CONTROL OF CERTAIN PARABOLIC MODELS FROM BIOLOGY AND SOCIAL SCIENCES

DOMÈNEC RUIZ-BALET AND ENRIQUE ZUAZUA

Abstract. These lecture notes address the controllability under relevant state constraints of reaction-diffusion equations. Typically the quantities modeled by reaction-diffusion equations in socio-biological contexts (e.g. population, concentrations of chemicals, temperature, proportions etc) are positive by nature. The uncontrolled models intrinsically preserve this nature thanks to the maximum principle. For this reason, any control strategy for such systems has to preserve these state constraints. We restrict our study in the case of scalar equations with monostable and bistable nonlinearities. The presence of constraints produces new phenomena such as a possible lack of controllability, or existence of a minimal controllability time. Furthermore, we explain general ways for proving controllability under state constraints. Among different strategies, we discuss how to use traveling waves and connected paths of steady states to ensure controllability. We devote particular attention to the construction of such connected paths of steady-states. Further recent extensions are presented, and open problems are settled. All the discussions are complemented with numerical simulations to provide intuition to the reader.

Keywords Reaction-diffusion, Control, Steady-states, Constraints, Mathematical Biology

Contents

0. Introduction

These lecture notes concern the control of reaction-diffusion equations from an application perspective. Many phenomena modeled by reaction-diffusion equations, such chemical reactions,

Funding: This work has been partially funded by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 694126-DyCon). The second author has also been funded by the Alexander von Humboldt-Professorship program, the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 765579-ConFlex, grant MTM2017-92996 of MINECO (Spain), ELKARTEK project KK-2018/00083 ROAD2DC of the Basque Government, ICON-ANR-16-ACHN-0014 of the French ANR and Nonlocal PDEs: Analysis, Control and Beyond, AFOSR Grant FA9550-18-1-0242.

or biological populations, carry the positivity of their quantities intrinsically. The temperature at a given point, the concentration of a chemical, or a population, for instance, are nonnegative quantities. A model that intends to predict their evolution must fulfill the positivity of such quantities. For this reason, the maximum principle for parabolic equations tells that the model, a side of its mathematical wellposedness, makes sense in the application level.

The purpose of control theory is to find ways to interact with a given system in order to achieve a specific purpose. One of them is the controllability problem: to ensure that there is a way to drive an initial condition to a target configuration. Furthermore, other objectives can be pursued, such as the stabilization problem (see [15, Part 3]).

Following the application philosophy, in this work, we explain the main features that one has to take care of when dealing with the controllability problem for reaction-diffusion equations with state constraints. In other words, we have to find a way to drive the system to our target configuration, taking care that our control interaction does not break the meaning of the quantities that the model represents.

The modern mathematical control theory of parabolic equations starts at the end of the last century with the works of [21, 24, 28, 29, 34, 52] (and references therein). However, the approach followed by the previous authors does not guarantee state constraints requirements such the positivity.

The study of controllability with state constraints for parabolic partial differential equations is a recent research topic. The core concern of these lecture notes is the discussion of different and specific phenomena that are important for, either achieving the controllability under state constraints, or, on the other hand, proving the lack of controllability.

The presence of constraints requires new methods for guaranteeing controllability. The staircase method is, by now, the most powerful tool for controlling under constraints. Furthermore, new relevant features arise such us the existence of a minimial controllability time. When state constraints are present, the controllability has a minimum positive time (whenever it can be achieved). This feature is not observed without constraints.

Another relevant complication is the appearance of intrinsic obstructions to the controllability, barrier functions. A barrier is a non-trivial steady-state that, due to the comparison principle, prevent any controlled trajectory from reaching specific targets.

The staircase method guarantees the controllability in large time to a specific steady-state under certain assumptions. One needs to have that the initial data is another steady-state, which is path-connected in an admissible way with the target. We will explain a way to construct such paths such that the constraints are not violated. The method consists of studying the phase-plane of the dynamical system associated to the elliptic equation that steady-states satisfy. Moreover, the discussion of the controllability for the one-dimensional bistable equation is also discussed [74].

These lecture notes mainly expose the one-dimensional case. However, the methods can be applied to more spatial dimensions and different nonlinearities. We will present briefly the multi-dimensional case [82], the case of spatially heterogeneous drifts [61], and a combination of the results of the Allee interaction [90] and the boundary interaction [74].

0.1. **Bibliographic notes on applications and control problems.**

0.1.1. *Applications.* Reaction-diffusion systems frequently appear in nature. Here we want to create an introductory remark on the different contexts where one can find them. The possible control interactions with the equation will strongly depend on the application context.

One can find reaction diffusion-equations in:

- In **magnetic systems** in material science [19]: Consider a material composed by magnetic dipole moments of atomic spins that can have two states. Consider that there is an infinite number of them placed in a lattice form. A natural model in the context is the Ising model. The physicist Roy J. Glauber proposed a dynamics to understand how the spins were shifting from up to down and vice versa depending on how many neighbor

spins were up or down [36]. Glauber's dynamics models a reaction between the spins and, considering a spin interchange, in [19] it is formally derived a reaction-diffusion system for the evolution of the magnetization of the material.
- In **chemical reactions** [85] [68]. In most of the cases, these models are systems which enjoy much richer dynamics. An example would be Turing patterns [93], which emerge surprisingly when one of the diffusivity constants is much smaller than the other one. The model proposed by Alan Turing [93] was originally proposed for morphogenesis, and it has been proven experimentally to be really successful [65].
- In **population dynamics**. The work of Kolmogorov [48] considers a model in which a population grows by means of a nonlinearity and diffuses in space. He found the existence of traveling waves studying this ecological model. A traveling wave is a profile $s(x)$ such that there exists a velocity c for which $s(x-ct)$ is a solution of the scalar reaction-diffusion equation in the whole $\mathbb{R}$. Vast literature can be found in traveling waves and convergence to them; we refer to [31]. For general diffusion models in ecology we refer to [3, 4, 13, 33, 62, 64, 68] (see [8] for an experimental case in which the authors see a traveling wave from data).
- In **evolutionary game theory**: In this field, one seeks to understand how players change their strategies depending on the strategy choice of the other players [18,42,44,67]. If one considers spatial diffusion of the players, one arrives at reaction-diffusion equations. [41, 43]
- In **neuroscience** for modeling the nerve impulse [22], the models can also show traveling waves.
- In **linguistics**: Also, in the work [78] they consider parabolic models for analyzing language shift.

0.1.2. *Introduction to control problems.* As well as sometimes some phenomena do not have a unique possible modelization, one may consider a large variety of interactions to a system for control purposes:

- *Interior Control*: This type of interaction finds applications, for example, in parabolic equations intending to model heat. If we consider a bar of length L and a subinterval $\omega \subset [0, L]$ where a heater/cooler is placed, one can write the following model:
$$\begin{cases} v_t - \partial_{xx} v = \chi_\omega a, \\ v(0,t) = v(L,t) = 0, \\ v(x,0) = v_0, \end{cases}$$
where v_0 is the initial temperature distribution; furthermore, it has been assumed that the temperature in the boundary is fixed. The heater interaction is modeled by a. From the control perspective, there are plenty of works in the literature, in addition to the other references we add [25, 26]. From the physical perspective, if one wants to use this model for heat, one should furthermore consider the positivity of the state as in [57].
- *Multiplicative and Bilinear Control*: This type of interaction amounts for being able to interact with the potential:
$$\begin{cases} v_t - \partial_{xx} v = av, \\ v(0,t) = v(L,t) = 0, \\ v(x,0) = v_0. \end{cases}$$
In this context, we refer to [12, 46, 47].
- *Allee Control*: In [90] the authors justify the control action via a micro-macro connection [19] [20]. Their proposal as a control action is to release sterile mosquitoes. They observe that this control interaction in the microscopic level amounts to consider in the macroscopic level the following equation:
$$\begin{cases} v_t - \partial_{xx} v = v(v-\theta(t))(1-v) \quad x \in \mathbb{R} \\ v(x,0) = v_0(x) \in L^\infty(\Omega; [0,1]), \end{cases}$$

where θ is their control.

- *Boundary Control*: The boundary control is closely related to the interior control. In this case, one can interact with the system by modifying the boundary values. In this situation, the way to act in the system is by moving the boundary condition of the system. These two approaches are intimately related through the extension-restriction [86]. This is the paradigm that will be studied in these lecture notes.
$$\begin{cases} v_t - \partial_{xx} v = 0, \\ v(0,t) = a_1(t), \quad v(L,t) = a_2(t), \\ v(x,0) = v_0. \end{cases}$$

Typical goals are:

(1) The controllability problem [15,92,100]: given an initial data can I find a control function such that the control solution reaches a specific target?
(2) The stabilization problem [15]: given an unstable equilibrium configuration. Can we find a control function depending on the state such that stabilizes the equilibrium?
(3) Optimal control. This problem is a minimization problem. One seeks to minimize a cost functional that depends on the state and on the control. Typically, the functional has an equation linking the state and the control as a restriction. This framework enables a wide variety of purposes, for instance [11, 53, 83, 91].

0.2. **Model problem.**

The main problem that will be treated here is the boundary control of some of the most common reaction-diffusion equations. To control from the boundary means that in our application, we do not have access inside the region. For example, when heating a body, the heat penetrates it from the boundary or when we seek to control an animal population that lives in a habitat that we cannot penetrate. A natural question is: How is the domain affecting the controllability?

Let $\Omega = [0, L] \subset \mathbb{R}$ bounded domain of positive measure and consider the following problem:

$$\begin{cases} v_t - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v(0,t) = a_1(t) \quad v(L,t) = a_2(t) & t \in (0,T), \\ 0 \le v(x,0) \le 1, & x \in [0,L], \end{cases} \tag{0.1}$$

where $f \in C^2$ satisfying $f(0) = f(1) = 0$. We will consider two types of f.

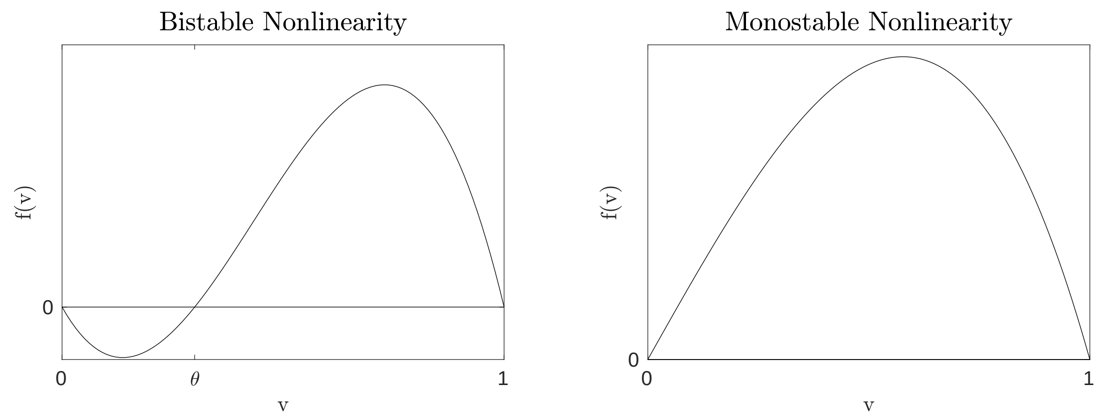

FIGURE 1. Representation of a bistable nonlinearity (left) and a monostable one (right).

- Monostable, $f'(0) > 0$ and $f'(1) < 0$ with $f(s) > 0$ for $s \in (0, 1)$ (see Figure 1 right). One example of a monostable nonlinearity is $f(v) = v(1 - v)$.

- Bistable, there is a number $\theta \in (0,1)$ such that $f(\theta) = 0$, $f'(\theta) > 0$ and $f'(0) < 0$ $f'(1) < 0$ with $f(s) < 0$ in $(0, \theta)$ and $f(s) > 0$ in $(\theta, 0)$ (see Figure 1 left). One example of bistable nonlinearity is $f(v) = v(1-v)(v-\theta)$.

We want to determine for which conditions we can drive the system to the steady-states $v = 1$ (with $a_1 = a_2 = 1$), $v = 0$ (with $a_1 = a_2 = 0$) and in the case of the bistable $v = \theta$ (with $a_1 = a_2 = \theta$).

Due to the nature that these models are aiming to capture, we must preserve certain restrictions on the state. This leads that the classical methodology for controlling semilinear equations (e.g. [29]) is not directly applicable because, in general, the control provided by [29] will violate the constraints. As we shall see, v normally represents a proportion or a population. For this reason, any control strategy has to take care that the state v has to fulfill this restriction for all time.

This fact will imply that the lack of controllability of the system for specific initial data and a big measure of the domain. We will see that the existence of nontrivial solutions of the elliptic problem

$$\begin{cases} -\partial_{xx} v = f(v), \\ v(0) = 0 \quad v(L) = 0, \end{cases} \tag{0.2}$$

will be critical for the controllability due to the parabolic comparison principle.

The existence of (0.2) depends basically on the measure of the domain. Thus the control results will vary depending on this measure. We will call positive solutions of (0.2) barriers.

In terms of applications, it is known that the survival of species depends on having minimum area requirements [80]. In other words, if the domain is big enough, we cannot expect that the population will be extinct even if all individuals that arrive at the boundary are being killed because the reproduction inside the domain can compensate it.

Mathematically one can see this feature with standard techniques. For instance, the stability of the 0 solution for monostable reaction-diffusion equations depends on the size of the domain.

From Matano's Theorem [60], we know that bounded solutions of one-dimensional reaction-diffusion equations converge to steady-states. This classical result, combined with the fact of the instability of 0 for large domains, is a way to understand the existence of a nontrivial solution of (0.2).

Previously this problem has already been tackled initially in [74] (and [17] without restrictions). Here we present the results of [74] for the controllability of these equations. We extend their work showing more variety of paths that can be generated. Moreover, the existence of nontrivial is tackled by means of variational techniques instead of phase-portrait ones. The results are extended, constructing a symmetric strategy, giving results for larger domains, and giving estimates from above for when obstructions are going to appear. Moreover, further discussions on recent results are done, such as the multi-dimensional extension [82] and the spatial heterogeneous drift [61].

0.3. **Organization.**

(1) In Section 1 we will present the development of a model such that its control interaction requires to fulfill state constraints.
(2) In Section 2, we will review some of the most important mathematical properties of parabolic and elliptic equations.
(3) In Section 3, firstly, we will give insights on the wellposedness of the control problem and to the controllability of parabolic equations. Also, in Section 3, we will present the staircase method [69].
(4) Section 4 is devoted to the understanding of the existence of nontrivial solutions of the elliptic problem (0.2). Furthermore, we give thresholds of their existence, depending on the measure of the domain.

(5) Section 5, concerns the graphical visualization of the energy functional associated with the elliptic problem (0.2) are presented in order to give a more intuition.
(6) Section 6 consists of explaining ways to construct admissible paths of steady-states in a way that the constraints are not violated. The problem will be the control towards the intermediate constant steady-state θ for the bistable nonlinearity. First, we introduce the strategy used in [74], and later, we extend the reasoning to obtain symmetry, and we discuss the length of the maximal path that can be taken with respect to the length of the domain.
(7) In Section 7, we summarize the results constructed throughout the previous sections.
(8) In Section 8, simulations of the control process and open paradigms are shown.
(9) The methods employed in Section 6 were employed to control the bistable equation in dimension one [74]. However, the result still holds in several dimensions [82]. In Section 9, a schematic way of the procedure in [82] together with the approach developed in [61] for dealing with spatially heterogeneous drifts is presented. The techniques before concern the boundary control of scalar bistable reaction-diffusion. In Section 9, we present a combination of the boundary control with the interaction with a parameter in the nonlinearity, the Allee-Control [90].
(10) In Section 10, we point out different results that one should take care of if one intends to deal with more general problems.
(11) In the end, in Section 11, the notes leave different open problems for the community to enlarge this new subfield.

1. Parabolic models

The goal of this section is to derive some of the models that will be considered and furthermore to justify their constraints.

1.1. **Monostable model.**

The first ODE models in population dynamics exhibit an exponential growth of the population:

$$\frac{P'}{P} = \beta,$$

where P is the population. Verhulst [95] already noticed that the competition for limited resources among individuals of the same population is more accurate and leads to a upper threshold of the population growth. Thus, the model formulated was:

$$P' = \beta P(1 - \frac{P}{\kappa}),$$

where κ is the capacity of the environment.

