p_∞ satisfies

$$-\nabla \cdot \left(N^2 \nabla p_\infty\right) = f(p_\infty) N^2.$$

Passing to the limit in

$$\forall k \in \mathbb{N}\,, p_k(0) = s_k$$

we get $p_\infty(0) = s$ and, finally, since for every $k \in \mathbb{N}$, p_k is radial, i.e

$$\forall k \in \mathbb{N}\,, \forall i,j \in \{1,\dots,d\}\,, x_j \frac{\partial p_k}{\partial x_i} - x_i \frac{\partial p_k}{\partial x_j} = 0,$$

we can pass to the limit in this identity to obtain that p_∞ is radial. In particular,

$$p_\infty = p_s$$

and so the continuity of the path holds.

□

To conclude the proof of Theorem 3, it suffices to apply the staircase method. □

5 Proof of Theorem 4: Blocking phenomenon

Proof of Theorem 4. The proof that there exists $L_\sigma(0) > 0$ such that, for any $L \geqslant L_\sigma$ a non-trivial solution to (10) exists is exactly the same as for the case without a drift a relies on an energy argument, as is done in the proof of Lemma 1.

Such energy arguments fail however when trying to prove that (11) has a solution, since the natural energy of the equation (11) satisfies, when N is the gaussian,

$$\mathscr{E}[L,u] = \int_{-L}^{L} e^{-\frac{x^2}{\sigma}} (u')^2 - \int_{-L}^{L} e^{-\frac{x^2}{\sigma}} F(u) \geqslant \mathscr{E}[L,z_1] \text{ with } z_1 \equiv 1,$$

because $F(u) \geqslant F(1)$.
To prove the existence of a non-trivial solution, we give a fine study of the phase portrait which will ensure that $L_\sigma(1)$ is well-defined and that $L_\sigma(1) > 0$.
Let $\alpha \in (0;\theta)$. Let $u_{\alpha,\sigma}$ be the solution, in $\mathbb{R}$, of

$$\begin{cases} -u''_{\alpha,\sigma} + 2\frac{x}{\sigma}u'_{\alpha,\sigma} = f(u_{\alpha,\sigma})\,, \\ u_{\alpha,\sigma}(0) = \alpha\,, \\ u'_{\alpha,\sigma}(0) = 0. \end{cases} \tag{32}$$

We first note that $u_{\alpha,\sigma}$ is an even function.
Our goal is to prove that

$$\exists \alpha_\sigma \in [0;\theta] \text{ such that } \Big(\exists x_{\alpha_\sigma} > 0\,,\text{ and } u_{\alpha,\sigma}(x_{\alpha_\sigma}) > 1\,, \forall t \in (0; x_{\alpha_\sigma})\,, 1 > u_{\alpha,\sigma}(t) > 0.\Big). \tag{33}$$

If this holds, choosing α_σ and defining $L_\sigma(1) := x_{\alpha_\sigma}$ automatically yields the desired conclusion.
Since we use in this proof energy arguments, let us define, for any $x \in \mathbb{R}_+$,

$$\mathscr{E}_\alpha(x) := \frac{1}{2}(u'_{\alpha,\sigma}(x))^2 + F(u_{\alpha,\sigma}(x))$$

and carry out some elementary computations.
First of all, we notice that

$$\begin{aligned} \frac{d}{dx}\mathscr{E}_\alpha(x) &= \frac{2x}{\sigma}u'_{\alpha,\sigma}(x)^2 \\ &= \frac{4x}{\sigma}\left(\mathscr{E}_\alpha(x) - F(u_{\alpha,\sigma}(x)\right) \\ &\geqslant \frac{4x}{\sigma}\left(\mathscr{E}_\alpha(x) - F(1)\right). \end{aligned}$$

Proof of (33). We start with the following result:

Claim 6. *For any $\sigma, \alpha > 0$, there exists $x_{\alpha,\sigma,\theta} > 0$ such that*

$$u_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) = \theta\,, 0 < u_{\alpha,\sigma} < \theta \text{ on } (0; x_{\alpha,\sigma,\theta})\,, u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) > 0\,, u'_{\alpha,\sigma} > 0 \text{ on } (0; x_{\alpha,\sigma,\theta}).$$

