Multiplicative controllability for parabolic equations

Giuseppe Floridia
Dipartimento di Patrimonio, Architettura, Urbanistica (PAU),
Università Mediterranea di Reggio Calabria

(joint works with U. Biccari, P. Cannarsa, A. Y. Khapalov, C. Nitsch, C. Trombetti)

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Outline

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1. Introduction
   - Control theory
2. 1-D reaction-diffusion equations
   - Main results: multiplicative controllability for sign changing states
3. Two applications
   - m-D reaction-diffusion equations with radial symmetry
     - An idea of the proof
   - Degenerate equations
     - Motivations: Energy balance models in climatology
4. Open problems

Outline

Controllability (linear heat equation)

$$\begin{cases} u_t = \Delta u + vu + h(x,t)\mathbb{1}_\omega \\ u_{|\partial\Omega} = 0 \qquad (\omega \subset \Omega) \\ u_{|t=0} = u_0 \end{cases} \qquad \begin{cases} u_t = \Delta u + vu \\ u_{|\partial\Omega} = g(t) \\ u_{|t=0} = u_0 \end{cases} \qquad \begin{cases} u_t = \Delta u + v(x,t)u \\ u_{|\partial\Omega} = 0 \\ u_{|t=0} = u_0 \end{cases}$$

Additive controls (locally distributed source terms)

Boundary controls

Bilinear controls (multiplicative controllability)

Definition (Exact controllability)

$\forall u_0 \in H_0, u^* \in H^*, (H_0, H^* \subseteq L^2(\Omega)), \exists$ *"a control function"*, $T > 0$ *such that* $u(\cdot, T) = u^*$.

Definition (Approximate controllability)

$\forall u_0 \in H_0, u^* \in H^*, (H_0, H^* \subseteq L^2(\Omega)), \forall \varepsilon > 0, \exists$ *"a control function"*, $T > 0$ *such that*

$$\|u(\cdot, T) - u^*\|_{L^2(\Omega)} < \varepsilon.$$

Regularizing effect of the heat equation and obstruction to exact controllability: $H^* \subset H_0 = L^2(\Omega)$:

$$u_0 \in L^2(\Omega) \Longrightarrow \exists! u \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega));$$

$$\Longrightarrow u(\cdot, t) \in H_0^1(\Omega), \forall t > 0.$$

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$$u_t - \Delta u = v(x,t)u + h(x,t)\mathbb{1}_\omega, \qquad \omega \subset \Omega$$

$\forall u_0 \in L^2(\Omega), u^* \in H$(“suitable”$H \subset L^2(\Omega)$)$, \exists \omega \subseteq \Omega, h, T > 0$ such that $u(\cdot, T) = u^*$.

Reference

H. Fattorini, D. Russell Exact controllability theorems for linear parabolic equations in one space dimension Arch. Rat. Mech. Anal., **4**, *(1971) 272–292*

Additive controllability by a duality argument (J.L. Lions, 1989): observability inequality and Hilbert Uniqueness Method (HUM).
In literature see also, many papers due to Komornik, Cannarsa, Haraux, Zuazua, ...

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- nonlinear problem: Nash-Moser theorem (Hörmander version);
- controllability of the linearized problem;
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Additive vs multiplicative controllability

Multiplicative controllability and Applied Mathematics

$$\Rightarrow \quad \text{rather than} \qquad\qquad \downarrow$$

$$u_t - \Delta u = \quad v(x,t) \quad u + h(x,t)$$

$$\text{use} \qquad \uparrow \qquad \text{as control variable}$$

Remark

$$\Phi : \text{ “control”} \longmapsto \text{“solution”}$$

Additive controls vs *Bilinear controls*

$\Phi : h \longmapsto u$ *is a linear map;* $\Phi : v \longmapsto u$ *is a nonlinear map.*

Some references on bilinear control of PDEs

- Ball, Marsden and Slemrod (1982) [rod and wave equation]
- Coron, Beauchard, Morancey, Gagnon [Scrödinger equation]
- Khapalov (2002–2010) [parabolic and hyperbolic equations, swimming models]