The ODE models where afterwards adapted to the PDE setting in order to model movement and invasion of species. Considering homogeneous diffusion and that the resources in space are space invariant, the equation can be formulated as follows:

$$\begin{cases} v_t - \partial_{xx} v = \beta v(1-v) & (x,t) \in (0,L) \times (0,T), \\ \partial_x v = 0 & (x,t) \in \{0,L\} \times (0,T), \\ 0 \le v(x,0) \in L^\infty(\Omega), \end{cases}$$

for $\beta > 0$. In general, the monostable models that we will consider fulfills the following

$$\begin{cases} v_t - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ \dfrac{\partial v}{\partial \nu} = 0 & (x,t) \in \{0,L\} \times (0,T), \\ 0 \le v(x,0) \le 1 & x \in [0,L], \end{cases}$$

with f fulfilling:

$$f(0) = f(1) = 0, \quad f(s) > 0 \text{ for } s \in (0,1), \quad f'(0) > 0, \quad f'(1) < 1.$$

In general, the constraint that $v(x,0) \le 1$ does not correspond to a physical meaning unless our quantity is a proportion. We impose this constraint in order to discuss in the same framework the bistable model and the monostable one.

1.2. **Bistable model.**

Consider a domain of positive measure $\Omega = [0,L] \subset \mathbb{R}$ and assume that we have two interacting quantities V and W. The interaction between V and W is nonlinear but it preserves the quantity $V + W$. We will discuss later possible contexts in which such situation might occur.

In addition, the quantities we model are nonnegative, $V(x,t), W(x,t) \in \mathbb{R}^+$. Moreover we consider that system has spatial diffusion:

$$\begin{cases} V_t - \partial_{xx} V = F(V,W) & (x,t) \in (0,L) \times (0,T), \\ W_t - \partial_{xx} W = -F(V,W) & (x,t) \in (0,L) \times (0,T), \\ \partial_x V = \partial_x W = 0 & (x,t) \in \{0,L\} \times (0,T), \\ 0 < V(x,0) \in L^\infty(\Omega), \\ 0 < W(x,0) \in L^\infty(\Omega), \end{cases} \tag{1.1}$$

where F is a Lipschitz continuous function satisfying $F(V,0) = F(0,W) = 0$ for every $V, W \in \mathbb{R}$.

1.2.1. *Approximation.* Our first step will be to find an approximation of (1.1) by means of a single equation.

In our motivation, we said that the total quantity of individuals or particles is preserved. Indeed, we observe that the quantity

$$P := V + W,$$

satisfies:

$$\begin{cases} P_t - \partial_{xx} P = 0 & (x,t) \in (0,L) \times (0,T), \\ \partial_x P = 0 & (x,t) \in \{0,L\} \times (0,T), \\ P(x,0) = V(x,0) + W(x,0). \end{cases} \tag{1.2}$$

We know that the limit behavior of equation (1.2) is the constant solution:

$$\lim_{t\to\infty} P(t,x) = \frac{1}{m(\Omega)} \int_\Omega V(x,0) + W(x,0) dx.$$

Therefore a reasonable approximation would be to assume that we are in the situation in which P is constant.

We ask ourselves how V and W are evolving in the context of this approximation.

Defining $v := \frac{V}{P}$ as the proportion of the V type in the whole population P, we can reduce (1.1) to:

$$\begin{cases} v_t - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ \partial_x v = 0 & (x,t) \in \{0,L\} \times (0,T), \\ 0 \le v(x,0) \le 1 & x \in [0,L]. \end{cases}$$

Note that, the fact of being a proportion now requires that $0 \le v(x,t) \le 1$ for all (x,t). The zeros of the original nonlinearity F guarantees that $f(0) = f(1) = 0$, then, by the comparison principle it is guaranteed that the state will remain for all time between 0 and 1 (see next section, Section 2).

It is said that f is bistable if:

$$f(0) = f(\theta) = f(1) = 0, \quad f'(0) < 0, \quad f'(1) < 0, \quad f'(\theta) > 0,$$
$$f(s) < 0 \text{ for } s \in (0,\theta), \quad f(s) > 0 \text{ for } s \in (\theta, 1).$$

As an example of the correspondence between F and f we claim that the nonlinearity

$$F(V,W) = \frac{VW}{(V+W)^2} \left((1-\theta)V + \theta W\right)$$

for $\theta \in (0,1)$ corresponds to:

$$f(v) = v(1-v)(v-\theta)$$

after doing the procedure of reducing to a proportion mentioned above.

The cubic nonlinearity has special interest in game theory as we will see in the following subsection.

1.2.2. *Interpretations of the model.*

- *Game theoretical interpretation.*

  This nonlinearity arises naturally in the context of evolutionary game theory in a coordination game.

  An example of a coordination game in the social context is the choice of language. Assume there are two bilingual individuals that meet each other and they establish a conversation in one of the two languages that they manage. However, people might have different preferences for languages, this is modeled by a payoff matrix.

  The replicator dynamics is nonlinear ODE that models the change of strategy in time of players. Players play continuously the game and they evaluate their success while comparing it with the average success of the other players. The replicator dynamics for a 2 strategy game is:

$$\frac{d}{dt}v = v\left((1,0)A\begin{pmatrix} v \\ 1-v\end{pmatrix} - (v,1-v)A\begin{pmatrix} v \\ 1-v\end{pmatrix}\right)$$

where $A \in M_2(\mathbb{R})$ is a payoff matrix.

In this model v is the proportion of players playing the first strategy and $1-v$ the proportion playing the remaining strategy. The first term in the right hand side evaluates the success of the first strategy while the second is the average success of the players. If the first strategy has less success than the average, the proportion of players playing the first strategy will diminish in favour of the second strategy.

The payoff matrix of a coordination two strategy game

$$A = \begin{pmatrix} (1-\theta) & 0 \\ 0 & \theta \end{pmatrix},$$

leads to $f(v) = v(1-v)(v-\theta)$. In this game we see that if $\theta < 1/2$ players prefer to play the first strategy rather than the second. However, the preference does not uniquely determine the behavior of the dynamics $\frac{d}{dt}v = f(v)$. One can see that both consensus $v = 0$ and $v = 1$ are stable. Indeed, $f'(0) < 0$ and $f'(1) < 0$, meaning that the less wanted strategy is stable if enough players are currently playing it.

For further details and other possible models we cite [18, 41, 42, 42, 43, 67]

- *Biological interpretation.*

  The interpretation in the biological context is the following: *if the population in one habitat is lower than a threshold θ they are not enough to survive.* This is the so-called Allee effect (see [88]).

  Game theoretical approach to population dynamics is also accepted. Assume that we have a population with two distinct genes. More successful genes will pass to next generations while the less successful ones will disappear. In this situation the replicator dynamics can be also applied.

  In ecology, spatial heterogeneities are omnipresent. Note that if A depends on x like

$$A(x) = \begin{pmatrix} (1-\theta(x)) & 0 \\ 0 & \theta(x) \end{pmatrix},$$

  $\theta(x)$ is determining which strategy is having an advantage depending on the conditions of the terrain. So that if $0 < s < \theta(x)$ then $f(x,s) < 0$ and positive if $\theta(x) < s < 1$. Then we obtain a model in which the nonlinearity depends on x. For general nonlinearities in ecological systems, one can think of two genes population, one gene is favorable dry environments, and the other is favorable in humid habitats. This case will be briefly discussed later on.

  One may consider another diffusivity constant, but the results will hold analogously after the absorption of the diffusivity constant by both, a change of the time scale and the nonlinearity.

- A chemical reaction with the conservation of the number of reactants [68, Ch.1], the reactants will be particles of the two different classes, one and two, and its respective quantities V and W.

1.3. **Boundary control.**

We will consider the situation in which we can not penetrate the domain and that we can only act on its boundary. In the boundary, we can act in the Neumann condition and setting the control

$$\begin{cases} v_t - \partial_{xx}v = f(v) & (x,t) \in (0,L)\times(0,T), \\ \dfrac{\partial v}{\partial \nu} = b(x,t) & (x,t) \in \{0,L\}\times(0,T), \\ 0 \le v(x,0) \le 1 & x \in [0,L]. \end{cases}$$

However, we can notice that a trajectory with Neumann control can will have a Dirichlet trace. For this reason we consider a Dirichlet control. Indeed, if we fix the control function $b(x,t)$, and we consider the function given by the Dirichlet trace $a(x,t)$ on the boundary for any $t \in [0,T]$ of v_b by uniqueness we have that $v_b = v_a$ and therefore the control problem is equivalent to:

$$\begin{cases} v_t - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v = a(x,t) & (x,t) \in \{0,L\} \times (0,T), \\ 0 \leq v(x,0) \leq 1 & x \in [0,L]. \end{cases} \tag{1.3}$$

Though the derivation of the equation endows naturally two constraints for being a proportion, v has to be positive and less than 1 in every point at any time. We will see in the next section that by a comparison principle, the constraints in the state will be automatically satisfied if we require that:

$$0 \leq a(x,t) \leq 1 \quad \text{for any } (x,t) \in \{0,L\} \times [0,T].$$

2. Review of some results on parabolic equations

In this section, we compile some useful results on semilinear parabolic equations that we will use later on.

2.1. **Convergence to steady states.**

Consider the following 1-D semilinear heat equation.

$$\begin{cases} v_t = (a(x)v_x)_x + f(x,v) & 0 < x < L, \quad 0 < t < T, \\ v(x,0) = v_0(x) & 0 \le x \le L, \\ v(0,t) = \beta_0, \quad v(L,t) = \beta_L, \end{cases} \tag{2.1}$$

for general nonlinearities, the blow up of (2.1) can occur. Let $t_e(v_0)$ be the maximum time of definition of (2.1).

Definition 2.1. Let $v_0 \in C([0,L];\mathbb{R})$ and suppose that the maximum time of definition depending on the initial datum is infinity, $t_e(v_0) = \infty$. The ω-limit set of v_0 is a subset of $C^1([0,L]) \cup C^2((0,L))$ defined as follows:

$$S(v_0) = \bigcup_{t>0} \overline{\{Q(\tau)v_0; \tau \ge t\}},$$

where $Q(t)$ is the operator defined as $Q(t) : C([0,L];\mathbb{R}) \to C([0,L];\mathbb{R})$ by $Q(t)v_0 = v(\cdot, t, v_0)$.

Matano's Theorem asserts that any bounded trajectory will converge to a steady-state (the result also holds for more general boundary conditions [59])

Theorem 2.2 (Matano, Theorems A and B from [60])**.** *The ω-limit set of any function $v_0 \in C([0,L];\mathbb{R})$ containts at most one element. For any initial data $v_0 \in C([0,L];\mathbb{R})$, one and only one of the following three holds:*

(1) *The solution blows up in finite time $t_e(v_0) < \infty$ and $\lim_{t \to t_e(v_0)} \|Q(t)v_0\|_{L^\infty([0,L];\mathbb{R})} = \infty$.*
(2) *The solution grows up as $t \to \infty$, $t_e(v_0) = \infty$ and $\lim_{t \to t_e(v_0)} \|Q(t)v_0\|_{L^\infty([0,L];\mathbb{R})} = \infty$.*
(3) *The solution converges to a solution of the elliptic problem.*

$$\begin{cases} 0 = (a(x)v_x)_x + f(x,v) & 0 < x < L, \quad 0 < t < T, \\ v(0) = \beta_0, \quad v(L) = \beta_L, \end{cases}$$

in the $C^1([0,L];\mathbb{R}) \cup C^2((0,L);\mathbb{R})$ topology.

2.2. **Comparison results.**

In addition to asymptotic dynamics, the comparison principle enables us to have very strong knowledge of the trajectories of the PDE. See [31, Chapter 5] [10, 32, 79]

Here, due to their formal similarity, we present Parabolic and Elliptic comparison principles together. We state the comparison principles in one dimension; however, they hold in several dimensions.

Definition 2.3 (Parabolic sub- and supersolutions)**.** Let $\Omega \in \mathbb{R}$ be a regular open set. Let $T > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ and $h : \partial\Omega \to \mathbb{R}$ be two smooth functions. Consider an elliptic operator:

$$\mathcal{L} := \partial_{xx} + k(x)\partial_x$$

where $k : \Omega \to \mathbb{R}^d$ is a smooth function. Consider the parabolic problem:

$$\begin{cases} v_t - \mathcal{L}v = f(v) & (x,t) \in \Omega \times (0,T) \\ v(t,\cdot) = h(t,\cdot) & (x,t) \in \partial\Omega \times (0,T) \\ v(0,\cdot) = v_0(\cdot) & x \in \Omega \end{cases} \tag{2.2}$$

A subsolution $\underline{v}$ of (2.2) satisfies:

$$\begin{cases} \underline{v}_t - \mathcal{L}\underline{v} \leq f(\underline{v}) & (x,t) \in \Omega \times (0,T) \\ \underline{v}(t,\cdot) \leq h(t,\cdot) & (x,t) \in \partial\Omega \times (0,T) \\ \underline{v}(0,\cdot) \leq v_0(\cdot) & x \in \Omega \end{cases}$$

A supersolution $\overline{v}$ of (2.2) satisfies:

$$\begin{cases} \overline{v}_t - \mathcal{L}\overline{v} \geq f(\overline{v}) & (x,t) \in \Omega \times (0,T) \\ \overline{v}(t,\cdot) \geq h(t,\cdot) & (x,t) \in \partial\Omega \times (0,T) \\ \overline{v}(0,\cdot) \geq v_0(\cdot) & x \in \Omega \end{cases}$$

Theorem 2.4 (Parabolic comparison principle[1])**.** *If $\underline{v}$ is a subsolution (respectively $\overline{v}$ a supersolution) to* (2.2) *and v is a solution such that $v \geq \underline{v}$ (respectively $v \leq \overline{v}$) on $(x,t) \in \partial\Omega \times [0,T) \bigcup \Omega \times \{0\}$ then $v \geq \underline{v}$ ($v \leq \overline{v}$) in $(x,t) \in \Omega \times (0,T)$*

Now we present the comparison principle for elliptic equations.

Consider $\Omega \subset \mathbb{R}$ to be a bounded open subset with boundary $C^{2,\alpha}$. Let $\mathcal{L} := a(x)\partial_{xx} + b(x)\partial_x + c(x)$ be an elliptic operator with coefficients in $C^{0,\alpha}(\overline{\Omega})$. Furthermore, let $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a locally Lipschitz function. Consider the problem:

$$\begin{cases} -\mathcal{L}v(x) = f(x, v(x)) & x \in \Omega \\ v(x) = 0 & x \in \partial\Omega \end{cases} \tag{2.3}$$

Definition 2.5 (Elliptic sub- and supersolutions)**.** We say that $\underline{v}$ (respectively $\overline{v}$) is a subsolution (respectively a supersolution) if $\underline{v}$ (respectively $\overline{v}$) belongs to $C^0(\overline{\Omega}) \cup C^2(\Omega)$ and verifies that:

$$\begin{cases} -\mathcal{L}\underline{v}(x) \leq f(x, \underline{v}(x)) & x \in \Omega \\ \underline{v} \leq 0 & x \in \partial\Omega \end{cases}$$

Respectively,

$$\begin{cases} -\mathcal{L}\overline{v}(x) \geq f(x, \overline{v}(x)) & x \in \Omega \\ \overline{v} \geq 0 & x \in \partial\Omega \end{cases}$$

Theorem 2.6 (Elliptic comparison principle, Theorem 5.17 in [51])**.** *Let us assume that there exist a subsolution and a supersolution to the problem* (2.3) *such that $\underline{v} \leq \overline{v}$.*

Then the problem (2.3) *admits a minimal solution $\underline{v}_*$ and a maximal $\overline{v}^*$ (possibly equal) such that, $\underline{v} \leq \underline{v}_* \leq \overline{v}^* \leq \overline{v}$ and there exist no solution u between $\underline{v}$ and $\overline{v}$ such that at a certain point $x \in \Omega$ satisfies either $v(x) \leq \underline{v}_*(x)$ or $v(x) \geq \overline{v}^*(x)$*

It is very well known that reaction-diffusion systems can blow up in finite time for certain initial data and specific nonlinearities [23, Ch.9, pp 547-550]. The following result guarantees the stability of our system if the initial data is between $0 \leq v_0(x) \leq 1$.

Theorem 2.7 (Stability)**.** *Assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is locally Lipschitz continuous with $f(0) = f(1) = 0$. Then, the solution of the problem*

$$\begin{cases} v_t = \partial_{xx}v + f(v) & (x,t) \in (0,L) \times (0,T), \\ v(x,t) = a(x) \in [0,1] & (x,t) \in \{0,L\} \times (0,T), \\ 0 \leq v(x,0) \leq 1, \end{cases}$$

is defined for all positive time.

Proof. (Idea) For the existence of a solution bounded by below by 0 and by 1 from above, we can notice that we can cut the nonlinearity outside $[0,1]$ and define it as zero. By the comparison principle, we know that the solution for this cut nonlinearity is between 0 and 1 and, therefore, a solution also for the original nonlinearity. Moreover, we have that this solution does not blow up in finite time (by the comparison principle with the constant functions 1 and 0 as super and subsolutions respectively) □

[1] as in [10]

2.3. **Travelling waves.**

In our model we put control in the whole boundary. Hence, the boundary conditions are not prescribed a priori. This allows us to make use of trajectories on bigger domains and eventually on unbounded domains.