Proof of Claim 6. Since $u_{\alpha,\sigma}$ is continuous and since $u_{\alpha,\sigma}(0) = \alpha < \theta$, there exists $\delta > 0$ such that

$$u_{\alpha,\sigma}([0;\delta]) \subset [0;\theta].$$

Let $x_{\alpha,\sigma,\theta}$ be defined as

$$x_{\alpha,\sigma,\theta} := \sup\{\delta > 0\,, u_{\alpha,\sigma}([0;\delta]) \subset [0;\theta]\} > 0.$$

Note that we might have $x_{\alpha,\sigma,\theta} = +\infty$, but we will exclude this case: we prove that $u_{\alpha,\sigma}$ is increasing on $[0; x_{\alpha,\sigma,\theta}]$ and, we show at the same time, that $x_{\alpha,\sigma,\theta} < +\infty$.

As a consequence, $u_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) = \theta$ and $0 < u_{\alpha,\sigma} < \theta$ on $[0; x_{\alpha,\sigma,\theta}]$.

$\underline{u_{\alpha,\sigma} \text{ is increasing on } [0; x_{\alpha,\sigma,\theta})}$: On $[0; x_{\alpha,\sigma,\theta})$, we have $f(u_{\alpha,\sigma}) < 0$, whence

$$u''_{\alpha,\sigma}(x) > \frac{2x}{\sigma}u'_{\alpha,\sigma}(x). \tag{34}$$

From the Grönwall inequality, it follows that, for every $x \in (0; x_{\alpha,\sigma,\theta})$, $u'_{\alpha,\sigma}(x) > 0$. Thus, $u_{\alpha,\sigma} \geqslant \alpha$ on $(0; x_{\alpha,\sigma,\theta})$ so

$$u_{\alpha,\sigma}(x) \underset{x \to x_{\alpha,\sigma,\theta}}{\to} \theta.$$

$\underline{x_{\alpha,\sigma,\theta} < \infty}$: Let $\overline{x} \in (0; x_{\alpha,\sigma,\theta})$. For any $x \in (\overline{x}; x_{\alpha,\sigma,\theta})$ the Grönwall inequality applied to Equation (34) gives

$$u'_{\alpha,\sigma}(x) \geqslant e^{\frac{2}{\sigma}(x^2-\overline{x}^2)} u'_{\alpha,\sigma}(\overline{x}) > u'_{\alpha,\sigma}(\overline{x}) > 0.$$

It follows that $x_{\alpha,\sigma,\theta} < \infty$ and so $u_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) = \theta$, $u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) > 0$ and $u_{\alpha,\sigma}(x) > 0$ on $(0; x_{\alpha,\sigma,\theta}]$. □

Claim 7. *There holds*

$$x_{\alpha,\sigma,\theta} \underset{\alpha\to 0\,,\alpha>0}{\to} +\infty.$$

Proof of Claim 7. This is a consequence of the Grönwall inequality applied to

$$\xi(x) := \frac{1}{2}(u_{\alpha,\sigma}^2 + u'^2_{\alpha,\sigma}).$$

First of all note that we know from Claim 6 that $u_{\alpha,\sigma}$, $u'_{\alpha,\sigma} > 0$ on $[0; x_{\alpha,\sigma,\theta}]$. Let $L > 0$ be defined as

$$L := \sup_{x\in[0,1]} \frac{-f(t)}{t} > 0.$$

Differentiating ξ gives

$$\begin{aligned}
\frac{d\xi}{dx} &= u'_{\alpha,\sigma}(x)\left(u_{\alpha,\sigma}(x) + u''_{\alpha,\sigma}(x)\right)\\
&= u'_{\alpha,\sigma}(x)\left(u_{\alpha,\sigma}(x) - f(u_{\alpha,\sigma}) + 2\frac{x}{\sigma}u'_{\alpha,\sigma}(x)\right)\\
&\leqslant u'_{\alpha,\sigma}(x)\left(u_{\alpha,\sigma}(x) + Lu_{\alpha,\sigma}(x) + 2\frac{x}{\sigma}u'_{\alpha,\sigma}(x)\right)\\
&\leqslant u'_{\alpha,\sigma}(x)u_{\alpha,\sigma}(x)\,(L+1) + 2\frac{x}{\sigma}u'_{\alpha,\sigma}(x)^2\\
&\leqslant \frac{L+1}{2}(u'_{\alpha,\sigma}(x)^2 + u_{\alpha,\sigma}(x)^2) + 2\frac{x}{\sigma}(u'_{\alpha,\sigma}(x)^2 + u_{\alpha,\sigma}(x)^2)\\
&\leqslant \xi(x)\left(L+1+4\frac{x}{\sigma}\right).
\end{aligned}$$