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Control theory & Reaction-diffusion equations

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$\Omega \subseteq \mathbb{R}^m$ bounded

$$\begin{cases} u_t \; = \; \Delta u \; + \; v(x,t)u \; + \; f(u) & \text{in} \quad Q_T := \Omega \times (0,T) \\ u_{|_{\partial\Omega}} = 0 & t \in (0,T) \\ u_{|_{t=0}} \; = u_0 & \end{cases} \tag{1}$$

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Strong maximum principle

$f(u)$ is differentiable at 0 and $f(0) = 0 \quad \Longrightarrow \quad \frac{f(u)}{u} \in L^\infty(Q_T)$

Thus, the strong maximum principle (SMP) can be extend to semilinear parabolic equation:

$$u_t \; = \; \Delta u \; + \; \left(v + \frac{f(u)}{u} \right) u$$

Well-posedness result

$$u_0 \in L^2(\Omega) \Longrightarrow \exists! u \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega));$$

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Strong Maximum Principle and obstruction to multiplicative controllability: $H^* \neq H_0^1(\Omega)$

$$u_0(x) = 0 \Longrightarrow u(x,t) = 0$$
$$u_0(x) \geq 0 \Longrightarrow u(x,t) \geq 0$$

If $u_0(x) \geq 0$ in Ω, then the SMP demands that the respective solution to (1) remains nonnegative at any moment of time, regardless of the choice of v. This means that system (1) cannot be steered from any nonnegative u_0 to any target state which is negative on a nonzero measure set in the space domain.

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H. Matano, Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **29**, *no. 2, (1982) 401-441.*

Controllability:

- Nonnegative states
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Changes of sign

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G. F., Approximate controllability for nonlinear degenerate parabolic problems with bilinear control, Journal of Differential Equations (JDE, Elsevier, 2014).

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We assume that $u_0 \in H_0^1(0,1)$ has *a finite number of points of sign change*, that is, there exist points $0 = x_0^0 < x_1^0 < \cdots < x_n^0 < x_{n+1}^0 = 1$ such that

$$\begin{aligned} u_0(x) = 0 \quad &\Longleftrightarrow \quad x = x_l^0, \quad l = 0,\ldots,n+1. \\ u_0(x)u_0(y) < 0, \quad &\forall x \in \left(x_{l-1}^0, x_l^0\right), \forall y \in \left(x_l^0, x_{l+1}^0\right). \end{aligned}$$

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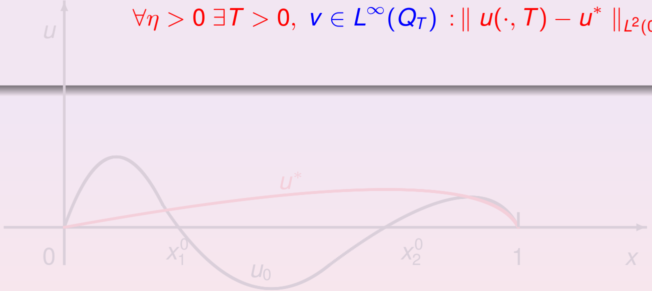
Theorem (P. Cannarsa, G.F., A.Y. Khapalov)

Consider any $u^* \in H_0^1(0,1)$, *whose amount of points of sign change is less or equal than to the amount of such points for* u_0. *Then,*

$$\forall \eta > 0 \; \exists T > 0, \; v \in L^\infty(Q_T) \; : \| \, u(\cdot, T) - u^* \, \|_{L^2(0,1)} \leq \; \eta.$$



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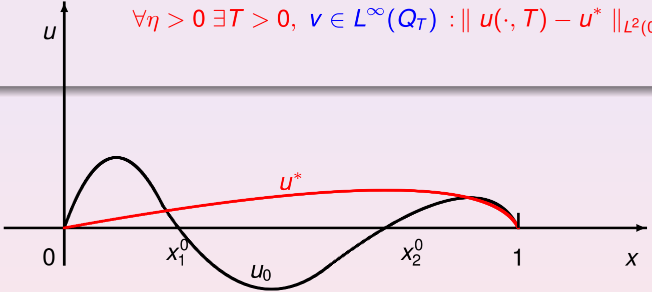