A particular example of trajectories in unbounded domains are traveling waves [31, 48].

Traveling waves are one of the most important phenomena among reaction-diffusion systems. In order to define them we need to consider the following Cauchy problem:

$$\begin{cases} v_t - \partial_{xx} v = f(v) & x \in \mathbb{R}, \\ v(0) = v_0(x) \in L^\infty(\mathbb{R}). \end{cases} \tag{2.4}$$

Definition 2.8 (Travelling waves)**.** A Travelling wave solution to (2.4) is a solution of the form $v(t, x) = w(x - ct)$ with $c \in \mathbb{R}$ being the wave speed.

A traveling wave is a solution of the Cauchy problem (2.4) that maintains the same profile as time evolves. The existence of such functions was discovered by Kolmogorov [48], and later much research has been done on them. Observe that if such profile exists, it must fulfill:

$$-cU'(s) + U''(s) = f(U(s)) \qquad s \in \mathbb{R} \tag{2.5}$$

where $s = x - ct$. From (2.5) by multiplying by $U'(s)$ and integrating over $\mathbb{R}$ one observes,

$$-c\int_{\mathbb{R}} (U'(s))^2 ds + \int_{\mathbb{R}} U''(s)U'(s)ds = \int_{\mathbb{R}} f(U(s))U'(s)ds.$$

One can see that:

$$c = \frac{F(1)}{\int_{\mathbb{R}} (U'(s))^2 ds}. \tag{2.6}$$

Equation (2.6) gives the direction of the traveling wave. Note that the term that matters is $F(1)$, if $F(1) > 0$, the traveling wave is moving to the right, while, if $F(1) = 0$, the profile must be stationary.

In addition one can obtain another relation for c:

$$-c\int_{\mathbb{R}} U'(s)ds + \int_{\mathbb{R}} U''(s)ds = \int_{\mathbb{R}} f(U(s))ds,$$

which implies that:

$$c = \int_{\mathbb{R}} f(U(s))ds.$$

2.3.1. *Bistable Nonlinearities.* The existence result for bistable nonlinearities is the following:

Theorem 2.9 (Travelling waves for the Bistable nonlinearity, Theorem 4.9 in [68])**.** *Assume that:*

$$f(0) = 0, \quad f'(0) < 0, \quad f(\theta) = 0, \quad f(1) = 0, \quad f'(1) < 0$$
$$f(v) < 0 \quad \text{for} \quad 0 < v < \theta, \quad f(v) > 0 \quad \text{for} \quad \theta < v < 1$$

There exist a unique traveling wave (c^*, v) of (2.4) with v decreasing and satisfying:

$$c^* > 0 \text{ for } F(1) > 0, \quad c^* < 0 \text{ for } F(1) < 0, \quad c^* = 0 \text{ for } F(1) = 0$$

For the cubic nonlinearity $f(t) = t(1-t)(t-\theta)$ the profile $U(x)$ has an explicit expression $U(x) = \frac{\exp\{-x/\sqrt{2}\}}{1+\exp\{-x/\sqrt{2}\}}$ shown in Figure 2 and its traveling speed is $c^* = \sqrt{2}\left(\frac{1}{2} - \theta\right)$.

Aside of connecting in a natural way stationary points, namely the steady-state 0 with the steady-state 1, another very important property is the exponential convergence to traveling waves.

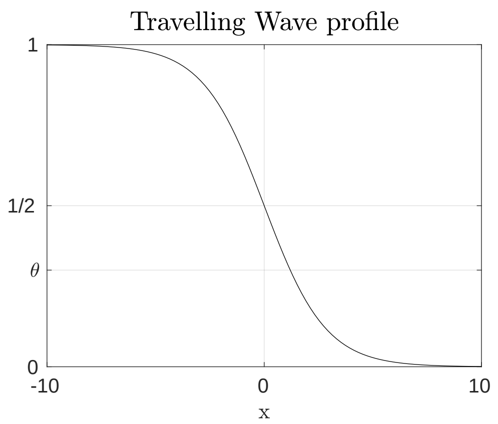

FIGURE 2. Profile for the cubic nonlinearity $f(t) = t(1-t)(t-\theta)$ in the interval $[-10, 10]$. The profile is independent of the value of θ.

Theorem 2.10 (Thm. 4.16 in [31]). *Assume that f is bistable. then if ϕ satisfies:*

$$\limsup_{x\to-\infty} \phi(x) < \theta \qquad \liminf_{x\to+\infty} \phi(x) > \theta.$$

Then there exist constants $C > 0$, $\mu > 0$ and $x_0 \in \mathbb{R}$ such that:

$$|u(x,t) - U(x - ct - x_0)| < Ce^{-\mu t}.$$

2.3.2. *Monostable Nonlinearities.* Now assume that f is monostable:

$$\begin{aligned} f(0) =& f(1) = 0, \quad f(s) > 0 \quad s \in (0,1) \\ & f'(0) > 0, \quad f'(1) < 0, \end{aligned}$$

In this situation there is also existence of traveling waves:

Theorem 2.11 (Theorem 4.15 in [31]). *For f monostable there exist a number $c^* > 0$ such that there exist a traveling wave solution with $U(-\infty) = 1$, $U(+\infty) = 0$ if and only if $c \geq c^*$.*

Note that, in this case, there exist infinitely many traveling waves. The most known case is the case of the Fisher-KPP equation from where the traveling waves were originally discovered [48].

$$\begin{cases} v_t - \partial_{xx} v = rv(1-v) \\ 0 \leq v(x,0) \leq 1 \end{cases} \tag{2.7}$$

Furthermore, for monostable nonlinearities such that $f'(0) \geq f'(u)$ the following result is well known:

Theorem 2.12 ([48]). *Let $f : \mathbb{R} \to \mathbb{R}$ be monostable fulfilling $f'(0) \geq f'(u)$ and consider the following Cauchy problem*

$$\begin{cases} v_t - \partial_{xx} v = f(v) \quad x \in \mathbb{R} \\ v(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} 0 \quad \text{for} \quad x > 0 \\ 1 \quad \text{for} \quad x < 0 \end{cases}$$

Then there exist a function $\psi \in C^1$ such that

$$|u(x,t) - U(x - c^*t - \psi(t))| \to 0 \quad \text{as} \quad t \to \infty \tag{2.8}$$

uniformly in x and:

$$\lim_{t\to\infty} \psi'(t) = 0$$

Other authors prove different convergence results for this type of nonlinearity and more general results. We refer also to [84] [81] [30].

The interest of traveling waves for control is that the restriction of a traveling wave to a bounded domain gives us a trajectory for going from a traveling wave arc near the steady-state 0 to approach asymptotically the steady-state 1. Furthermore, they can naturally be used to construct sub-solutions.

3. Wellposedness of the control problem and controllability

3.1. Comments on the wellposedness.

First, we give some insight into the wellposedness main control problem considered in this manuscript:

$$\begin{cases} v_t - \partial_{xx} v = f(v), \\ v(0,t) = a_1(t), \quad v(L,t) = a_2(t), \\ v(x,0) = v_0(x) \in L^\infty([0,L];[0,1]). \end{cases} \tag{3.1}$$

Definition 3.1 (Weak solution)**.** Consider the space:

$$\mathcal{T} := \{\varphi \in C^\infty([0,T]\times[0,L]) : \quad \varphi(\cdot,T)=0, \quad \varphi(x,t)=0 \quad (x,t)\in\{0,L\}\times[0,T]\},$$

for $v_0 \in L^\infty([0,L];[0,1])$ and for $a_i \in L^\infty((0,T))$ for $i=1,2$, $v \in C^0([0,T],H^{-1}([0,L])) \cap L^\infty([0,L]\times(0,T))$ is a weak solution of system (3.1) if for every $\varphi \in \mathcal{T}$ one has that:

$$\int_0^T \int_0^L v\big(-\varphi_t - \varphi_{xx}\big) + f(v)\varphi dx dt = \int_0^L v_0 \varphi(x,0) dx + \int_0^T a_1(t)\varphi_x(0,t) - a_2(t)\varphi_x(L,t) dt$$

Theorem 3.2. *For f Lipschitz, there exists a unique weak solution of* (3.1) *for all $T>0$.*

The proof of Theorem 3.2 is given in [69] consists of splitting the problem into two subproblems, let $v = w + y$ to satisfy:

$$\begin{cases} w_t - \partial_{xx} w = 0, \\ w(0,t) = a_1(t), \quad w(L,t) = a_2(t), \\ w(x,0) = 0, \end{cases} \tag{3.2}$$

and

$$\begin{cases} y_t - y_{xx} = f(y+w), \\ y(0,t) = 0, \quad y(L,t) = 0, \\ y(x,0) = v_0(x) \in L^\infty([0,L];[0,1]). \end{cases} \tag{3.3}$$

For discussing problem (3.2) one should first introduce the notion of transposition solution [54, Ch.13] the second one (3.3) will follow from Banach fixed point.

First, we give some formal insights into the proof of the wellposedness of (3.2). Due to the low regularity assumed on the boundary data, we should provide the proper notion to understand a solution of (3.2). From what it follows, we will use the following notation:

$$\Omega = [0,L], \quad Q = [0,T]\times\Omega, \quad \Sigma = [0,T]\times\partial\Omega.$$

Let us introduce the adjoint problem (3.4), for every $\phi \in L^2([0,T];L^2((0,L)))$ consider

$$\begin{cases} -p_t - p_{xx} = \phi, \\ p(0,t) = 0, \quad p(L,t) = 0, \\ p(x,T) = 0. \end{cases} \tag{3.4}$$

Equation (3.4) after a change on the time direction is a heat equation which is wellposed in the space:

$$u \in L^2([0,T];H_0^1((0,L))), \quad \frac{d}{dt}u \in L^2([0,T];H^{-1}((0,L))),$$

the wellposedness follows from Galerkin approximations see [23, Ch.7 pp.378]. Take the solution of (3.2) multiply it by ϕ and integrate over Q:

$$\begin{aligned}\int_0^T\int_0^L w\phi dxdt &= \int_0^T\int_0^L w(-p_t - p_{xx})dxdt = \int_0^T w\frac{\partial p}{\partial \nu}\Big|_{x=0}^{x=L} dt\\ &= \int_0^T a_2(t)\frac{\partial p}{\partial \nu}(L,t) - a_1(t)\frac{\partial p}{\partial \nu}(0,t)dt\\ &= \langle a, \Lambda\phi\rangle_{L^2(\Sigma)}\end{aligned}$$

The map $\Lambda : L^2(Q) \to L^2([0,T]; L^2(\partial\Omega))$ that takes a function ϕ and gives the trace of the adjoint (3.4) is linear and continuous.

A transposition solution is a distribution v such that satisfies the following relationship for every $\phi \in L^2(Q)$

$$\int_0^T\int_0^L v\phi = \langle a, \Lambda\phi\rangle_{L^2(\Sigma)}. \tag{3.5}$$

For details see [54, Ch.4 S. 8, 12.3, 13 and S. 15 Example 1] and [53, page 202] for details.

Theorem 3.3 (Existence and Uniqueness of the nonhomogeneous boundary value problem Theorem 13.1 [54])**.** *There exists a unique weak solution of problem* (3.2)

The wellposedness of problem (3.3) is an application of Banach Fixed-point [69]. Considering the problem:

$$\begin{cases} y_t - y_{xx} = f(\xi + w)\\ y(0,t) = y(L,t) = 0\\ y(x,0) = v_0(x)\end{cases} \tag{3.6}$$

Defining the map $\psi(\xi)$ as the solution of (3.6), we are defining a map $\psi : B_R \longrightarrow B_R$ where $B_R \subset L^\infty(Q)$ is a ball of radius R.

Using the variations of constants formula with the semigroup of the heat equation one can obtain the contraction for T small enough.

3.2. Null Controllability for the linear problem.

In this subsection, we will discuss the main features of the null controllability of parabolic equations. We do not intend to give all the details in the proofs but to provide the main arguments and references on the topic. The first approach to controllability for the heat equation in one dimension is due to [25]. However, their techniques are limited to dimension one. We will also expose the argument in dimension one, but the arguments presented easily extend to several dimensions.

Hereafter, we present the extension restriction principle for translating the boundary control problem into an interior control problem. The argument consists of extending the system to a bigger domain and consider a control in a source form inside the extended part. Then, control the system and then use the uniqueness of the solution for arguing that the restriction into the boundary of the controlled state mentioned before is indeed a boundary control for the original control problem.

First of all, let us extend the domain $\Omega = [0,L]$ to $\Omega_E = [-1, L+1]$, let $\omega \subset (-1,0)$ (see Figure 3) and consider the following linear parabolic control problem:

$$\begin{cases} v_t - \partial_{xx}v - b(x,t)v = \chi_\omega h & (x,t) \in \Omega_E \times [0,T],\\ v(-1,t) = 0, \quad v(L+1,t) = 0 & t \in (0,T],\\ v(x,0) = v_0(x) & x \in \Omega_E,\end{cases} \tag{3.7}$$

where abusing notation we consider that the initial data is equal to v_0 inside $[0,L]$ and goes smoothly to 0 outside $[0,L]$, $b \in L^2([0,T], L^2(\Omega_E))$.

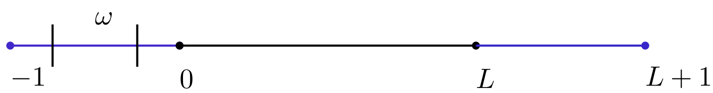

FIGURE 3. Original domain $\Omega = [0, L]$, extended domain $\Omega_E = [-1, L+1]$ and the control region ω.

We multiply the extended heat equation (3.7) by the solution following adjoint problem:

$$\begin{cases} p_t + p_{xx} - b(x,t)p = 0 & (x,t) \in \Omega_E \times [0,T], \\ p(-1,t) = 0, \quad p(L+1,t) = 0 & t \in (0,T], \\ p(x,T) = p^T(x) & x \in \Omega_E, \end{cases} \tag{3.8}$$

and we integrate by parts to obtain:

$$0 = \int_{\Omega_E} v(T)p(T)dx - \int_{\Omega_E} v(0)p(0)dx - \int_0^T \int_{\Omega_E} v\left(p_t + p_{xx} - bp\right) dxdt - \int_0^T \int_\omega hpdxdt$$

hence,

$$0 = \int_{\Omega_E} v(T)p(T)dx - \int_{\Omega_E} v(0)p(0)dx - \int_0^T \int_\omega hpdxdt$$

therefore, our goal is to find a function $\overline{h}$ such that:

$$-\int_{\Omega_E} v(0)p(0)dx - \int_0^T \int_\omega \overline{h}pdxdt = 0$$

for all p^T such that the solution of (3.8) satisfies that $\int_0^T \int_\omega p^2 < +\infty$, this will automatically imply that $v(T) = 0$. Note that one is not requiring that $p^T \in L^2(\Omega_E)$, in fact, p^T belongs to a much larger space [29].

How can we construct such $\overline{h}$? The crucial observation was to observe that if the functional

$$J : H \longrightarrow \mathbb{R},$$

$$p^T \longrightarrow J(p^T) = \int_0^T \int_\omega p^2 dxdt + \int_{\Omega_E} p(0)v(0)dx,$$

where,

$$H := \left\{ p^T \text{ such that } p \text{ solves (3.8) and } \int_0^T \int_\omega p^2 dxdt < +\infty \right\},$$

has a minimizer $p^*(T)$, then the Gateaux derivative at $p^*(T)$ will be zero for any $\xi^T \in H$,

$$DJ(p^*(T))[\xi^T] = \int_0^T \int_\omega \xi p^* dxdt + \int_{\Omega_E} \xi(0)v(0)dx = 0$$

note that if we take $\overline{h} = p^*$ we have found the desired internal control. Moreover, this control is the minimal control with respect to the L^2-norm.

The previous argument work provided that J has a minimizer. J is continuous and convex. The hardest point to prove is the coercivity of J. Note that the coercivity is equivalent to prove the so-called observability inequality for the adjoint system:

$$\|p(0)\|^2_{L^2(\Omega_E)} \leq C \int_0^T \int_\omega p^2 dxdt, \quad \forall p^T \in H. \tag{3.9}$$

In [34], the observability inequality (3.9) is proved by means of Carleman estimates (see also [27]).