Since $\xi(0) = \frac{1}{2}\alpha^2$ we conclude from Grönwall's lemma that

$$\xi(x) \leqslant \frac{\alpha^2}{2}e^{(L+1)x+2\frac{x^2}{\sigma}}.$$

As a consequence, at $x_{\alpha,\sigma,\theta}$, we must have

$$\theta \leqslant \sqrt{\xi(x_{\alpha,\sigma,\theta})} \leqslant \frac{\alpha}{\sqrt{2}}e^{\frac{L+1}{2}x_{\alpha,\sigma,\theta}+\frac{x^2_{\alpha,\sigma,\theta}}{\sigma}}.$$

Thus, we have

$$x_{\alpha,\sigma,\theta} \underset{\alpha\to 0}{\to} +\infty,$$

as claimed. □

Claim 8. *Let $x_{\alpha,\sigma,\theta}$ be defined as*

$$x_{\alpha,\sigma,\theta} := \inf\left\{x\,, x \textit{ satisfies the conclusion of Claim 6}\right\}.$$

For any $\sigma > 0$, there exists $\alpha \in (0;\theta)$ such that

$$u'_\alpha(x) \underset{x\to\infty}{\to} +\infty\,, u'_{\alpha,\sigma}(x) > 0 \textit{ on } [x_{\alpha,\sigma,\theta}; +\infty).$$

Proof of Claim 8. We first notice that

$$\frac{d}{dx}u'_{\alpha,\sigma} = -f(u_{\alpha,\sigma}) + \frac{2x}{\sigma}u'_{\alpha,\sigma}(x) =: g(x) \tag{35}$$

The non-linearity changes sign at $x_{\alpha,\sigma,\theta}$.

We note that $g(x_{\alpha,\sigma,\theta}) = u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta})) > 0$. We want to ensure that the right-hand side of (35) enjoys some monotonicity property.

To guarantee this, we first note that, on $[0; x_{\alpha,\sigma,\theta}]$,

$$\frac{d}{dx}\mathscr{E}_\alpha(x) = \frac{2x}{\sigma}u'_{\alpha,\sigma}(x)^2 \geqslant 0$$

and so

$$\mathscr{E}_\alpha(x) \geqslant \mathscr{E}_\alpha(0) = F(\alpha).$$

Thus,

$$u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) \geqslant \sqrt{2\left(F(\alpha) - F(\theta)\right)}.$$

We hence assume that $\alpha \leqslant \frac{\theta}{2}$, which implies $F(\alpha) \geqslant F\left(\frac{\theta}{2}\right)$. This gives a uniform lower bound of the form

$$u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) \geqslant c_0 > 0.$$

Now, regarding the monotonicity of the right-hand side of (35), we note that

$$\begin{aligned}
g'(x) &= -u'_{\alpha,\sigma}f'(u_{\alpha,\sigma}) + \frac{2}{\sigma}u'_{\alpha,\sigma} + \frac{2x}{\sigma}\left(\frac{2x}{\sigma}u'_{\alpha,\sigma} - f(u_{\alpha,\sigma})\right) \\
&= u'_{\alpha,\sigma}\left(-f'(u_{\alpha,\sigma}) + \frac{2}{\sigma} + 4\frac{x^2}{\sigma^2} - \frac{f(u_{\alpha,\sigma})}{u'_{\alpha,\sigma}}\right) \\
&= u'_{\alpha,\sigma}G(x, u_{\alpha,\sigma}, u'_{\alpha,\sigma})
\end{aligned}$$

with

$$G(x,u,v) := -f'(u) + \frac{2}{\sigma} + 4\frac{x^2}{\sigma^2} - \frac{f(u)}{v}.$$

We now want to ensure the following condition:

$$\forall v \geqslant c_0\,, \forall x \geqslant x_{\alpha,\sigma,\theta}\,, G(x,u,v) \geqslant 0. \tag{36}$$

Extending if need be f into a $W^{1,\infty}$ function outside of $[0,1]$, we see that this condition is guaranteed if, for any $x \geqslant x_{\alpha,\sigma,\theta}$ we have

$$||f'||_{L^\infty} + \frac{||f||_{L^\infty}}{c_0} \leqslant \frac{2}{\sigma} + 4\frac{x^2}{\sigma^2}. \tag{37}$$

This is turn implies a condition on $x_{\alpha,\sigma,\theta}$, and we need to guarantee that $x_{\alpha,\sigma,\theta}$ can be chosen arbitrarily large as $\alpha \to 0$. However this is a consequence of Claim 7.