Let $u_0 \in H_0^1(0,1)$ have *a finite number of points of sign change.*

Theorem (P. Cannarsa, G.F., A.Y. Khapalov)

Consider any $u^* \in H_0^1(0,1)$ *which has exactly as many points of sign change in the same order as* u_0. *Then,*

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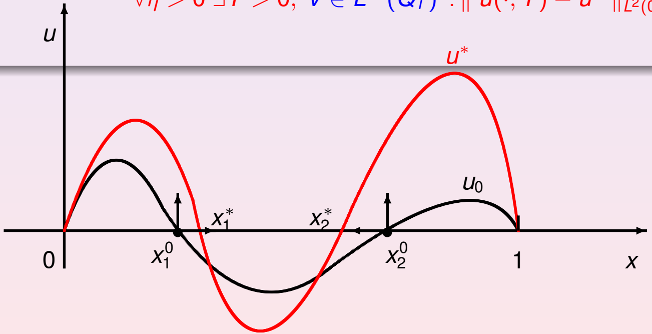

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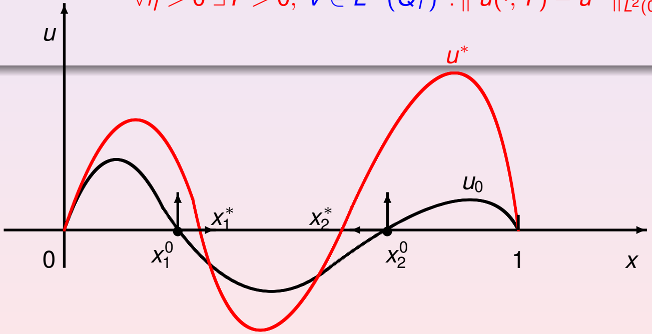
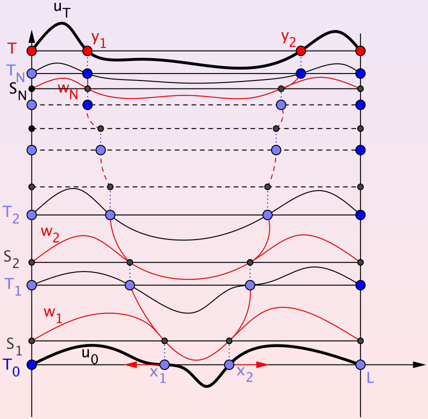


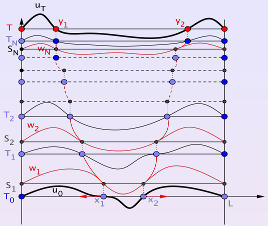
Control strategy

Given $N \in \mathbb{N}$ (N will be determined later) we consider the following partition of $[0, T]$ in $2N + 1$ intervals:

$$[0, S_1] \cup [S_1, T_1] \cup \cdots \cup [T_{k-1}, S_k] \cup [S_k, T_k] \cup \cdots \cup [T_{N-1}, S_N] \cup [S_N, T_N] \cup [T_N, T]$$

$$v_1 \neq 0 \qquad 0 \qquad \cdots \qquad v_k \neq 0 \qquad 0 \qquad \cdots \qquad v_N \neq 0 \qquad 0 \qquad v_{N+1} \neq 0$$





Construction of the zero curves

- On $[S_k, T_k]$ $(1 \leqslant k \leqslant N)$ we use of the Cauchy datum $w_k \in C^{2+\theta}([0,1])$ in

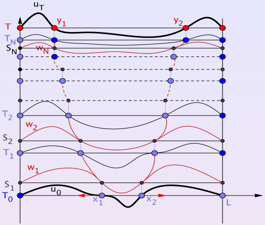
$$\begin{cases} w_t = w_{xx} + f(w), & \text{in } (0,1)\times[S_k,T_k], \\ w(0,t) = w(1,t) = 0, & t \in [S_k, T_k], \\ w\,|_{t=S_k} = w_k(x), & w_k''(x)\,|_{x=0,1} = 0, \end{cases}$$

as a control parameter to be chosen to generate and to move the curves of sign change.