Note that once one has found a control function for the system (3.7) then we know that the trajectory $v(x, T; h) = 0$, taking the restriction of the controlled solution in the original domain $[0, L]$:

$$a(0,t) = v(0,t;h) \quad a(L,t) = v(L,t;h)$$

we obtain by uniqueness a null-control for the boundary control problem:

$$\begin{cases} v_t - \partial_{xx} v - b(x,t)v = 0 & (x,t) \in \Omega \times [0,T] \\ v(0,t) = a(0,t), \quad v(L,t) = a(L,t) & t \in (0,T] \\ v(x,0) = v_0 & x \in \Omega \end{cases}$$

3.3. **Null controllability for the semilinear problem and further comments.**

Up to now, we have given an insight on how to control linear heat equations. For nonlinear problems, a common approach is to use a fixed point argument, using either Schauder or Kakutani fixed point [21]. Originally, the idea was employed for the semilinear wave equation, see for instance [97, 98] and references therein. The idea is to consider for any function $\xi \in L^2(Q)$, the bounded potential b_ξ, defined in the following form:

$$b_\xi(x,t) = \begin{cases} \dfrac{f(\xi(x,t))}{\xi(x,t)} & \text{if } \xi(x,t) \neq 0, \\ f'(0) & \text{otherwise.} \end{cases}.$$

Then, one considers the linear controlled problem

$$\begin{cases} v_t - \partial_{xx} v - b_\xi(x,t)v = \chi_\omega h_\xi & (x,t) \in \Omega_E \times (0,T), \\ v(-1,t) = v(L+1,t) = 0 & t \in (0,T], \\ v(x,0) = v_0, \quad v(x,T) = 0 & x \in \Omega_E. \end{cases} \tag{3.10}$$

The map $\psi : L^2(Q) \to L^2(Q)$ defined as $\xi \to \psi(\xi) = v$ solution of (3.10) is continuous and compact, then by Schauder fixed point one obtains the desired result (see [21]).

Other references for the control of heat and semilinear heat equations are the approach by Lebeau and Robiano [52], the control of weakly blowing up semilinear heat equations, and its control cost [28, 29].

Regarding the control cost, it should be mentioned that the cost of control of the linear heat equation is the following:

$$\|a\|_{L^2([0,T],\mathbb{R}^2)} \leq \exp\left\{C\left(\frac{1}{T} + \|b\|_{L^\infty} T + \|b\|_{L^\infty}^{2/3}\right)\right\}$$

The reader should notice that when the time horizon T is very small, the cost of control is very large. This means that the control can oscillate a lot.

Big oscillations near the final time occur do to the fact that p^T might not be in L^2.

In Figure 4 one can see an oscillating control. Moreover, we observe that the control strategy is breaking the important constraints that we want to maintain. This is the reason why this classical methodology is not enough and one has to discuss more sophisticated methods.

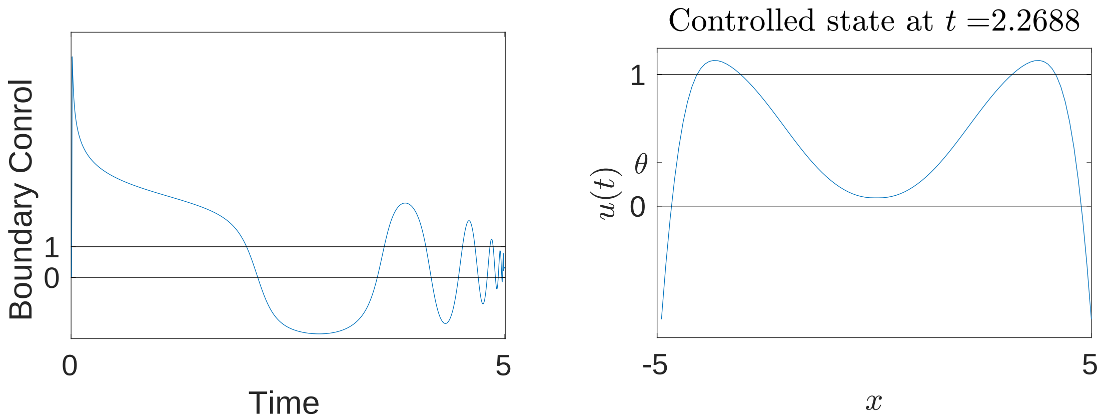

FIGURE 4. (Left) Control function $a(0,t)$ that steers the cubic bistable equation to the steady-state $w \equiv \theta$ in time $T = 5$. (Right) Snapshot at time $t = 2.288$ of the controlled trajectory $u_a(\cdot,t)$ violating the constraints.

3.4. **Constrained controllability results.**

In this subsection we present the main Theorem given in [69] which ensure that under certain assumptions the problem of high amplitude oscillations observed in Figure 4 can be overpassed. However, this will require much more knowledge on the dynamics of the system, namely to know paths of steady-states to be able to "jump" from one to another with small oscillations until the is reached (Figure 5).

Consider

$$
\begin{cases} v_t - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v = a(x,t) & (x,t) \in \{0,L\} \times (0,T), \\ 0 \le v(0,x) = v_0(x), \end{cases} \tag{3.11}
$$

we say that v_1 and v_0 are path connected steady-states if there exist a continuous function from $[0,1]$ to the set of admissible steady-states $\gamma : [0,1] \to S$ such that $\gamma(0) = v_0$ and $\gamma(1) = v_1$. Denote by $\overline{v}^s := \gamma(s)$.

Theorem 3.4 (Theorem 1.2 in [69]). *Let be v_0 and v_1 be path connected bounded steady-states. Assume there exists $\nu > 0$ such that:*

$$
\overline{v}^s(x) \ge \nu \quad \text{for } x \in \{0,L\}, \tag{3.12}
$$

for any $s \in [0,1]$. Then, if T is large enough, there exist a control function $a \in L^\infty\left((0,T);\mathbb{R}^2\right)$ such that the problem (3.11) with initial datum v_0 and control a admits unique solution verifying $v(T,\cdot) = v_1$ and $a \ge 0$ on $(0,T) \times \{0,L\}$

Figure 5 shows qualitatively the strategy. Once one has a connected path of steady-states, in short time intervals one can use local controllability to move from one steady-state to another close steady-state in the path. The strategy is based on using an L^∞ bound on the control in terms of the L^∞ norm of the difference between the initial datum and the target. This allows to extract a finite number of steady-states along the path and apply local controllability from one to another without breaking the constraints.

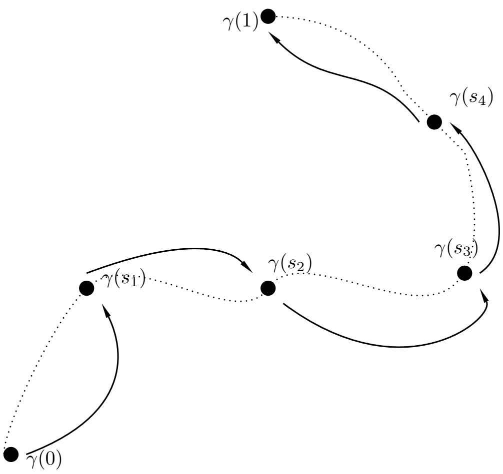

FIGURE 5. Qualitative representation of the Staircase strategy. The dashed blue line represents the connected path of steady-states and the red line the state under the control.

Remark 3.5. This strategy by construction requires a large time, while the controllability of the linear and semilinear heat equation can be achieved in arbitrarily small time.

3.5. **Minimal controllability time.**

Let us consider the problem (3.11) where f is Lipschitz. As mentioned before, when state constraints are not present, the controllability problem to some steady-state can be achieved in arbitrary small time [29].

Hence if one seeks to control in $T << 1$ the cost is becoming exponentially large. However, the results of [29] concern the dynamics without constraints in the state. Whenever one has state constraints such positivity there exists a minimal controllability time, for the linear heat equation see [57] and for the semilinear see [69]. Here, we will provide a schematic proof of the existence of a minimal control time in [69]

In the following we prove the existence of a minimal controllability time when bilateral state constraints are present.

Theorem 3.6 (Positivity of the minimal controllability time for the semilinear equation [69])**.** *Let us consider a steady-state target* $\overline{v}$*:*

$$\begin{cases} -\overline{v}_{xx} = f(\overline{v}) & x \in (0, L), \\ \overline{v}(0) = a_1, & \\ \overline{v}(L) = a_2, & \\ 0 < \overline{v} < 1 & x \in (0, L). \end{cases}$$

Furthermore consider the boundary control problem (3.11) *with target function* $\overline{v}$ *and* $v_0 \neq \overline{v}$*. Then the controllability cannot be achieved in arbitrary small time if the control function satisfies* $0 \leq a \leq 1$*.*

Proof. The result is an application of the comparison principle.

Without loss of generality we will consider only the case in which there exist a subset of positive measure $\omega \subset [0, L]$ in which $v_0(x) \geq \overline{v}(x)$.

Denote the trajectory of (3.11) associated with the control function $a \equiv 0$ by $v^{(1)}$.

By hypothesis we have that there exists a nonnegative test function $0 \neq \phi \in H_0^1(0, L)$ such that:

$$\int_0^L (v_0 - \overline{v}_0)\phi dx > 0.$$

Then since the solution is continuous in H^{-1}, there exists T_0 such that:

$$\int_0^L (v^{(1)}(\cdot,T) - \overline{v}(\cdot,T))\phi > 0 \quad \text{for all } T < T_0.$$

By the comparison principle we have that $v \geq v^{(1)}$ for all times lower than T_0. Then we have that $\int_0^L (v(\cdot,T) - \overline{v}(\cdot,T))\phi > 0 \quad$ for all $T < T_0$.

□

Remark 3.7. If we have unilateral constraints, the result is still true. However, it requires to employ duality techniques and to use the continuity of the normal derivative of the adjoint equation (see [69]).

4. Barriers and multiplicity of steady-states

4.1. Barriers.

The first result is of negative nature: *The lack of controllability for large L.* The main problem that we encounter for being able to control to stationary states is that multiple solutions for the same boundary conditions can act as a "barrier".

The comparison principles presented in Subsection 2.2 ensures that a solution of the elliptic equation:

$$\begin{cases} -\partial_{xx} w = f(w) & x \in (0, L), \\ w(0) = w(L) = 0, & \\ 0 < w(x) < 1 & \forall x \in (0, L) \end{cases} \tag{4.1}$$

will always be below the solution of the following parabolic problem

$$\begin{cases} v_t - \partial_{xx} v = f(v) & (x, t) \in (0, L) \times (0, T], \\ v(x, t) = a(x, t) \geq 0 & (x, t) \in \{0, L\} \times [0, T], \\ v(x, 0) \geq w(x). & \end{cases}$$

Then we see that $w(x)$ is an intrinsic obstruction to the controllability. We say that $w(x)$ acts as a barrier since

$$w(x) \leq v(x, t).$$

As a consequence, we can not reach any state below $w(x)$ with control functions $a(x, t)$ that are equal or above 0 in the boundary.

Recall that our control a physically means to be able to steer a proportion in the boundary, and for that reason, we have the constraint $0 \leq a(x, t) \leq 1$.

Hence, the existence of solutions for the Dirichlet condition equals to zero (4.1) is problematic in terms of the controllability to 0 (see Figure 6). We will see that the existence of multiple solutions depends basically on the length of the domain L.

The dependence on the domain L can be understood intuitively from the application level. There is a population that reproduces in the interior of a domain while being killed at the boundary. If the domain is big, the fact that the individuals that reach the boundary die might not compensate the reproduction of the population inside the domain.

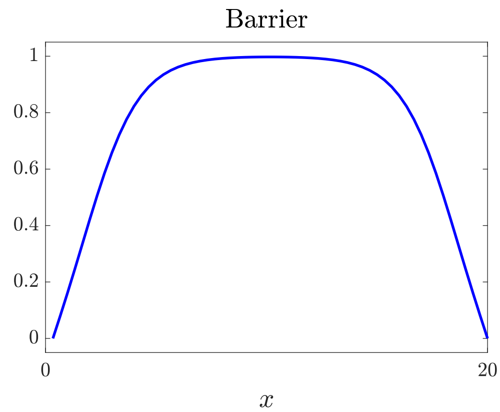

FIGURE 6. Simulation of the semilinear heat equation with cubic nonlinearity $f(y) = y(1 - y)(y - 1/3)$ in the interval $[0, 20]$ finding the barrier.

Note that nontrivial solutions with a boundary value 1 will have the same effect.

This section is devoted to studying the existence and nonexistence of nontrivial solutions.

We will restrict the study in the following section in the one-dimensional case, even though the results also hold in multi-D by the same techniques.

4.2. **Rescaling.**

Consider an interval $[0, L]$.

$$\begin{cases} v_t - \partial_{xx} v = f(v) & \text{in } [0, L], \\ v(0,t) = a_1(t), \quad v(L,t) = a_2(t) & \\ 0 \leq v(x,0) \leq 1 & \forall x \in [0, L] \end{cases} \tag{4.2}$$

First, let us reparameterize equation (4.2) for considering it into the interval $[0, 1]$. For every regular domain $[0, L]$ in which we consider the reaction-diffusion equation (4.2) under the spatial transformation $s(x) = x/L$ and the time transformation $\tau(t) = t/L^2$, and setting $v(L^2\tau, Ls) = u(\tau, s)$ the problem reads:

$$\begin{cases} u_\tau - \partial_{ss} u = L^2 f(u) & x \in [0, 1], \\ u(0,t) = a_1(t) \in [0, 1], & \\ u(1,t) = a_2(t) \in [0, 1], & \\ u(x,0) \in H_0^1([0, 1]). & \end{cases} \tag{4.3}$$

Following [74], we study the steady-states of (4.3).

We will omit the subscript s, and we will use t instead of τ and x instead of s whenever it is clear from the context.

In this section we denote $\lambda = L^2$. In [55] the existence of positive solutions for the semilinear elliptic problem and its multiplicity is studied, we collect here the results of the work [55] that can be applied in our steady-state problem (4.4)

Consider the interval $[0, 1]$ and consider the following problem:

$$\begin{cases} -\partial_{xx} u = \lambda f(u) & x \in [0, 1], \\ 0 < u < 1 & x \in [0, 1], \\ u(0) = u(1) = 0 & \end{cases} \tag{4.4}$$

where $\lambda = L^2$

4.3. **Variational formulation.**

The variational formulation of:

$$\begin{cases} -\partial_{xx} v = \lambda f(x, v), \\ v(0) = v(1) = 0, \end{cases} \tag{4.5}$$

is:

$$\int_0^1 v_x h_x - \lambda f(x, v) h dx = 0 \qquad \forall h \in H_0^1([0, 1]).$$

which corresponds to look for critical points of the following functional:

$$I : H_0^1([0, 1]) \longrightarrow \mathbb{R}$$
$$v \longrightarrow I[v] := \int_0^1 \frac{1}{2} v_x^2 - \lambda F(x, v) dx$$

where $F(x, v) = \int_0^{v(x)} f(x, s) ds$. The extrema of I correspond to weak solutions of (4.5).

Theorem 4.1 (Coercivity of I)**.** *Assume that:*

$$\limsup_{|s|\to\infty} \lambda \frac{f(x,s)}{s} < \lambda_1([0,1]), \tag{4.6}$$

where $\lambda_1([0,1]) = \pi^2$, the first eigenvalue of the Dirichlet Laplacian. Then I is coercive

Proof. For simplicity we will take care only on the case in which $s \to +\infty$, the case $s \to -\infty$ follows similarly. Since

$$\limsup_{s\to\infty} \lambda \frac{f(x,s)}{s} < \lambda_1([0,1])$$

we know that there exist $R > 0$ such that

$$\lambda \frac{f(x,s)}{s} < \lambda_1 \quad \forall s \geq R$$

using also the fact that $f(x,0) = 0$ we can write:

$$\begin{aligned}
\int_0^v \lambda f(s)ds &= \int_0^R \lambda f(s)ds + \int_R^v \lambda \frac{f(s)}{s} sds \\
&\leq \int_0^R \lambda f(s)ds + \frac{p}{2}\left(u^2 - R^2\right) \\
&\leq \frac{p}{2}u^2 + C(R,f,\lambda)
\end{aligned}$$

for a certain $p < \lambda_1([0,L])$. So

$$\begin{aligned}
I[u] &= \int_0^1 \frac{1}{2}u_x^2 - F(u)dx \\
&\geq \int_0^1 \frac{1}{2}u_x^2 - \frac{p}{2}u^2 - C(R,f,\lambda)dx \\
&\geq \frac{1}{2}\int_0^1 \left(1 - \frac{p}{\lambda_1([0,1])}\right) u_x^2 - C(R,f,\lambda)dx \\
&\geq \frac{1}{2}\left(1 - \frac{p}{\lambda_1([0,1])}\right)\int_0^1 u_x^2 dx - C(R,f,\lambda) \\
&\geq c\|u\|^2_{H_0^1([0,1])} - C(R,f,\lambda)
\end{aligned}$$

where $c > 0$ because $p < \lambda_1([0,1])$. So I is coercive.