As a consequence, coming back to (35), we see that, since g is locally positive because $g(x_{\alpha,\sigma,\theta}) = u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) > c_0$ we can define

$$A_1 := \sup\{A \in \mathbb{R}_+^*\,, u'_{\alpha,\sigma} \geqslant u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) \text{ in } [x_{\alpha,\sigma,\theta}; x_{\alpha,\sigma,\theta} + A]\} > 0$$

and we now show that

$$A_1 = +\infty.$$

We first note that g is non-decreasing on $[x_{\alpha,\sigma,\theta}; x_{\alpha,\sigma,\theta}+A]$, by (36) and since $g' = u'_{\alpha,\sigma} G(x, u_{\alpha,\sigma}, u'_{\alpha,\sigma})$.

That $A_1 = \infty$ is now an easy consequence of this fact: indeed, we have

$$\frac{d}{dx}(u'_{\alpha,\sigma}) = g(x) \geqslant g(x_{\alpha,\sigma,\theta}) > 0$$

and so we have

$$u'_{\alpha,\sigma}(x) \underset{x\to\infty}{\to} +\infty.$$

□

As a consequence of these two claims, there exists $x_{\alpha,\sigma,1}$ such that

$$u_\alpha(x_{\alpha,\sigma,1}) = 1\,, u_{\alpha,\sigma} > 0 \text{ on } [0; x_{\alpha,\sigma,1}].$$

This concludes the proof of (33).

□

This concludes the proof of the existence part of Theorem 4 by setting $L_\sigma^{(1)} = x_{\alpha,\sigma,1}$. Let us set

$$L^*_\sigma(1) := \inf\{L > 0\,, (11) \text{ has a non-trivial solution in } [-L, L]\} > 0.$$

We now prove that $L^*_\sigma(1) \underset{a\to\infty/0}{\to} \infty/0$. The analysis when $\sigma \to \infty$ is quite easy, while the case $\sigma \to 0$ is harder to tackle.

Claim 9. *It holds*

$$L^*_\sigma(1) \underset{\sigma\to\infty}{\to} +\infty.$$

Proof of Claim 9. Let u_1 be a non-trivial solution of (11). We can assume that u_1 is a radially symmetric solution, i.e $u_1(x) = u_1(-x)$. Assume this is not the case, and set x_1 such that

$$u_1(x_1) = \min u_1 < \theta$$

by the maximum principle. We can assume without loss of generality that $x_1 > 0$. Define

$$\varphi_1 : [0; L] \ni x \mapsto u_1(x_1)\mathbb{1}_{[0;x_1]} + u_1(x)$$

and extend it by parity to $[-L, L]$. Then φ_1 is a radially symmetric supersolution of the equation, and $z_0 \equiv 0$ is a subsolution of the same equation. The constructive iterative procedure of the construction of sub and super solutions gives the existence of a radially symmetric solution to the equation.

Thus we assume that u_1 is even.

Let $\alpha := u_1(0)$, we then have $u_1 = u_{\alpha,\sigma}$, where $u_{\alpha,\sigma}$ was constructed in the first part of the proof of the Theorem.

We start by noticing that integrating Equation (32) we have, for every $x \in [0; L]$,

$$\begin{aligned} e^{-\frac{x^2}{\sigma}} u'_{\alpha,\sigma}(x) &= \int_0^x (-f(u_{\alpha,\sigma}))(t) e^{-\frac{t^2}{\sigma}}\,dt \\ &\leqslant \int_0^x ||f||_{L^\infty} \end{aligned}$$

Thus, for any $x > 0$, we have

$$u'_{\alpha,\sigma}(x) \leqslant e^{\frac{x^2}{\sigma}} x ||f||_{L^\infty}.$$

Integrating this inequality between 0 and L, we get

$$1 \leqslant \frac{e^{\frac{L^2}{\sigma}} L^2 ||f||_{L^\infty}}{2}.$$

As a consequence, $L^*_\sigma(1)$ can not stay bounded as $\sigma \to \infty$, which concludes the proof. □