- The ℓ-th curve of sign change $(1 \leqslant \ell \leqslant n)$ is given given by solution ξ_ℓ^k

$$\begin{cases} \dot{\xi}_\ell^k(t) = -\frac{w_{xx}(\xi_\ell(t),t)}{w_x(\xi_\ell(t),t)}, & t \in [S_k, T_k] \\ \xi_\ell^k(S_k) = x_\ell^k \end{cases}$$

where the x_ℓ^k's are the zeros of w_k and so $w(\xi_\ell^k(t),t) = 0$



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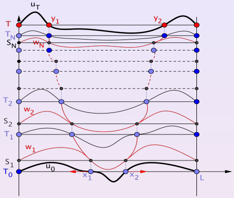

The control parameters w_k's will be chosen to move the curves of sign change in the following way

$$\begin{cases} \dot{\xi}_\ell(t) = -\frac{w_{xx}(\xi_\ell(t),t)}{w_x(\xi_\ell(t),t)}, & t \in [S_k, T_k] \\ \xi_\ell(S_k) = x_\ell^k, \end{cases} \qquad w(\xi_\ell^k(t), t) = 0 \Longrightarrow$$

$$\Longrightarrow \dot{\xi}_\ell(S_k) = -\frac{w_{xx}(\xi_\ell(S_k), S_k)}{w_x(\xi_\ell(S_k), S_k)} = -\frac{w_k''(\xi_\ell(S_k))}{w_k'(\xi_\ell(S_k))} = \operatorname{sgn}(y_l - x_l^0)$$

- To fill the gaps between two successive $[S_k, T_k]$'s, on $[T_{k-1}, S_k]$ we construct the bilinear control v_k that steers the solution of

$$\begin{cases} u_t \;=\; u_{xx} \;+\; v_k(x,t)u \;+\; f(u), & \text{in} \quad (0,1)\times[T_{k-1}, S_k], \\ u(0,t) = u(1,t) = 0, & t \in [T_{k-1}, S_k], \\ u \;|_{t=T_{k-1}} = u_{k-1} + r_{k-1}, & \end{cases}$$

from $u_{k-1} + r_{k-1}$ to w_k, where u_{k-1} and w_k have the same points of sign change, and $\|r_{k-1}\|_{L^2(0,1)}$ is small. $v_k(x,t)$ piecewise static

Sketch of the proof. In the particular case: $r_{k-1} = 0$ and

$$\exists \delta^* > 0: \; \delta^* \leq \frac{w_k(x)}{u_{k-1}(x)} < 1, \qquad \forall\, x \in (0,1) \setminus \bigcup_{l=1}^{n} \{x_l\},$$

let us consider $v_k(x,t) \;:=\; \frac{1}{T}\bar{v}_k(x)$, where

$$\bar{v}_k(x) \;=\; \begin{cases} \log\left(\frac{w_k(x)}{u_{k-1}(x)}\right), & \text{for} \;\; x \neq 0, 1, x_l \;\; (l = 1\ldots, n) \\ 0, & \text{for} \;\; x = 0, 1, x_l \;\; (l = 1\ldots, n). \end{cases}$$

$$u(x,T) \;=\; e^{\bar{v}_k(x)} u_{k-1}(x) \;+\; \int_0^T e^{\bar{v}_k(x)\frac{(T-\tau)}{T}} \left(u_{xx}(x,\tau) + f(x,\tau,u(x,\tau))\right) d\tau$$

$$\Rightarrow \|u(\cdot,T) - w_k(\cdot)\|^2_{L^2(0,1)} \leq \; T \,\|\; u_{xx} + f(\cdot,\cdot,u)\;\|^2_{L^2(Q_T)}.$$