□

Theorem 4.2 (Existence of a minimizer)**.** *Under the assumptions of Theorem 4.1, I has a minimizer.*

Proof. Note that the Lagrangian $L(p,z,x) = \frac{1}{2}|p|^2 - F(z)$ is convex with respect to the variable p. Therefore I is weakly lower-semicontinuous (Theorem 1 Ch.8 pp.468 in [23]) and the functional has a minimizer $u \in H_0^1([0,L])$. □

Remark 4.3. The hypothesis (4.6) is only needed for proving that the functional is coercive. In our case, it does not really matter. For the monostable nonlinearity, we require $f(s)$ to have a positive zero, which means that we can extend f by zero afterwards.

Note that for both the monostable case and the bistable case we can redefine f to be constant before $s = 0$ ($f(x,s) = 0$ for $s \leq 0$) and after $s = 1$ ($f(x,s) = 0$ for $s \geq 1$). In general we are interested on minimizers such that $0 \leq v(x) \leq 1$. After redefining f (if needed) by

$$\tilde{f}(s) := \begin{cases} f(s) \text{ if } s \geq 0, \\ 0 \text{ if } s < 0, \\ 0 \text{ if } s > 1, \end{cases}$$

we define the functional:

$$I : H_0^1([0,1]) \longrightarrow \mathbb{R}$$
$$v \longrightarrow I[v] := \int_0^1 \frac{1}{2}v_x^2 - \lambda \tilde{F}(v)dx$$

where $\tilde{F}(v) = \lambda \int_0^v \tilde{f}(s)ds$

Indeed $I[0] = 0$. Thanks to the redefinition of f, we ensure that the minimizer satisfies $0 < v < 1$ and satisfies the Euler-Lagrange equations:

$$\begin{cases} -\partial_{xx}u = \lambda\tilde{f}(u) & x \in (0,1) \\ u > 0 & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

since $0 < u < 1$, $\tilde{f} = f$ in this range and therefore is a solution of our original problem as well.

Definition 4.4. We define λ^* as the infimum value $\lambda \in \mathbb{R}^+$ for which there is a solution of:

$$\begin{cases} -\partial_{xx}v = \lambda f(v) & x \in (0,1), \\ 0 < v < 1 & x \in (0,1), \\ v(0) = v(1) = 0. \end{cases} \tag{4.7}$$

The following classical result [55] makes use of subsolutions to prove that if for a certain λ_1, there exists a nontrivial solution to (4.7) then, for any $\lambda > \lambda_1$ there will exist a nontrivial solution. Recall that $\lambda = L^2$.

Proposition 4.5. *For every $\lambda > \lambda^*$ there exist a nontrivial solution to* (4.7)

Proof. Assume that there exist a solution to:

$$\begin{cases} -(v_1)_{xx} = f(v_1) & x \in (0,L_1), \\ 0 < v_1 < 1 & x \in (0,L_1), \\ v_1(0) = v_1(L) = 0. \end{cases}$$

To prove that for any $L > L_1$ there exists a nontrivial solution, one can construct a subsolution of:

$$\begin{cases} -\partial_{xx}v = f(v) & x \in (0,L), \\ 0 < v < 1 & x \in (0,L), \\ v(0) = v(L) = 0, \end{cases}$$

using v_1. One can extend v_1 by zero in $[L_1, L]$,

$$\tilde{v}(x) = \begin{cases} v_1(x) & \text{if } x \in [0,L_1], \\ 0 & \text{if } x \in [L_1,L]. \end{cases}$$

$\tilde{v}$ is a weak subsolution for the problem in $[0,L]$, note that 1 is always a supersolution, therefore we have that if a nontrivial solution exists for $[0,L_1]$ for some $L_1 > 0$, then for every $L > L_1$ there will exist a nontrivial solution. ☐

4.4. **Monostable nonlinearity.**

4.4.1. *Dirichlet condition* 0. In this section, we will develop a variational approach to the boundary value problem. Making use of the variational structure of the problem, we will be able to give estimates from above for λ^* in any dimensions. Furthermore, the existence of positive solutions and some estimates can also be given for the case in which nonlinearities are depending on x, i.e. $f : [0,1] \times \mathbb{R} \to \mathbb{R}$.

The following Theorem was proven in [7, Theorem II.1], the proof given below is making use of a variational argument rather than comparison.

Theorem 4.6 (One Upper bound of λ^*). [2] *Assume that*

$$\lim_{s\to 0^+} \lambda \frac{f(x,s)}{s} > \lambda_1([0,1]) \text{ uniformly in } x \in [0,1]$$

and

$$\overline{\lim_{s\to\infty}} \lambda \frac{f(x,s)}{s} < \lambda_1([0,1]) \quad \forall x \in [0,1]$$

Then, for all $\lambda > \frac{\lambda_1([0,1])}{\min_{x\in[0,1]} f'(0,x)}$ *there exist a solution of the problem* (4.4)*. Therefore:*

$$\lambda^* \leq \frac{\lambda_1([0,1])}{\min_{x\in[0,1]} f'(0,x)}.$$

Proof. The main idea of the proof is simple: To prove the existence of a local minimizer for I that takes values in $0 \leq v \leq 1$ and to prove that 0 is not a minimizer. **Zero is not a minimizer**. Let e_1 be the first eigenfunction of the operator $A = (-\partial_{xx})$. We know that this function is positive. Set $v = \epsilon e_1$ for $\epsilon > 0$ to be chosen later on.

By hypothesis, we assumed that:

$$\lim_{s\to 0^+} \lambda \frac{f(x,s)}{s} > \lambda_1([0,1]) \text{ uniformly in } x \in [0,1]$$

this means that there exists $r \in \mathbb{R}^+$ such that

$$\frac{\lambda f(x,s)}{s} > \lambda_1([0,1]) \quad \forall s \in [0,r]$$

choose ϵ small enough such that $\epsilon e_1 < r$ and now evaluate the functional:

$$\begin{aligned} I[\epsilon e_1] &= \int_0^1 \frac{\epsilon^2}{2} \partial_x e_1^2 - \int_0^{\epsilon e_1} \lambda \frac{f(x,s)}{s} s ds dx \\ &\leq \int_0^1 \frac{\epsilon^2}{2} \partial_x e_1^2 - p\frac{\epsilon^2 e_1^2}{2} dx \end{aligned}$$

for some $p > \lambda_1([0,1])$. Then integrating by parts $\partial_x e_1^2$ and using the fact that $-\partial_{xx} e_1 = \lambda_1 e_1$:

$$I[\epsilon e_1] \leq \frac{\epsilon^2}{2} \int_0^1 (\lambda_1([0,1]) - p)\, e_1^2 dx < 0$$

Therefore we know that for all $\lambda > \frac{\lambda_1([0,1])}{\inf_{x\in[0,L]} f'(0,x)}$ there exists a positive solution, this means that

$$\lambda^* \leq \frac{\lambda_1([0,1])}{\inf_{x\in[0,L]} f'(0,x)}.$$

□

Remark 4.7 (Interpretation for applications). Note that the nonlinearities also depend on x. From the modeling perspective, this result allows us to discuss the situations in which every point in the space has different characteristics for the reproduction of species or different velocity for the reaction.

The following Theorem is a lower bound for this splitting in the case in which the nonlinearities are concave and twice differentiable.

Theorem 4.8 (A Lower bound of λ^*). *Let f be twice differentiable such that $f'(0) > 0$ and concave, i.e. $f''(t) \leq 0$. Then if:*

$$\lambda < \frac{\lambda_1([0,1])}{f'(0)}$$

[2] Here we provide a weaker version of the original theorem and moreover the proof given is a variational argument. In [7] one can find also a proof that does not rely on a variational argument

there cannot be any positive solution to Problem (4.4)*, Therefore,*

$$\lambda^* \geq \frac{\lambda_1([0,1])}{f'(0)} > 0$$

Proof. Multiply the equation by v and integrate over the domain and integrate by parts:

$$\int_0^1 -v\partial_{xx}v dx = \lambda \int_0^1 f(v)v dx,$$

$$\int_0^1 v_x^2 - \lambda \int_0^1 f(v)v = 0.$$

By the Poincaré inequality,

$$\int_0^1 \lambda_1([0,1])v^2 - \lambda f(v)v \leq 0$$

Now consider the Taylor formula on f

$$f(v) = f(0) + f'(0)v + \int_0^v f''(t)(v-t)dt$$

Due to the fact that $f(0) = 0$ we end up with

$$\int_0^1 \left(\lambda_1([0,1]) - \lambda f'(0)\right) v^2 - \lambda v \int_0^v f''(t)(v-t)dtdx \leq 0$$

since $v > 0$ we have that $v - t \geq 0$ moreover by assumption $f''(t) \leq 0$ we obtain that the second term is bigger or equal than zero, hence we can conclude that a necessary condition to have a positive solution is:

$$\int_0^1 \left(\lambda_1([0,1]) - \lambda f'(0)\right) v^2 \leq 0$$

which concludes the proof.

□

Remark 4.9 (Space dependent nonlinearity)*.* After a subtle change in the proof of 4.8, one can see that indeed the result also holds for the following problem:

$$\begin{cases} -\partial_{xx}v = \lambda f(v,x) \\ v(0) = v(L) = 0 \\ v > 0 & x \in [0,1] \end{cases} \tag{4.8}$$

where $f(v,x)$ is twice differentiable with respect to v, concave with respect to v. Following the argument we will end up with

$$\int_{[0,L]} \left(\lambda_1([0,1]) - \lambda f'(0,x)\right) v^2 \leq 0$$

If we assume that

$$f'(0,x) \geq \inf_{x\in[0,L]} f'(0,x) =: C > 0$$

We see that in this case we obtain also a lower bound for λ^*:

$$\lambda^* \geq \frac{\lambda_1([0,1])}{\max_{x\in[0,1]} f'(0,x)} > 0$$

Theorem 4.10 (λ^* for monostable concave nonlinearities)**.** *When f does not depend on x we have:*

$$\lambda^* = \frac{\lambda_1([0,1])}{f'(0)} > 0$$

Otherwise,

$$\frac{\lambda_1([0,1])}{\max_{x\in[0,1]} f'(x,0)} \leq \lambda^* \leq \frac{\lambda_1([0,1])}{\min_{x\in[0,1]} f'(x,0)}$$

Remark 4.11 (Uniqueness of positive solutions for concave nonlinearities). When f is concave and a positive solution exists it is unique [55] [6].

Remark 4.12 (Non uniqueness of positive solutions for non concave nonlinearities). When f is not concave we might not have uniqueness. Assuming that $f'(0) > 0$, in this case if $\lambda^* < \frac{\lambda_1([0,1])}{f'(0)}$ we have that for all λ such that $\lambda^* < \lambda < \lambda_1([0,1])$ there exists a second positive solution. This is proven using topological degree arguments [55].

4.4.2. *Phase Portrait.* In one dimension the elliptic equation

$$-\partial_{xx}u = f(u),$$

can be interpreted as a dynamical system Consider the following dynamical system:

$$\frac{d}{dx}\begin{pmatrix} u \\ u_x \end{pmatrix} = \begin{pmatrix} u_x \\ -f(u) \end{pmatrix}. \tag{4.9}$$

For the monostable nonlinearity we have

$$\begin{cases} u_x = v \\ v_x = -f(u) \end{cases}$$

we notice as before that 1 is a topological saddle for the nonlinear system and 0 is a center for the linearized system, the differential matrix is:

$$DF(u,v) = \begin{pmatrix} 0 & 1 \\ -\partial_u f(u) & 0 \end{pmatrix}$$

so since the condition of being monostable is to have only two equilibria, 0 and 1 for which $\partial_u f(u)\big|_{u=1} < 0$ and $\partial_u f(u)\big|_{u=1} > 0$ we have that

$$DF(1,0) = \begin{pmatrix} 0 & 1 \\ -\partial_u f(u)\big|_{u=1} > 0 & 0 \end{pmatrix}, \qquad DF(0,0) = \begin{pmatrix} 0 & 1 \\ -\partial_u f(u)\big|_{u=0} < 0 & 0 \end{pmatrix}$$

by symmetry with respect to the horizontal axis, we can also conclude that 0 is a center for the nonlinear system (also due to the fact that the system is Hamiltonian).

Moreover, by the first integral of the system, we know that the separatrix of the saddle is given by:

$$u_x = \pm\sqrt{2(F(1) - F(u))}.$$

The following Figure 7 is a representation of the phase portrait in the monostable case

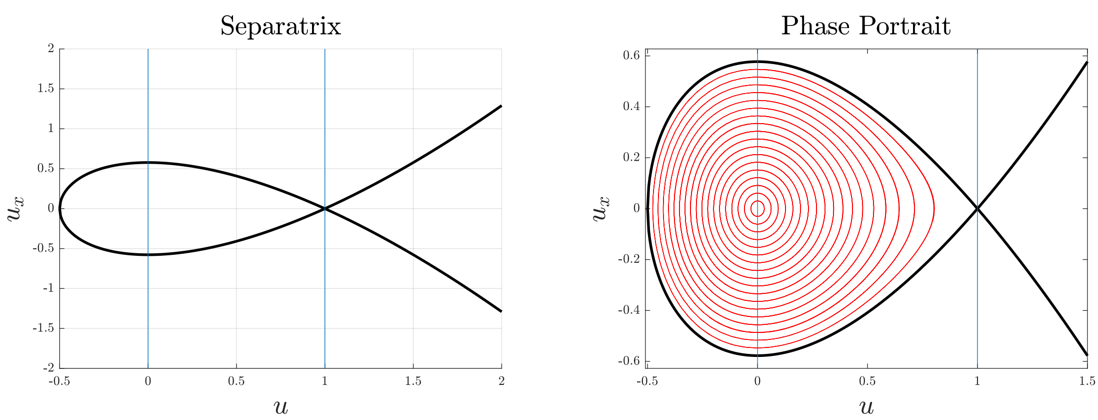

Figure 7. Left, separatrix of the system where the blue lines limit the admissible region. Right, the phase portrait inside limit curves of the separatrix.

4.4.3. *Bifurcation diagram.* In Figure 8 (left), one can qualitatively visualize the bifurcation diagram for a concave nonlinearity, while in Figure 8 (right) the case in which it is not concave. In the horizontal axis, $\lambda = L^2$ is represented, and in the vertical one, the infinity norm of the nontrivial solutions when they exist (for further examples see [55]).

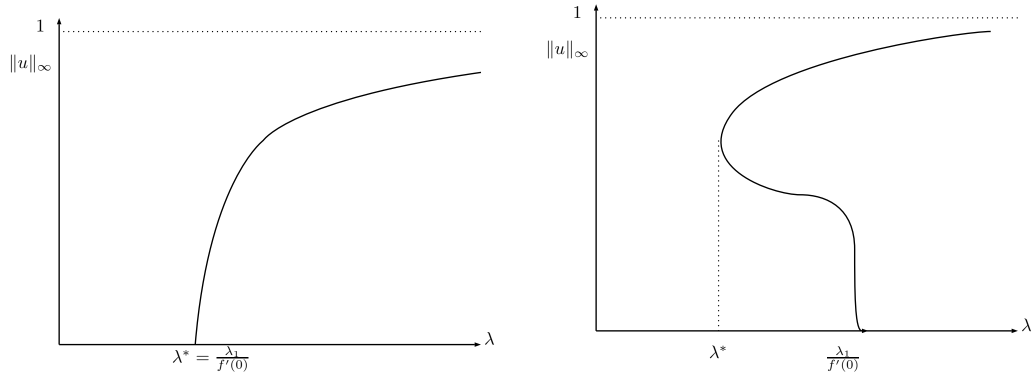

FIGURE 8. Qualitative Bifurcation diagram for the stationary solutions

4.4.4. *Dirichlet condition* 1. As said before, a non-trivial solution around the boundary value 1 would have the same blocking effect. However, we shall see that in this context, such solution does not exist.

To prove the nonexistence of a solution to the problem:

$$\begin{cases} -\partial_{xx}u = \lambda f(u) & x \in (0,1), \\ 0 < u < 1 & x \in (0,1), \\ u(0) = u(L) = 1. \end{cases} \tag{4.10}$$

we will show that for any initial datum between 0 and 1 the solution of the parabolic problem:

$$\begin{cases} u_t - \partial_{xx}u = \lambda f(u) & (x,t) \in (0,1) \times (0,T) \\ 0 < u < 1 & (x,t) \in (0,1) \times (0,T) \\ u(0,t) = u(1,t) = 1 & t \in (0,T) \\ 0 \leq u(x,0) \leq 1 \end{cases}$$

goes asymptotically to $u(x) = 1$. This will imply that there is no solution to (4.10). Indeed we can write the semilinear parabolic problem as a gradient system:

$$u_t = -\nabla_u I[u],$$

where I is:

$$I[u] = \int_0^1 \frac{1}{2}u_x^2 - \lambda F(u)dx.$$

So, convergence to 1 for any initial data implies that $I[u]$ has no critical point for any $u \in H^1([0,1])$ satisfying $0 < u < 1$ with Dirichlet trace equals to 1, and therefore it does not exist any weak solution of (4.10).