We now pass to the proof of the following Claim:

Claim 10. *There holds*

$$L^*_\sigma(1) \underset{\sigma \to 0}{\to} 0.$$

Proof of Claim 10. We argue by contradiction. Assume that there exists a sequence $\{\sigma_k\}_{k\in\mathbb{N}}$ such that

$$L^*_{\sigma_k}(1) \underset{k\to\infty}{\not\to} 0 \,, \sigma_k \underset{k\to\infty}{\to} 0$$

Let

$$\underline{L} := \underline{\lim}_{k\to\infty} L^*_{\sigma_k}(0) > 0.$$

Let $\alpha > 0$ be fixed. From Claim 6 we know that, for every $\sigma > 0$, there exists $x_{\alpha,\sigma,\theta} > 0$ such that

$$u_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) = \theta \,, u_{\alpha,\sigma} \text{ is increasing on } [0; x_{\alpha,\sigma,\theta}].$$

Let

$$u_k := u_{\alpha,\sigma_k} \,, \quad x_k := x_{\alpha,\sigma_k,\theta}.$$

We reach a contradiction by distinguishing two cases:

1. <u>0 is an accumulation point of $\{x_k\}$</u>: Up to a subsequence, we can assume that

   $$x_k \underset{k\to\infty}{\to} 0.$$

   From the mean value theorem, there exists $\{y_k\}_{k\in\mathbb{N}}$ such that

   $$y_k \underset{k\to\infty}{\to} 0 \,, u_k'(y_k) = \frac{\theta - \alpha}{x_k} \underset{k\to\infty}{\to} +\infty.$$

   This implies that $u_k' \to +\infty$ "uniformly on $[y_k; y_k + \varepsilon]$ for every ε small enough" as made precise in the following statement:

   $$\forall \varepsilon > 0 \,, \forall M \in \mathbb{R}^*_+ \,, \exists k_M \in \mathbb{N} \,, \forall k \geqslant k' \,, u_k' \geqslant M \text{ on } [y_k; y_k + \varepsilon].$$

   This is once again an application of the Grönwall Lemma: we have

   $$\frac{d}{dx} u_k' \geqslant -||f||_{L^\infty} + \frac{2x}{\sigma} u_k'.$$

   This implies

   $$\forall t \geqslant 0 \,, u_k'(y_k + t) \geqslant (u_k'(y_k) - ||f||_{L^\infty} t)\, e^{\frac{(y_k+t)^2 - y_k^2}{\sigma}},$$

   giving the desired conclusion.
   Fixing $\varepsilon = \frac{\underline{L}}{2}$ and using $u_k'(y_k) \underset{k\to\infty}{\to} +\infty$ gives the desired conclusion.

   It immediately follows that, for k large, enough, there exists $x_{k,1}$ such that $|x_{k,1} - y_k| \leqslant \frac{\underline{L}}{2}$, $u_k(x_{k,1}) = 1$ and $u_k > 0$ on $(0; x_{k,1})$, which is obviously a contradiction.

2. <u>0 is not an accumulation point of $\{x_k\}$</u>: Assuming 0 is not an accumulation point of $\{x_k\}_{k\in\mathbb{N}}$, a contradiction ensues in the following manner: we now that there thus exists a point $\underline{x} > 0$ such that
$$y \leqslant \underline{\lim}_{k\to\infty} x_k \,,\, \lim_{k\to\infty} u_k(y) \leqslant \theta - \delta$$
for some $\delta > 0$. Then we note that, by integration of (32) we get
$$u_k'(y) = e^{\frac{y^2}{\sigma}} \int_0^y e^{\frac{-t^2}{\sigma}}(-f(u_k(t))dt. \tag{38}$$
Since $y \in [0; x_k]$ for every k large enough, we have $\alpha \leqslant u_k \leqslant \theta - \delta$ for every $t \in [0; y]$, so that
$$\exists \delta' > 0 \,,\, f(u_k) \leqslant -\delta' \text{ on } [0; y].$$
Plugging this in the integral formulation (38) gives the lower bound
$$u_k'(y) \geqslant \delta e^{\frac{y^2}{\sigma}} \int_0^y e^{-\frac{t^2}{\sigma}} dt.$$
We estimate the right hand side using Laplace's method:
$$\int_0^y e^{-\frac{t^2}{\sigma}} dt \underset{\sigma\to 0}{\sim} C\sqrt{\sigma}$$
for some $C > 0$, which immediately gives
$$u_k'(y) \underset{k\to\infty}{\to} +\infty.$$
We conclude as in the first case.