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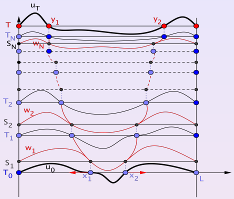
Closing the loop



- The distance-from-target function satisfies the following estimate, for some $C_1, C_2 > 0$ and $0 < \theta < 1$,

$$0 \leq \sum_{\ell=1}^{n} |\xi_\ell^N(T_N) - y_\ell| \leq \sum_{\ell=1}^{n} |x_\ell^0 - y_\ell| + C_1 \sum_{k=1}^{N} \frac{1}{k^{1+\frac{\vartheta}{2}}} - C_2 \sum_{k=n+1}^{N} \frac{1}{k} \overset{N\to\infty}{\longrightarrow} -\infty$$

- So the distances of each branch of the null set of the solution from its target points of sign change decreases at a linear-in-time rate while the error caused by the possible displacement of points already near their targets is negligible
- This ensures, by contradiction argument, that $\sum_{\ell=1}^{n} |\xi_\ell^N(T_N) - y_\ell| < \epsilon$ within a finite number of steps.

Outline

m-d radial case

$$\Omega = \{x \in \mathbb{R}^m : |x| = \sqrt{x_1^2 + \ldots + x_m^2} \leq 1\}$$

$$\begin{cases} u_t = \Delta u + v(x,t)u + f(u) & \text{in } Q_T := \Omega \times (0,T) \\ u_{|\partial\Omega} = 0 & t \in (0,T) \\ u_{|t=0} = u_0 \end{cases}$$

u_0 and $v(\cdot, t)$ radial functions. Moreover, all possible hypersurface (lines) of sign change of u_0 are hyperspheres (circles) with center at the origin.

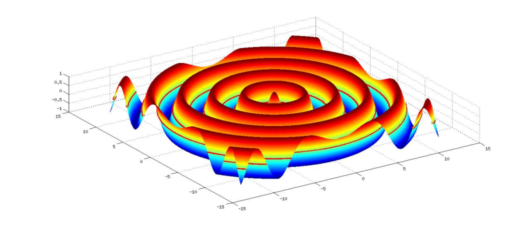


Figure: $u_0(x,y) = \cos(2\sqrt{x^2+y^2})$, initial state

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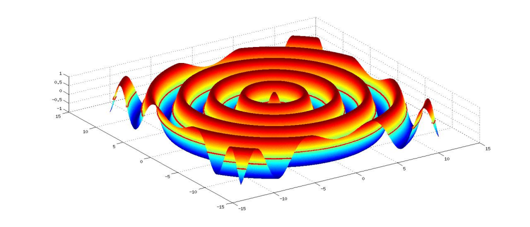

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Main results

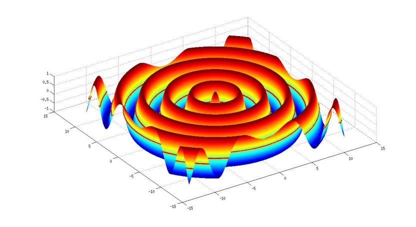
Theorem (G.F.)

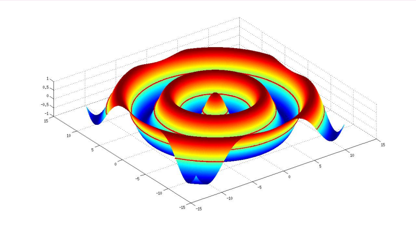
Let $u_0 \in H^2(\Omega) \bigcap H_0^1(\Omega)$. Assume that $u^* \in H^2(\Omega) \bigcap H_0^1(\Omega)$ has as many lines of sign change in the same order as $u_0(x)$. Then,

$$\forall \varepsilon > 0 \, \exists T > 0, v \in L^\infty(Q_T) \text{ such that } \| u(\cdot, T) - u^* \|_{L^2(\Omega)} < \varepsilon.$$

Corollary (G.F.)