For monostable nonlinearities a Lyapunov functional exists [73]:

$$V(t) := \int_0^1 u - 1 - \log(u)dx$$

indeed,

$$\frac{d}{dt}V(t) = -\int_0^1 \frac{u_x^2}{u^2}dx - \int_0^1 \lambda f(u)\frac{1-u}{u}dx \leq 0$$

Remark 4.13 (Comparison with Travelling waves)*.* Another way to check it is by using the comparison principle with the traveling wave solution for the Cauchy problem. Indeed we know that a decreasing traveling wave function exists for monostable nonlinearities [68, Ch.4 Th. 4.5 pp.67]. Since it is decreasing and connecting 0 and 1 for any initial data $u(x, 0) > 0$ of the parabolic problem, we can choose a section of the traveling wave that is strictly under $u(x, 0)$. Then, the boundary conditions of this section of the traveling wave will be below 1, so this restriction of the traveling wave is a subsolution of the parabolic problem with Dirichlet data equals to 1. This argument enables us to conclude that the solution of the parabolic problem will converge to 1.

4.5. **Bistable nonlinearity.**

4.5.1. *Dirichlet condition* 0*.* Now we turn our attention to bistable nonlinearities. The structure of the proofs estimates will be similar.

Theorem 4.14 (A Lower bound for λ^*)**.** *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$, assume that f is bounded uniformly with respect to x. Assume furthermore that $f(x, 0) = 0$ Consider:*

$$\begin{cases} -\partial_{xx} v = \lambda f(x, v) & x \in (0, 1), \\ v(0) = v(1) = 0, & \\ 1 > v > 0 & (x, t) \in (0, 1). \end{cases}$$

Then,

$$\frac{\lambda_1([0, 1])}{\max_{(x,s)\in[0,1]\times[0,1]} \frac{f(x,s)}{s}} \le \lambda^*$$

Proof. For proving it, we will make use of the parabolic problem:

$$\begin{cases} v_t - \partial_{xx} v = \lambda f(x, v) & (x, t) \in (0, 1) \times (0, T), \\ v(0, t) = v(1, t) = 0 & t \in (0, T), \\ 1 > v(x, 0) > 0 & x \in [0, 1]. \end{cases} \tag{4.11}$$

We generate a supersolution in the following way: first observe that if v is a solution of (4.11) then we have:

$$\lambda f(v) = \lambda \frac{f(x, v)}{v} v \le \lambda P v,$$

since $\frac{f(x,v)}{v} \le P$ for some P. Then we know that the solution of problem (4.11) is lower than y solution to:

$$\begin{cases} y_t - y_{xx} = \lambda P y & (x, t) \in (0, 1) \times (0, T), \\ y(0, t) = y(1, t) = 0 & t \in (0, T), \\ 1 > y(x, 0) > 0 & x \in [0, 1]. \end{cases} \tag{4.12}$$

Consider the eigenfunctions of the Laplacian $-\partial_{xx} e_n = \lambda_n e_n$. One can express the solution of the problem above by the separation of variables technique: Let $c_n(t) = (y(t), e_n)_{L^2([0,1])}$, and take its time derivative:

$$\begin{aligned} \frac{d}{dt} c_n(t) &= (y_t, e_n)_{L^2([0,1])} = (y_{xx} + \lambda P y, e_n)_{L^2([0,1])} \\ &= (y, \partial_{xx} e_n)_{L^2([0,1])} + \lambda P\, (y, e_n)_{L^2([0,1])} \\ &= (\lambda P - \lambda_n)\, c_n(t) \end{aligned}$$

So,

$$y(t, x) = \sum_{n=1}^{\infty} (v(x, 0), e_n)_{L^2([0,1])}\, e^{(\lambda P - \lambda_n)t} e_n$$

As we said we know by the parabolic comparison principle that:

$$\begin{aligned}\|v(t)\|_{L^2}^2 \leq \|y(t)\|_{L^2}^2 &= \sum_{n=1}^{\infty} \left|(v(x,0), e_n)_{L^2([0,1])}\right|^2 e^{2(\lambda P - \lambda_n([0,1]))t} \\ &\leq \|v_0\|^2 e^{2(\lambda P - \lambda_1([0,1]))t}\end{aligned}$$

If $\lambda P < \lambda_1([0,1])$ then $v(t)$ will go to zero independently of the initial datum.

For concluding we come back to the elliptic problem to see a contradiction with the existence of a positive solution.

Assume that $v(x,0)$ is stationary and solves the semilinear elliptic problem. By the estimate just shown above, we know that differentiating with respect to time, we should get zero from one side and the estimate on the right-hand side from the parabolic problem.

$$0 = 2v(\partial_{xx} v + \lambda f(x,v)) \leq 2(\lambda P - \lambda_1)\|v(x,0)\|^2 e^{2(\lambda P - \lambda_1)t}$$

But the quantity on the right is negative if $v(x,0)$ is not identically zero. So the proof is concluded. This estimate is:

$$\frac{\lambda_1([0,1])}{\max_{(x,s)\in[0,1]\times[0,1]} \frac{f(x,s)}{s}} \leq \lambda^*$$

□

Theorem 4.15 (An upper bound for λ^*). *Assume that $f(0) = f(\theta) = f(1) = 0$, and that $f'(0) < 0$, $f'(1) < 1$, $f'(\theta) > 0$. Moreover consider $F(v) = \int_0^v f(s)ds$ and assume that $F(1) > 0$. Consider the interval $[0,1]$. The following problem:*

$$\begin{cases} -\partial_{xx} u = \lambda f(u) & x \in (0,1) \\ u(0) = u(1) = 0 & \\ u > 0 & x \in [0,1] \end{cases} \tag{4.13}$$

has a solution for every $\lambda \geq 8\frac{F(1)-F(\theta)}{F(1)^2}$.

This implies an upper bound of λ^*:

$$\frac{\pi^2}{\max_{s\in[0,1]} \frac{f(s)}{s}} \leq \lambda^* \leq 8\frac{F(1)-F(\theta)}{F(1)^2}$$

Proof. We know that 0 is a solution of the Euler-Lagrange equations of the corresponding functional.

$$I[u] = \frac{1}{2}\int_0^1 u_x^2 dx - \lambda \int_0^1 F(v)dx.$$

The strategy of the proof is the following, as in the other Theorem 4.6, we proof that the functional admits a minimizer independently of λ.

Then we construct a family of functions $v_\delta \in H_0^1((0,L))$. This family is zero in the boundary and increase linearly to 1 (where $F(1) > 0$) and δ is related with the measure of the set in which the function is not constant.

The idea is: first ensure under which conditions on δ we have that:

$$\int_0^1 F(v(x)) > c > 0,$$

once we have this, we choose λ in order to dominate the term:

$$\frac{1}{2}\int_0^1 v_x^2 dx.$$

that will depend only on the δ chosen before and thus we constructed v such that:

$$I[v] < 0.$$

We define v_δ in the following way:

$$v_\delta(x) = \begin{cases} \frac{2}{\delta}x & \text{if } x \in \left[0, \frac{\delta}{2}\right] \\ 1 & \text{if } x \in \left(\frac{\delta}{2}, 1 - \frac{\delta}{2}\right) \\ -\frac{2}{\delta}\left(x + \frac{2}{\delta}\right) & \text{if } x \in \left[1 - \frac{\delta}{2}, 1\right]. \end{cases}$$

One can see the function v_δ in Figure 9.

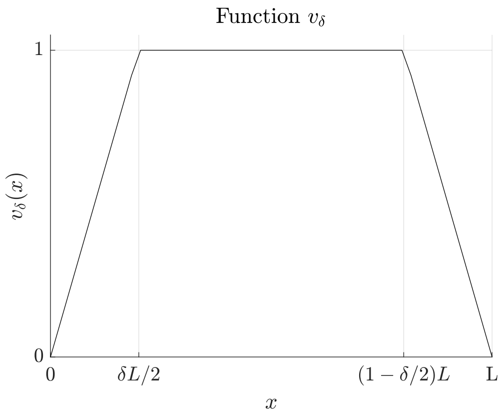

FIGURE 9. Function v_δ

Note that $v_\delta \in H_0^1((0,1))$. Then we have that:

$$(\partial_x v_\delta)^2 = \begin{cases} 0 & \text{if } x \in \left(\frac{\delta}{2}, 1 - \frac{\delta}{2}\right) \\ \frac{4}{\delta^2} & \text{if } x \in \left[0, \frac{\delta}{2}\right] \cup \left[1 - \frac{\delta}{2}, 1\right] \end{cases}$$

we want to find a pair (λ, δ) for which:

$$I[v] = \int_0^1 \frac{1}{2}|\partial_x v|^2 - \lambda \int_0^{v(x)} f(s)dsdx < 0$$

For doing so, first we choose $\delta > 0$ to be small enough such that:

$$\int_0^1 \int_0^{v(x)} f(s)dsdx > c > 0$$

we split the space integral in two parts:

$$\begin{aligned} \int_0^1 \int_0^{v(x)} f(s)dsdx &= \int_{\frac{\delta}{2}}^{1-\frac{\delta}{2}} \int_0^{v(x)} f(s)dsdx + \int_{[0,1]\setminus\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)} \int_0^1 f(s)dsdx \\ &\geq \int_{[0,1]\setminus\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)} F(\theta)dx + F(1)(1-\delta) \\ &= F(1)(1-\delta) + F(\theta)\delta \end{aligned}$$

So, it will suffice if we require that:

$$0 < \delta < \frac{F(1)}{F(1) - F(\theta)}. \tag{4.14}$$

We fix δ fulfilling (4.14) and now or goal is to choose λ big enough so that the space integral on $F(v(x))$ dominates the gradient part.

$$\begin{aligned} I[v_\delta] &= \int_0^1 \frac{1}{2}|\partial_x v_\delta|^2 - \lambda F(v_\delta(x))dx \\ &\leq \int_0^1 \frac{1}{2}|\partial_x v_\delta|^2 dx - \lambda\left(F(1)(1-\delta) + \delta F(\theta)\right) \\ &= \int_{[0,1]\setminus\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)} \frac{2}{\delta^2}dx - \lambda\left(F(1)(1-\delta) + \delta F(\theta)\right) \\ &= \frac{2}{\delta} - \lambda\left(F(1)(1-\delta) + \delta F(\theta)\right) \end{aligned}$$

so, it will be sufficient if:

$$\lambda > \frac{2}{\delta\left(F(1)(1-\delta) + F(\theta)\delta\right)} \tag{4.15}$$

Hence, any pair (λ, δ) that satisfies both (4.14) and (4.15) will guarantee the existence of a nontrivial solution. Now the question is: For which choice of δ can we obtain the minimum upper bound in terms of λ?

We choose the δ in the interval $0 < \delta < \frac{F(1)}{F(1)-F(\theta)}$ such that maximizes the denominator:

$$\delta\left(F(1)(1-\delta) + F(\theta)\delta\right).$$

We note that $-F(1)+F(\theta)$ is negative, hence we have a polynomial in δ that attains its maximum in:

$$\delta^* = \frac{F(1)}{2(F(1)-F(\theta))}$$

This optimal δ^* satisfies the requirement on δ:

$$\delta^* = \frac{F(1)}{2(F(1)-F(\theta))} < \frac{F(1)}{F(1)-F(\theta)}$$

So we have that:

$$\lambda > \frac{8(F(1)-F(\theta))}{F(1)^2},$$

will be enough.

□

Remark 4.16. Notice that the structure of the proof of Theorem 4.15 also works for the monostable case. When bounding by above the integral of the primitive, we will have $F(1)$ instead of $F(1) - F(\theta)$ because the primitive in the monostable case is monotone.

Corollary 4.17 (Unexpected Corollary)**.** *Let f by any $C^2(\mathbb{R};\mathbb{R})$ function satisfying:*

- *$f(0) = f(\theta) = f(1) = 0$ with $0 < \theta < 1$*
- *consider $F(v) = \int_0^v f(s)ds$ and suppose that $F(1) > 0$*
- *$f'(0) < 0$, $f'(\theta) > 0$ and $f'(1) < 0$*
- *$f < 0$ in $(0,\theta)$ and $f > 0$ in $(\theta, 1)$*

Then,

$$\pi^2 \leq 8\frac{F(1)-F(\theta)}{F(1)^2}\max_{s\in[0,1]} f'(s) := Q$$

Proof. Indeed this is a corollary from the theorem proven before.

$$\max_{s\in[0,1]} \frac{f(s)}{s} = \max_{s\in[0,1]} f'(s)$$

if f is differentiable and the derivative at 0 exists. Applying the mean value theorem we obtain that

$$\frac{f(s)}{s} = \frac{f(s) - f(0)}{s} = f'(\xi) \quad \text{for some } \xi \in [0, s]$$

□

Remark 4.18 (Open question). The remarkable issue here is that Corollary 4.17 has a very mild reminiscence of the PDE theory, $\lambda_1 = \pi^2$ in $[0, 1]$. How can this general estimate be proven without the PDE theory?

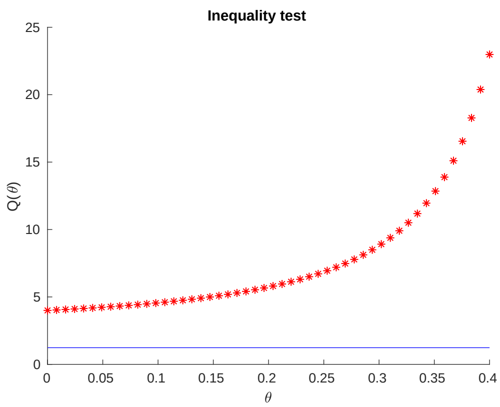

FIGURE 10. The blue line is π, the red dots is the quantity Q depending on θ for the prototypical example $f(s) = s(1 - s)(s - \theta)$. For this nonlinearity $F(1) > 0$ is equivalent to $\theta < 1/2$. The values between 0.4 and 0.5 are not shown because the function blows up since for $\theta = 1/2$, one has that $F(1) = 0$.

Remark 4.19 (Existence of positive solutions for nonlinearities depending on space). Similar arguments can be applied to show the existence of positive solutions with space-dependent nonlinearities. Consider:

$$\begin{cases} -\partial_{xx} u = f(x, u) & x \in (0, 1), \\ u(0) = u(1) = 0, & \\ 1 > u > 0 & x \in [0, 1]. \end{cases}$$

Consider that exist smooth functions $\theta, K : [0, 1] \longrightarrow \mathbb{R}^+$ such that:

- $\theta(x) < K(x)$ for all x
- $f(\theta(x), x) = 0$ for all x
- $f(K(x), x) = 0$ for all x
- $f(s, x) < 0$ for all $0 < s < \theta(x)$
- $f(s, x) > 0$ for all $\theta(x) < s < K(x)$
- There exist $x^* \in [0, 1]$ such that $F(K(x^*), x^*) > 0$.

Then there is a finite λ for which the positive solution exists.

Remark 4.20 (Survival of the gene). The physical interpretation of the previous result is the following: Consider a a population with two characteristics. Each characteristic is advantageous only in some parts of the environment. If the set in which one characteristic is advantageous is big enough we can not control to zero this characteristic by means of a boundary control.

Remark 4.21 (Double blocking phenomenon). Note that constructing spatial heterogeneities like this one, one can generate nontrivial solutions with boundary 1 and 0. Setting different values of $\theta(x)$ above and below $\frac{1}{2}$ for the nonlinearity $f(y) = y(1 - y)(y - \theta(x)$ one could apply these methods to prove the existence of both nontrivial solutions. By the comparison principle, this example leads to a double blocking phenomenon, already observed in [61].

Proposition 4.22 (Maximum of positive solutions)**.** *Let u be a solution to:*

$$\begin{cases} -\partial_{xx}u = \lambda f(u) & x \in (0,1), \\ u(0) = u(1) = 0, & \\ 1 > u > 0 & x \in (0,1), \end{cases}$$

with f being bistable then the maximum of u in $[0,1]$ is above θ:

$$\max_{x\in[0,1]} u(x) > \theta.$$

Proof. The proof follows by contradiction. Assume that the maximum of u is lower or equal than θ, then the energy estimate gives us the contradiction:

$$0 < \int_0^1 u_x^2 dx = \int_0^1 uf(u) < 0$$

where the strict inequality in the left-hand side comes from the assumption that the solution is not trivial, and the right-hand side inequality comes from the fact that f is negative in $(0,\theta)$. □

4.5.2. *Dirichlet condition θ.* Note that these results imply the thresholds for λ_θ^* for the bistable nonlinearities i.e. when the nontrivial solution to:

$$\begin{cases} -\partial_{xx}v = \lambda f(v) \\ v(0) = v(1) = \theta \\ v > \theta \end{cases}$$

By hypothesis, we know that $f > 0$ in $(\theta, 1)$, $f'(\theta) > 0$, $f'(1) < 0$ and $f(\theta) = f(1) = 0$. f is monostable in $[\theta, 1]$, indeed $w = v - \theta$ does satisfy the criteria of the previous theorems.