□

We use the same type of arguments to analyse the behaviour of $L^*_\sigma(0)$ as $\sigma \to 0$. Obviously, when $\sigma \to \infty$, $L^*_\sigma(0)$ goes to L^*, the threshold for existence of a non-trivial solution to (10) already studied in [33]. We now prove that adding a gaussian drift yields the existence of a non-trivial drift with a small variance σ leads to the existence of non-trivial solutions around 0, even when the length of the interval is quite small.

Claim 11. *There holds*
$$L^*_\sigma(0) \underset{\sigma\to 0}{\to} 0.$$

Proof of Claim 11. Here we only need to prove
$$\forall L > 0 \,,\, \exists \sigma_L > 0 \,,\, \forall \sigma \leqslant \sigma_L \,,\ (10) \text{ has a non-trivial solution,}$$
which is stronger that what we require.

To do so, we use an energy argument similar to that of the proof of Theorem 2: introduce, for a given $L > 0$, the energy functional of Equation (10):
$$\mathscr{E}^\sigma : W^{1,2}_0((-L, L]) \ni u \mapsto \frac{1}{2}\int_L^L |\nabla u|^2 e^{-\frac{x^2}{\sigma}} - \int_L^L F(u) e^{-\frac{x^2}{\sigma}}.$$
We now consider a smooth function $\varphi \in \mathscr{C}^\infty((-L; L))$ with compact support and with
$$0 \leqslant \varphi \leqslant \varphi(0) = 1 \,,\, \varphi \equiv 1 \text{ on } (-\frac{L}{2}; \frac{L}{2}).$$

We apply the Laplace method to

$$\mathscr{E}^{\sigma}[\varphi].$$

We first note that we immediately have, since φ is is a fixed test function with a zero derivative on a neighborhood of 0,

$$\int_{L}^{L} |\nabla u|^2 e^{-\frac{x^2}{\sigma}} = \underset{\sigma\to 0}{o}(\sqrt{\sigma}).$$

On the other hand, the right hand side satisfies, thanks to the Laplace methode,

$$\int_{L}^{L} F(u) e^{-\frac{x^2}{\sigma}} \underset{\sigma\to 0}{\sim} CF(1)\sqrt{\sigma}$$

where $C > 0$ is a positive constant. Thus, for σ small enough, we have

$$\mathcal{E}^{\sigma}[\varphi] < 0,$$

hence the existence of a non-trivial solution for (10).

□

□

6 Conclusion

6.1 Obtaining the results for general coupled systems

As explained in Section 1.2 of the Introduction, the equations considered in this article correspond to some scaling limits for more general coupled systems of reaction-diffusion equations, and it seems interesting to investigate whether or not the results we obtained in this article might be generalized to encompass the case of such general systems. As was explained in Section 1.2, these models can be used to control populations of infected mosquitoes and arise in evolutionary dynamics. Obtaining a finer understanding of the real underlying dynamics rather than the simplified version under scrutiny here seems, however, challenging. Indeed, although controllability results for linear systems of equations exist (see for instance [23]), the non-linear case has not yet been completely studied.

However, given that, as explained in the Introduction, gene-flow models and spatially heterogeneous models are limits in a certain scaling of such systems, it would be interesting to see whether or not our perturbation arguments, that were introduced to pass from the spatially homogeneous model the the slowly varying one, could work to pass from this scaling limit to the whole system in a certain regime.

6.2 Open problem: the minimal controllability time and spatial heterogeneity

Let us now list a few questions which, to the best of our knowledge, are still open and seem worth investigating.

- The qualitative properties of time optimal controls:

  As suggested in [33] one might try to optimize the control with respect to the controllability time. Indeed, its is known that, under constraints on the control, parabolic equations have a minimal controllability time, see for instance [40, 32].

For constrained controllability it is known that there exists a minimal controllability time to control, for instance, from 0 to θ (see [33]). We may try to optimize the control strategies so as to minimize the controllability time. In our case, that is, the spatially heterogeneous case, are these controls of bang-bang type? Another qualitative question that is relevant in this context is that of symmetry: in the one dimensional case, when working on an interval $[-L, L]$, are time-optimal controls symmetric? In the multi-dimensional case, when the domain Ω is a ball, is it possible to prove radial symettry of time optimal controls?