The result of Theorem extends to the case when u^* has a lesser amount of lines of sign change which can be obtained by merging those of u_0.





u_0 and $v(\cdot, t)$ radial functions:

$$u_0(x) = z_0(r) \quad \text{and} \quad v(x,t) = V(r,t) \qquad \forall x \in \Omega, \ \forall t \in [0, T]$$

where $r = |x|$. Then,

$$\begin{cases} z_t \ = \ z_{rr} \ + \ \frac{m-1}{r} z_r + \ V(r,t)z \ + \ f(z) & \text{in} \quad (0,1) \times (0,T) \\ \lim\limits_{r \to 0^+} r^{\frac{m-2}{2}} z_r(0,t) = 0 = z(1,t) & t \in (0,T) \\ u_{|_{t=0}} = z_0. & \end{cases}$$

z_0 has finitely many points of change of sign in $[0,1]$, that is, there exist points

$$0 = r_0^0 < r_1^0 < \cdots < r_n^0 < r_{n+1}^0 = 1$$

such that $\lim\limits_{r \to 0^+} r^{\frac{m-2}{2}} z_0'(r) = 0$ and

$$z_0(r) = 0 \quad \Longleftrightarrow \quad r = r_l^0, \quad l = 1, \ldots, n+1$$

$$z_0(r) z_0(s) < 0, \quad \forall r \in \left(r_{l-1}^0, r_l^0\right), \forall s \in \left(r_l^0, r_{l+1}^0\right) \qquad l = 1, \ldots, n.$$

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Outline

Semilinear degenerate problems

$$
\begin{cases}
u_t - (a(x)u_x)_x = \alpha(x,t)u + f(x,t,u) \quad \text{in } Q_T := (-1,1)\times(0,T) \\
\begin{cases}
\begin{cases}
\beta_0 u(-1,t) + \beta_1 a(-1)u_x(-1,t) = 0 & t\in(0,T) \\
\gamma_0\, u(1,t) + \gamma_1\, a(1)\, u_x(1,t) = 0 & t\in(0,T)
\end{cases} & (\text{for } WDP) \\
a(x)u_x(x,t)|_{x=\pm 1} = 0 \qquad t\in(0,T) & (\text{for } SDP)
\end{cases} \\
u(0,x) = u_0(x) \qquad x\in(-1,1)\,.
\end{cases}
$$

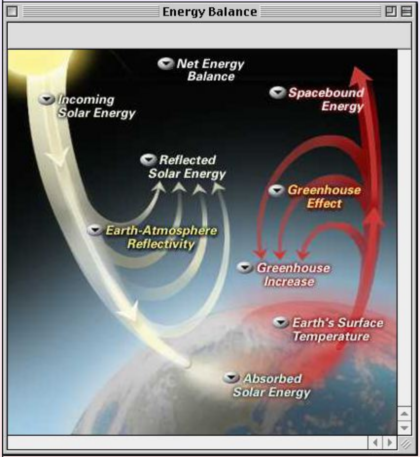
$$\mathbf{a}\in \mathbf{C^0}([-\mathbf{1},\mathbf{1}])\cap \mathbf{C^1}(]-\mathbf{1},\mathbf{1}[)\ :\ \mathbf{a}(\mathbf{x})>\mathbf{0}\ \forall\,\mathbf{x}\in(-\mathbf{1},\mathbf{1}),\ \mathbf{a}(-\mathbf{1})=\mathbf{a}(\mathbf{1})=\mathbf{0}.$$

We distinguish two cases:

- ⋆ $\frac{1}{a}\in L^1(-1,1)$ (WDP), e.g. $a(x)=\sqrt{1-x^2}$, $a\notin C^1([-1,1])$ $(\beta_0\beta_1\le 0,\ \gamma_0\gamma_1\ge 0)$;
- ⋆ $\frac{1}{a}\notin L^1(-1,1)$ (SDP), e.g. $a(x)=1-x^2$ (see later Budyko-Sellers climate model) $a\in C^1([-1,1])$ (assume that $\int_0^x \frac{1}{a(s)}\,ds\in L^{q_\vartheta}(-1,1)$, for same $q_\vartheta\ge 1$).