Theorem 4.23 (The estimates of λ_θ^* for bistable nonlinearities)**.**

$$\frac{\pi}{\max_{s\in[\theta,1]} \frac{f(s+\theta)}{s-\theta}} \leq \lambda_\theta^* \leq \min\left\{\frac{2}{F(1)-F(\theta)}, \frac{\pi}{f'(\theta)}\right\}$$

If f is convex in $(0,\theta)$ and concave in $(\theta,1)$ then:

$$\lambda_\theta^* = \frac{\pi}{f'(\theta)}$$

Leaving aside the estimates given above, one, in general, has the following result.

Theorem 4.24 (Order in the thresholds)**.** *When $F(1) > 0$ we have that:*

$$\lambda_\theta^* \leq \lambda^*$$

and fixing $\lambda > \lambda^*$ we denote by v_θ and v_0 the maximum nontrivial solution bounded by 1 of the elliptic problem with Dirichlet boundary conditions equal to θ and 0 respectively, then:

$$v_\theta \geq v_0$$

Proof. The result follows from the elliptic comparison principle, together with the fact that any nontrivial solution of the boundary value problem has its maximum above θ. □

4.5.3. *Dirichlet condition* 1. Here we will consider the problem:

$$\begin{cases} -\partial_{xx}u = \lambda f(u) & x \in (0,1), \\ 1 > u > 0 & x \in (0,1), \\ u(0) = u(1) = 1 & \end{cases} \tag{4.16}$$

As mentioned before, by using comparison principles to sections of traveling waves, we can prove that we converge to 1 for any size of the domain.

Proposition 4.25 (Convergence to 1 for any domain)**.** *For any bounded domain $[0, L]$, the solution of the reaction diffusion system* (4.2) *with Dirichlet data equals to* 1 *converges to* 1*.*

Proof. We know that the problem:

$$\begin{cases} u_t - \partial_{xx}u = f(u) & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ 0 \le u(0,x) \le 1 & x \in \mathbb{R}. \end{cases}$$

Has a traveling wave solution, and by Theorem 2.9, we know that the traveling wave profile is a monotone function decreasing in the direction of the velocity vector. The idea is to use a segment of the traveling wave as a parabolic subsolution to our problem. Now we come back to our (parabolic) problem;

$$\begin{cases} u_t - \partial_{xx}u = f(u) & \text{in } (0,T] \times [0,L], \\ 0 < u(0,x) < 1 & x \in [0,L], \\ u(t,x) = 1 & \text{on } [0,T] \times \{0,L\}. \end{cases} \tag{4.17}$$

Since the traveling wave profile is monotone decreasing we can consider a segment of the traveling wave such that is below to $u(0,x)$ in $[0,L]$, let us denote by $TW(x)$ the maximum profile of traveling wave that satisfies:

$$TW(x) \le u(x,0) \quad \forall x \in [0,L].$$

Now we note that the following problem:

$$\begin{cases} u_t - \partial_{xx}u = u(1-u)(u-\theta) & \text{in } (0,T] \times [0,L], \\ u(0,x) = TW(x) & x \in [0,L], \\ u(t,x) = TW(x-ct) & \text{on } [0,T] \times \{0,L\} \end{cases} \tag{4.18}$$

is a subsolution of (4.17), then by the parabolic comparison principle we have that the solution of (4.17) will be above (4.18) and therefore the solution of (4.17) will converge to 1.

□

4.5.4. *Phase Portrait.* As it has been already said, in one dimension the elliptic equation

$$-\partial_{xx}u = f(u)$$

can be interpreted as a dynamical system Consider the following dynamical system:

$$\frac{d}{dx}\begin{pmatrix} u \\ u_x \end{pmatrix} = \begin{pmatrix} u_x \\ -f(u) \end{pmatrix}$$

Here we study the phase portrait of system (4.9). First of all we notice that the points $(0,0)$ (corresponding to the stationary solution $u(x)=0$) and $(1,0)$ (corresponding to the stationary solution $u(x)=1$) are saddles for all values of $\theta \in (0,1)$. Indeed our nonlinearity fulfills:

$$\frac{\partial}{\partial u}f(u)\big|_{u=0} < 0, \qquad \frac{\partial}{\partial u}f(u)\big|_{u=1} < 0, \qquad \frac{\partial}{\partial u}f(u)\big|_{u=\theta} > 0$$

The corresponding linearized system around $(0,0)$ and the one around $(1,0)$ have the following matrices:

$$\begin{pmatrix} 0 & 1 \\ -\frac{\partial}{\partial u}f(u)\big|_{u=0} & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -\frac{\partial}{\partial u}f(u)\big|_{u=1} & 0 \end{pmatrix}$$

The eigenvalues of those matrices are real, and with a different sign. Then we know that for the nonlinear system, we have topological saddles. In the other hand, for $(\theta,0)$, we have that the corresponding linearized system is a center since the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 \\ -\frac{\partial}{\partial u}f(u)\big|_{u=\theta} & 0 \end{pmatrix}$$

lie in the imaginary axis. This is not enough to conclude, but observing that the system is symmetric with respect to the horizontal axis, we have that $(\theta,0)$ is a center for the nonlinear system.

One can easily find a first integral of the system:

$$E(u,v)=\frac{1}{2}v^2+F(u)$$

where $F(u)=\int_0^u f(s)ds$.

For $F(1)>0$, notice that the separatrix of the saddle in 0 is the same trajectory and encloses $(\theta,0)$. Indeed, $E(0,0)=0$, hence we have the curves $v=\pm\sqrt{-2F(u)}$ below and above the horizontal axis. At the point $0<\theta_1<1$ that fulfills $F(\theta_1)=0$ these curves meet. One can see this since $F(1)>0$ and since $F(u)<0$ for all $0<u<\theta$, we know that there exist a $1>\theta_1>\theta$ such that $F(\theta_1)=0$.

Notice that when $F(1)=0$ one has that $F(u)<0$ for all $0<u<1$. This means that the separatrix split into two trajectories that connect 0 and 1 (These are the traveling wave profiles for $F(1)=0$).

We will call Γ the region in the phase-plane that the separatrix $v_{E=0}(u)=\pm\sqrt{-2F(u)}$ encloses. Notice that Γ is included in $[0,1]\times\mathbb{R}$ which means that all arcs of any length inside Γ are admissible for our constraints.

Doing the same procedure for finding the separatrix that exit from $(1,0)$ we end up with the curves $v_{E=1}(u)=\pm\sqrt{2(F(1)-F(u))}$. At the vertical axis, $u=0$, they take values $v_{E=1}(0)=\pm\sqrt{2F(1)}$.

Notice that, in the case of the cubic nonlinearity $f(u)=u(1-u)(u-\theta)$, as we increase θ towards $\frac{1}{2}$, $v_{E=1}(0)$ is decreasing until arriving to 0 for $\theta=\frac{1}{2}$, while, at the same time θ_1 goes to 1.

Moreover, we can also find the separatrix outside our admissible domain. We can see that

$$v_{E=1}=\pm\sqrt{2(F(1)-F(u))}$$

is well defined for $u>1$ since $F(u)$ is a strictly decreasing function so $-F(u)$ is increasing. This means that both separatrix do not cross the horizontal axis anymore after $u=1$. This is valid also for $F(1)=0$.

Furthermore, the separatrix going out from 0 towards $-\infty$,

$$v_{E=0}=\pm\sqrt{-2F(u)}$$

is well defined for $u<0$. Notice that this argument also holds for $F(1)=0$.

See Figure 11 and 12 for the phase-portrait representation and the graphical representation of the separatrix.

The above discussion is the proof of the following significant result.

Proposition 4.26 (Admissible boundary conditions, admissible steady-states)**.** *Consider the problem:*

$$\begin{cases}-\partial_{xx}u=f(u)\\ u(0)=a\\ u(L)=a\\ 0\le a\le 1\end{cases}\tag{4.19}$$

with a admissible, i.e. $0\le a\le 1$. *Then any solution $u(x)$ of* (4.19) *is in the admissible region.*

Proof. The solution of (4.19) is symmetric, and hence in the phase plane, the initial datum lies above the horizontal axis and the final datum below it. Then by the uniqueness of the ODE system, we know that we cannot cross the separatrix. Hence, if the solution attaches a value above 1 or below 0, it would not be able to cross the separatrix that is growing towards $\pm\infty$. Noticing that the sign of the horizontal component of the vector field $\begin{pmatrix}v\\-f(u)\end{pmatrix}$ only changes when we cross the horizontal axis, we see that, in the case that we go out from our admissible region, we are not going to enter in it again. □

Remark 4.27. This proposition will be crucial for proving the controllability towards θ when $F(1)=0$ with the same control in both boundaries.

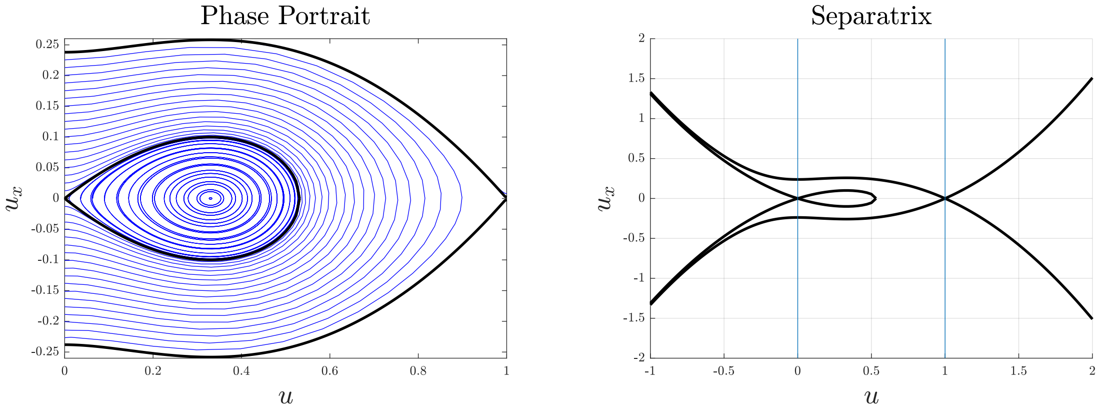

FIGURE 11. Phase portraits for $f(s) = s(1-s)(s-\theta)$, when $\theta < 1/2$ ($F(1) > 0$), left in the admissible region, right separatrix. The blue lines in the right hand side plot denote the admissible region. Down, the modulus of the velocity in the plane

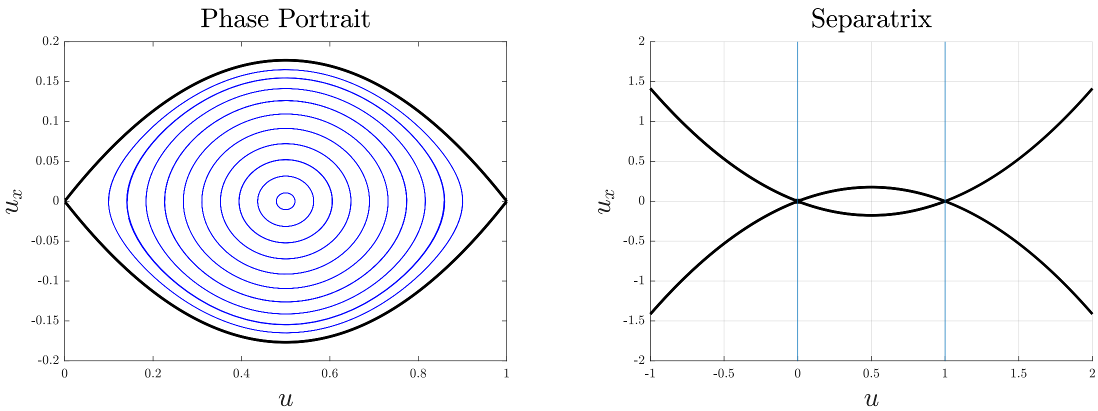

FIGURE 12. Phase portraits for $f(s) = s(1-s)(s-\theta)$ when $\theta = 1/2$ ($F(1) = 0$), left in the admissible region, right separatrix. The blue lines in the right hand side plot denote the admissible region. Down, the modulus of the velocity in the plane

4.5.5. *An expression for L in the phase plane.* In the particular case of one dimension, one can obtain an expression for the length L (time L in the ODE setting) in the phase portrait using the first integral of the system. We restrict our study in the curves that cross the vertical axis. The parameter $\alpha \in (0, F(1))$ is introduced. The time needed for a trajectory that starts in the vertical axis until it returns to it (the length of the stationary solution).

Notice that any trajectory starting from the vertical axis until it reaches is maximum is strictly increasing. This means that it is a C^1 diffeomorphism from the time interval $[0, 1/2L(\alpha)]$ to its altitude $[0, u_{max}(\alpha)]$. Let us denote this diffeomorphism (that depends on α) by U :

$[0, 1/2L(\alpha)] \to [0, u_{max}(\alpha)]$. Then noticing that $u_{max}(\alpha) = Y(L(\alpha)/2)$ and that $0 = Y(0)$:

$$\begin{aligned} L(\alpha) &= 2\int_0^{L(\alpha)/2} dz = 2\int_{U^{-1}(0)}^{U^{-1}(L(\alpha)/2)} dz \\ &= 2\int_0^{u_{max}(\alpha)} \left(U^{-1}\right)'(u)du = 2\int_0^{u_{max}(\alpha)} \frac{1}{U'(U^{-1}(u))}du. \end{aligned}$$

Now, we make use of the first integral, notice that here x plays the role of time, so the term $v = u_x = U'$. Moreover, $u_{max}(\alpha) = F^{-1}(\alpha)$ and $Y' = \sqrt{2}\sqrt{\alpha - F(u)}$. For the bistable nonlinearity F is not invertible but in the case of looking for trajectories that cross the vertical axis and that are below the separatrix going out from 1 we notice that in that set is invertible,

$$F : [\theta_1, \sqrt{2F(1)}] \to [0, F(1)]$$

is monotone increasing.

$$L(\alpha) = \sqrt{2}\int_0^{F^{-1}(\alpha)} \frac{1}{\sqrt{\alpha - F(U(U^{-1}(u))}}du = \sqrt{2}\int_0^{F^{-1}(\alpha)} \frac{1}{\sqrt{\alpha - F(u)}}du$$

Changing the way of parameterizing an expression for L^* can be written in the following way:

$$L^* = \inf_{\beta\in(\theta_1,1)} \sqrt{2}\int_0^{\beta} \frac{du}{\sqrt{F(\beta) - F(u)}}$$

Using the aforementioned expression in [74] the authors prove the following thresholds

Proposition 4.28 (Proposition 4 in [74]). *If f is C^2, monostable and the following holds:*

$$f^2 \geq 2Ff'$$

then,

$$L^* \geq \frac{\pi}{\sqrt{\frac{\pi^2}{f'(0)}}}$$

4.5.6. *Bifurcation diagrams.* In this subsection, we show bifurcation diagrams for some bistable nonlinearities and graphics of the bounds obtained before.

The blue and the red lines in Figures 13 and 14 represent the nontrivial solutions for Dirichlet boundary 0 and θ respectively. For the blue curve, the vertical axis is the infinity norm. For the red curve, the vertical axis is the infinity norm for when the curve is above θ and $\theta - \|u - \theta\|_\infty$ when the curve is under θ.In this way it is showing the minimum value taken since it corresponds to solutions for the following problem:

$$\begin{cases} -\partial_{xx}v = \lambda f(v) & x \in (0,1) \\ 0 < v < \theta & x \in (0,1) \\ v(0) = v(1) = \theta \end{cases}$$

Figures 15 and 16 show the bounds for the nonlinearity $f(s) = s(1-s)(s-\theta)$ for different values of θ.