- The influence of spatial heterogeneity on controllability time:

  Adding a drift (which corresponds to the spatially heterogeneous model) modifies the controllability time. As we have seen, such heterogeneities might lead to a lack of controllability. However, it is also suggested in the numerical experiments shown below that adding a drift might be beneficial for the controllability time. It might be interesting to consider the following question: given L^∞ and L^1 bounds on the spatial heterogeneity N, which is the drift yielding the minimal controllability time? In other terms: how can we design the domain so as to minimize the controllability time? In the simulation below, we thus considered the following optimization problem: letting, for any drift N, $T(N)$ be the minimal controllability time from 0 to θ of the spatially heterogeneous equation (2) (with $T(N) \in (0; +\infty]$), solve

$$\inf_{-M \leqslant M \leqslant 1\,,\int_{-L}^{L} N=0} T(N).$$

  We obtain the following graph with $M = 250$ and $L = 2.5$:

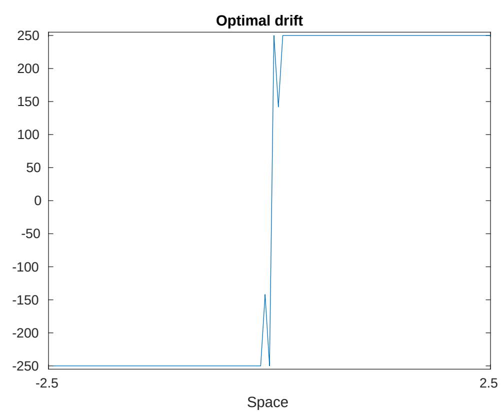

Figure 11: Time optimal spatial heterogeneity.

A Proof of Lemma 1

Proof of Lemma 1. Let us first remark that (13) has a variational structure. Indeed, u is a solution of

$$-\Delta u + \varepsilon\langle \nabla n\,, \nabla u\rangle = f(u)\,, u \in W_0^{1,2}(\Omega)$$

if and only if

$$-\nabla \cdot (e^{\varepsilon n}\nabla u) = f(u)e^{\varepsilon n}\,, u \in W_0^{1,2}(\Omega). \tag{39}$$

Following the arguments of [6, Remark II.2], we introduce the energy functional associated with (39): let

$$\mathscr{E}_1 : W_0^{1,2}(\Omega) \ni u \mapsto \frac{1}{2}\int_\Omega e^{\varepsilon n}|\nabla u|^2 - \int_\Omega e^{\varepsilon n} F(u),$$

From standard arguments in the theory of sub and super solutions [6], if there exists $v \in W_0^{1,2}(\Omega)$ such that

$$\mathscr{E}_1(v) < 0 \tag{40}$$

then there exists a non-trivial solution to (13). We now prove that there exists $v \in W_0^{1,2}(\Omega)$ such that (40) holds, by adapting the construction of [35]: let $\mathbb{B}(\overline{x}; \rho_\Omega)$ be one of the ball of maximum radius inscribed in Ω. Up to a translation, we assume that $\overline{x} = 0$.
Let $\delta > 0$. We define v_δ as follows

$$v_\delta : \begin{cases} x \in \mathbb{B}(0; \rho_\Omega - \delta) \mapsto 1, \\ x \in \mathbb{B}(0; \rho_\Omega)\backslash\mathbb{B}(0; \rho_\Omega - \delta) \mapsto \frac{\rho_\Omega^2 - ||x||^2}{\rho_\Omega^2 - (\rho_\Omega - \delta)^2}, \\ x \in \Omega\backslash\mathbb{B}(0; \rho_\Omega) \mapsto 0. \end{cases}$$

An explicit computation yields

$$\int_\Omega |\nabla v|^2 \sim_{\delta \to 0} C\rho_\Omega^d$$

for some constant $C > 0$, and

$$\int_\Omega F(v) = F(1)\rho_\Omega^d + \underset{\delta \to 0}{\mathscr{O}}(\rho_\Omega^{d-1}).$$

Hence, since n is bounded, the conclusion: as $\rho_\Omega \to \infty$ ad $\delta \to 0$ the energy of v_1 is negative.

□

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