Energy balance models

The effect of solar radiation on climate



heat variation

$$= R_a - R_e + D$$

- R_a = absorbed energy
- R_e = emitted energy
- D = diffusion

The Budyko-Sellers model (1969)

$\mathcal{M}$ compact surface without boundary (typically S^2)

$$u_t - \Delta_{\mathcal{M}} u = R_a(t,x,u) - R_e(t,x,u)$$

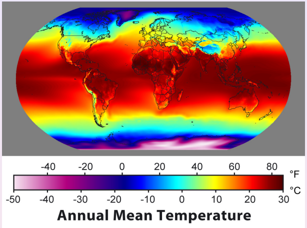
where $u(t,x)$ = temperature distribution

- $R_a(x,u) = Q(t,x)\beta(u)$ $\begin{cases} Q = \text{insolation function} \\ \beta = \text{coalbedo} = 1 - \text{albedo} \end{cases}$
-
  - $R_e(x,u) = A(t,x) + B(t,x)u$ Budyko
  - $R_e(x,u) \simeq c\, u^4$ Sellers

One-dimensional BS model

on $\mathcal{M} = \Sigma^2$, $\quad \Delta_{\mathcal{M}} u = \dfrac{1}{\sin\phi}\left\{ \dfrac{\partial}{\partial\phi}\left(\sin\phi \dfrac{\partial u}{\partial\phi}\right) + \dfrac{1}{\sin\phi}\dfrac{\partial^2 u}{\partial\lambda^2}\right\}$

ϕ = colatitude $\qquad \lambda$ = longitude



taking average at $x = \cos\phi$ BS model reduces to

$$\begin{cases} u_t - \left((1-x^2)u_x\right)_x = g(t,x)\, h(u) + f(t,x,u) & x \in]-1,1[\\ (1-x^2)u_x\big|_{x=\pm 1} = 0 \end{cases}$$

Reference

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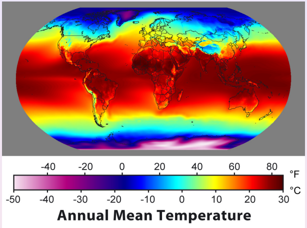

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A prophecy by J. von Neumann

Reference

P. Cannarsa, P. Martinez, J. Vancostenoble, Memoirs AMS, 2016

Nature (1955):

Microscopic layers of colored matter spread on an icy surface, or in the atmosphere above one, could inhibit the reflection-radiation process, melt the ice and change the local climate.

⇒ rather than ↓

$$u_t - \Delta_{\mathcal{M}} u = g(t, \quad x \quad, u) + f(t,x)$$

use ↑ as control variable

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Charles L. Epstein, Rafe Mazzeo, Degenerate Diffusion Operators Arising in Population Biology, book by Princeton University Press, 2011

Semilinear degenerate problems

$$\begin{cases} u_t - (a(x)u_x)_x = \alpha(x,t)u + f(x,t,u) \quad \text{in } \; Q_T := (-1,1)\times(0,T) \\ \begin{cases} \begin{cases} \beta_0 u(-1,t) + \beta_1 a(-1)u_x(-1,t) = 0 & t\in(0,T) \\ & \qquad (\text{for } \; WDP) \\ \gamma_0\, u(1,t) \, + \, \gamma_1\, a(1)\, u_x(1,t) = 0 & t\in(0,T) \end{cases} \\ a(x)u_x(x,t)|_{x=\pm1} = 0 \qquad\qquad t\in(0,T) \quad (\text{for } \; SDP) \end{cases} \\ u(0,x) = u_0(x) \qquad\qquad x\in(-1,1)\,. \end{cases}$$

$$\mathbf{a \in C^0([-1,1]) \cap C^1(]-1,1[) \; : \; a(x) > 0 \; \forall\, x \in (-1,1), \; a(-1) = a(1) = 0.}$$

We distinguish two cases:

- $\star$ $\frac{1}{a} \in L^1(-1,1)$ (WDP), e.g. $a(x) = \sqrt{1-x^2}$, $a \notin C^1([-1,1])$ $(\beta_0\beta_1 \le 0, \; \gamma_0\gamma_1 \ge 0)$;
- $\star$ $\frac{1}{a} \notin L^1(-1,1)$ (SDP), e.g. $a(x) = 1 - x^2$ (see later Budyko-Sellers climate model) $a \in C^1([-1,1])$ (assume that $\int_0^x \frac{1}{a(s)}\, ds \in L^{q_\vartheta}(-1,1)$, for same $q_\vartheta \ge 1$).

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$\alpha \in L^\infty(Q_T)$, (bilinear control), $u_0 \in L^2(-1,1)$;
$f : Q_T \times \mathbb{R} \to \mathbb{R}$ is such that

- $(x,t,u) \longmapsto f(x,t,u)$ is a Carathéodory function; $u \longmapsto f(x,t,u)$ is differentiable at $u=0$; $t \longmapsto f(x,t,u)$ is locally absolutely continuous;
- $\exists \gamma_* \geq 0, \vartheta \geq 1$ and $\nu \geq 0$ such that, for a.e. $(x,t)\in Q_T, \forall u,v\in\mathbb{R}$, we have
$$|f(x,t,u)| \leq \gamma_* \, |u|^\vartheta,$$
$$-\nu\left(1+|u|^{\vartheta-1}+|v|^{\vartheta-1}\right)(u-v)^2 \leq \left(f(x,t,u)-f(x,t,v)\right)(u-v) \leq \nu(u-v)^2,$$
$$f_t(x,t,u)\, u \geq -\nu\, u^2\,;$$

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$$(SD) \quad H_a^1(-1,1) := \{u \in L^2(-1,1) : u \text{ loc. abs. continuous in } (-1,1), \sqrt{a}u_x \in L^2\}\,;$$

$$(WD) \quad H_a^1(-1,1) := \{u \in L^2(-1,1) : u \text{ absolutely continuous in } [-1,1], \sqrt{a}u_x \in L^2\}\,.$$

$$H_a^2(-1,1) := \{u \in H_a^1(-1,1) |\, au_x \in H^1(-1,1)\}$$

Given $T > 0$, let us define the function spaces

$$\mathcal{B}(Q_T) := C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H_a^1(-1,1))$$

$$\mathcal{H}(Q_T) := L^2(0,T; D(A)) \cap H^1(0,T; L^2(-1,1)) \cap C([0,T]; H_a^1(-1,1))$$

Theorem

For all $u_0 \in H_a^1(-1,1)$ there exists a unique strict solution $u \in \mathcal{H}(Q_T)$ to (3).

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Reference

G. F., Approximate controllability for nonlinear degenerate parabolic problems with bilinear control, Journal of Differential Equations (2014).

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Theorem (G.F., JDE 2014)

Consider any $u_0, u^* \in L^2(-1,1)$ *nonnegative. Then,*

$$\forall \eta > 0\ \exists T > 0,\ v \in L^\infty(Q_T)\ : \| u(\cdot, T) - u^* \|_{L^2(-1,1)} \leq \eta.$$

Let $u_0 \in H^1_a(-1,1)$ have *a finite number of points of sign change.*

Theorem (G. F., C. Nitsch, C. Trombetti)

Consider any $u^* \in H^1_0(-1,1)$ *which has exactly as many points of sign change in the same order as* u_0. *Then,*

$$\forall \eta > 0\ \exists T > 0,\ v \in L^\infty(Q_T)\ : \| u(\cdot, T) - u^* \|_{L^2(-1,1)} \leq \eta.$$

Open problems

- Initial states that change sign: to investigate problems in higher space dimensions on domains with specific geometries
  - m-D non-degenerate case without radial symmetry and initial condition that change sign
- To extend this approach to other evolution equations:
  - Fractional heat equations (with Umberto Biccari)
  - Porous media equation
  - p-Laplacian operators
- Exact bilinear/multiplicative controllability (with P. Cannarsa and C. Urbani)
- 1-D degenerate reaction-diffusion equations on networks

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Thank you for your attention!