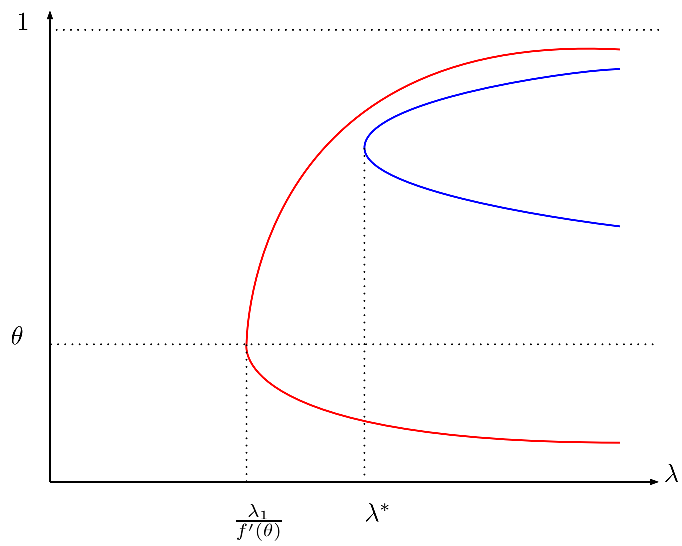

Figure 13. Qualitative Bifurcation diagram for the stationary solutions for a bistable nonlinearity which is convex in $(0, \theta)$ and concave in $(\theta, 1)$

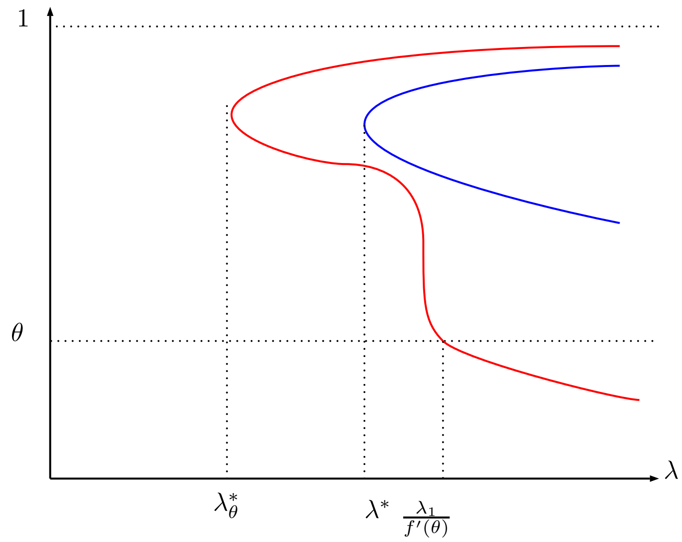

Figure 14. Qualitative Bifurcation diagram for a bistable nonlinearity that is convex in $(0, p)$ with $p > \theta$ and concave in (p, θ) as $f(s) = s(1-s)(s-\theta)$ for $\theta < 1/2$.

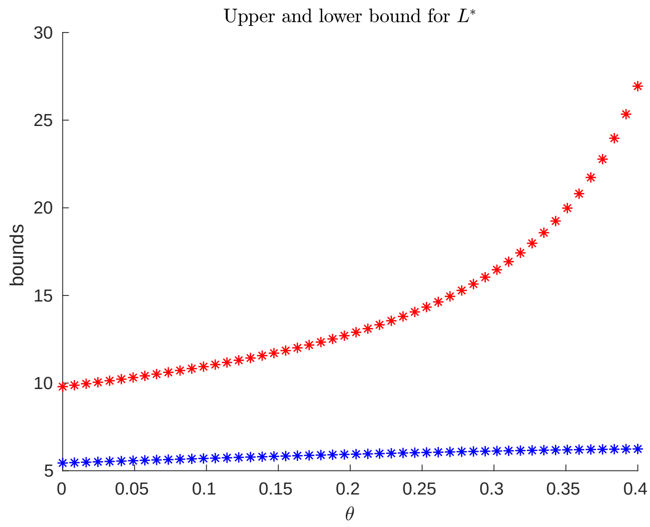

FIGURE 15. Bounds on L^* for different values of θ for the nonlinearity $f(s) = s(1-s)(1-\theta)$. The red dots represent upper bounds and the blue dots lower bounds.

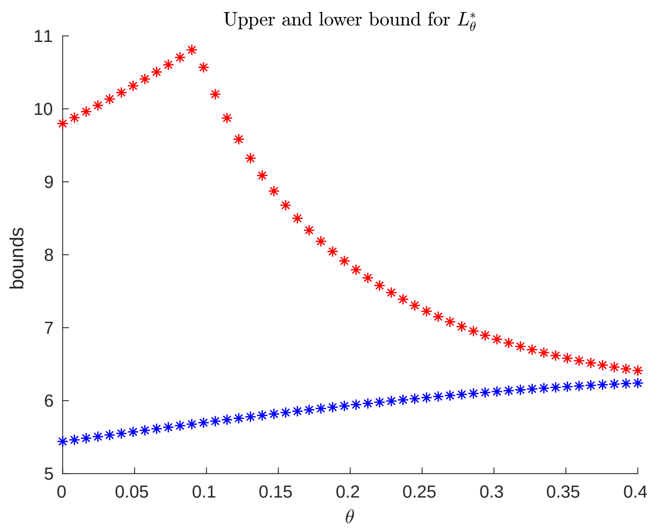

FIGURE 16. Bounds on L^*_θ for different values of θ for the nonlinearity $f(s) = s(1-s)(1-\theta)$. The red dots represent upper bounds and the blue dots lower bounds.

Remark 4.29. Note that the lower bound proven for λ^* and λ^*_θ is the same, in general, to our knowledge, we do not know if $\frac{\lambda_1([0,1])}{f'(\theta)}$ is smaller or bigger than λ^*.

Remark 4.30 (Further bifurcations). More bifurcations for the boundary value θ can occur increasing λ, the solutions that bifurcate and have oscillations around θ are not represented in the diagrams. Though, it has to be said that those solutions will appear after the nontrivial solution above θ, and the nontrivial solution below θ has both appeared. The reason is that an oscillatory solution of:

$$\begin{cases} -\partial_{xx}u = \lambda f(u) & x \in (0,1), \\ 0 < u < 1 & x \in (0,1), \\ u(0) = u(1) = \theta, \end{cases}$$

is above/below θ in a set smaller measure than the original set. Therefore, the domain should be big enough for this smaller subset to be above the thresholds. This reasoning holds for any dimension using the monotonicity of the eigenvalues. Indeed, we have that if $D \subset \Omega$ then $\lambda_1(\Omega) \leq \lambda_1(D)$ (see [40, Section 1.3.2])

Remark 4.31 (Harmonic Oscillator)*.* Note that if we linearize the ODE dynamics associated with the elliptic problem around $(\theta, 0)$, one obtains the harmonic oscillator. Indeed the linearized system

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -f(\theta)u \end{pmatrix},$$

corresponds to

$$\partial_{xx}u = -f'(\theta)u \tag{4.20}$$

We observe that $f'(\theta)$ corresponds to the frequency of the oscillations. In this way, one can intuitively understand how nontrivial solutions that are close to $v \equiv \theta$ can exist or not. The general solution of (4.20) is:

$$u(x) = A\sin\left(\sqrt{f'(\theta)}x\right) + B\cos\left(\sqrt{f'(\theta)}x\right).$$

Asking for Dirichlet conditions, lead us to the choice of $B = 0$, we see, then, that a critical length is $L = \frac{\pi}{\sqrt{f'(\theta)}}$.

This length is the critical length for which the stationary solution $v \equiv \theta$ becomes unstable in the first eigenfunction. Consider

$$\begin{cases} u_t - \partial_{xx}u = f(u) & (x,t) \in (0,L) \times (0,T) \\ u(0) = u(L) = \theta \end{cases}$$

and linearize around $u \equiv \theta$:

$$\begin{cases} \tilde{u}_t - \tilde{u}_{xx} = f'(\theta)\tilde{u} & (x,t) \in (0,L) \times (0,T), \\ \tilde{u}(0) = \tilde{u}(L) = 0. \end{cases} \tag{4.21}$$

The first eigenvalue of (4.21) is $\frac{\pi^2}{L^2 f'(\theta)}$, which becomes unstable when $\frac{\pi^2}{L^2 f'(\theta)} > 0$.

The same reasoning can be applied for further bifurcations while linking it with the unstability.

5. Numerical visualization of the barriers for the 1-D problem

In this section we aim to illustrate with figures the results obtained in the previous section. We will consider this specific 1-D problem:

$$\begin{cases} -\partial_{xx}u = \lambda u(1-u)(u-\theta) & x \in (0,1) \\ u(0) = u(1) = a & \\ 0 \le u \le 1 & x \in (0,1) \end{cases}$$

doing the change of variables $v = u - a$, we want to see the evolution of the energy functional

$$J_\lambda(v,a) = \int_0^1 \frac{1}{2} v_x^2 - \lambda \left\{ \int_0^{v(x)} f(s+a)ds \right\} dx$$

when moving the parameter λ (the measure of the domain) and the Dirichlet condition a.

For representing these energy functionals, we consider the energy functional along the first and third eigenvector of the Dirichlet Laplacian. We will plot J in the finite-dimensional subspace V generated by:

$$e_1(x) := \sin(\pi x), \qquad e_3(x) := \sin(3\pi x).$$

The functional $J : V \to \mathbb{R}$ can be expressed in terms of the coordinates with respect to the eigenfunctions showed before:

$$J_\lambda^{e_1,e_3}(\alpha,\beta,a) := J_\lambda(\alpha e_1 + \beta e_3, a) = \int_0^1 \frac{1}{2} \left|\alpha \partial_x e_1(x) + \beta \partial_x e_3(x)\right|^2 - \lambda \left\{ \int_0^{\alpha e_1 + \beta e_3} f(s+a)ds \right\} dx$$

α and β are the coordinates with respect to the e_1 and e_3 direction respectively.

Moreover, we observe how other critical points may appear for the Dirichlet condition 0 as increasing the measure of the domain $\lambda = L^2$ (Figure 17)[3]. We have to emphasize that these plots are only showing the functional along with two directions, and they are just illustrative.

In Figure 19[4], one can see two local minima three Mountain Passes. One of the Mountain Passes is hidden as a local maximum. One should note that the origin is a maximum for the components represented. However, any steady-state can be a maximum since there will always exist a high frequency perturbation for which the steady-state will be stable.

Remark 5.1. Note that for certain values of λ given in Figure 18, we have already multiplicity of solutions even though the representation in the e_1 direction does not show the issue clearly. The intention of Figure 18 is to represent how the functional evolves when one changes λ for understanding why the solution of:

$$\begin{cases} -\partial_{xx}u = \lambda f(u) & x \in (0,1), \\ 1 > u > \theta & x \in (0,1), \\ u = \theta, & \end{cases}$$

appears for a smaller value of λ than the solution of:

$$\begin{cases} -\partial_{xx}u = \lambda f(u) & x \in (0,1), \\ 0 < u < \theta & x \in (0,1) \\ u = \theta. & \end{cases} \tag{5.1}$$

Note that in Figure 18 one can observe how the non-symmetry of f plays a role in the energy functional.

[3]In the figures of the section a change of scale has been applied to represent in a more visual way the shape of the functional

[4]In the figure a different scale from Figure 18 has been applied to represent in a more visual way the shape of the functional for large domains.

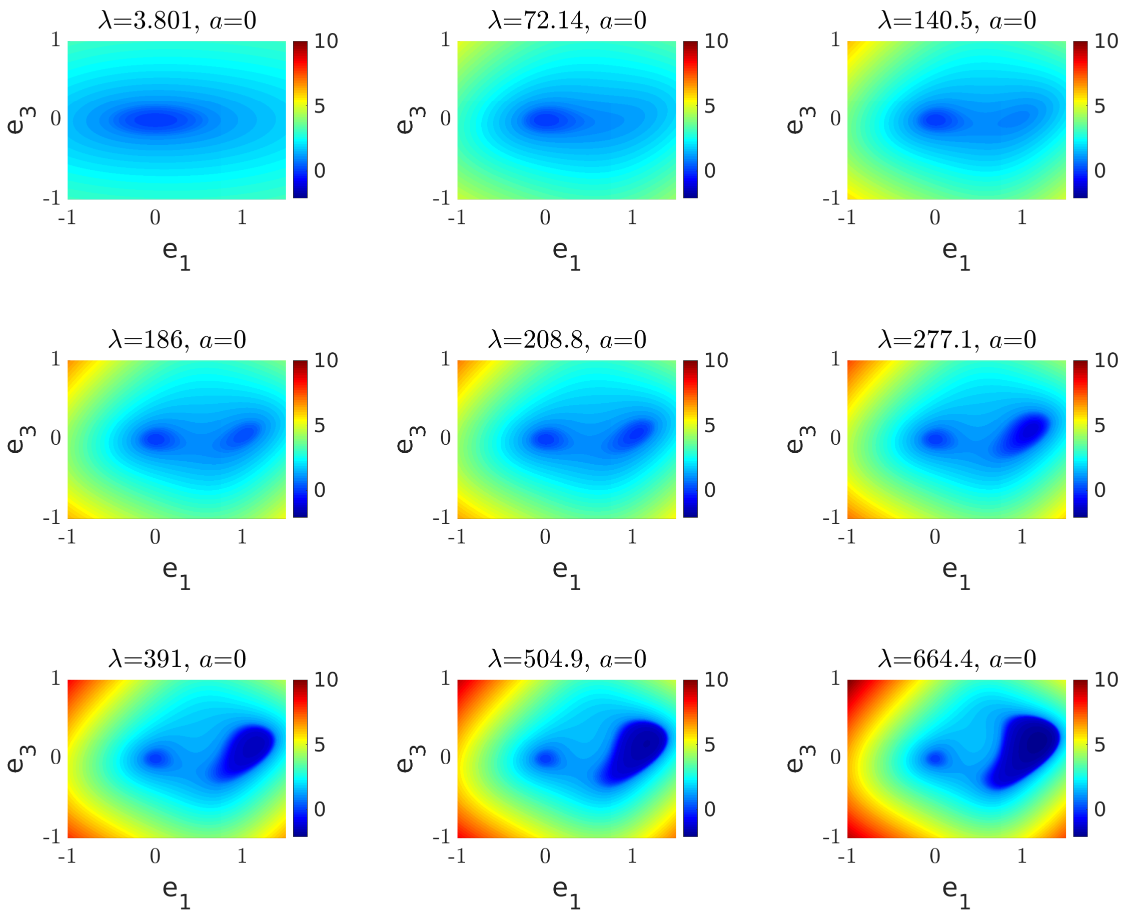

FIGURE 17. Functional $J_{\lambda}^{e_1,e_3}(\alpha,\beta,0)$ for different values of λ

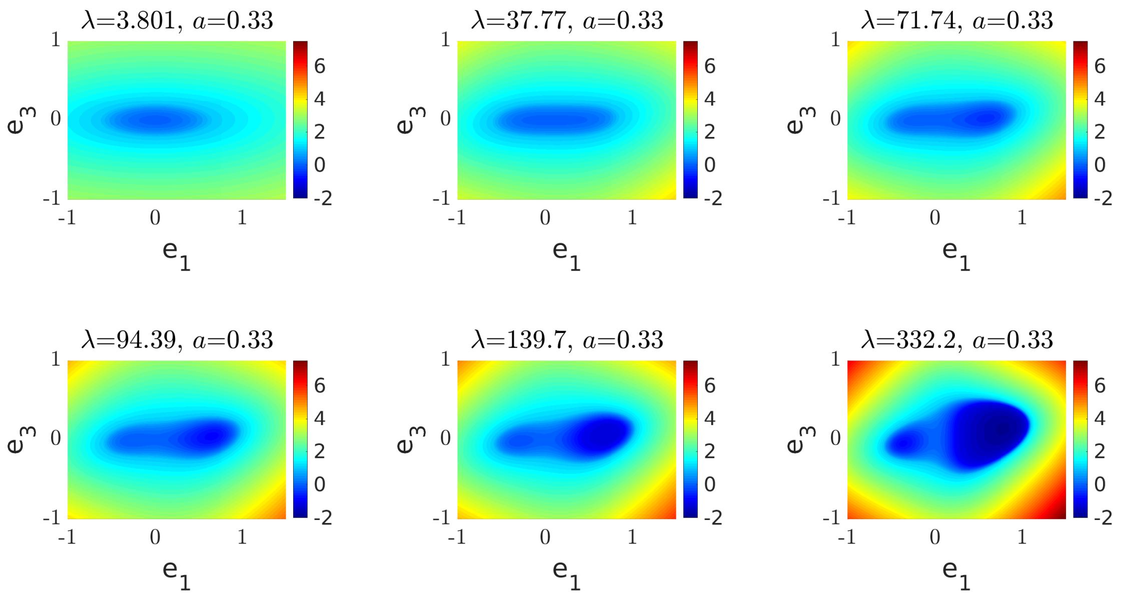

FIGURE 18. In these figures the value of $J_{\lambda}^{e_1,e_3}(\alpha,\beta,\theta)$ is contrasted against different pairs of α,β for different λ

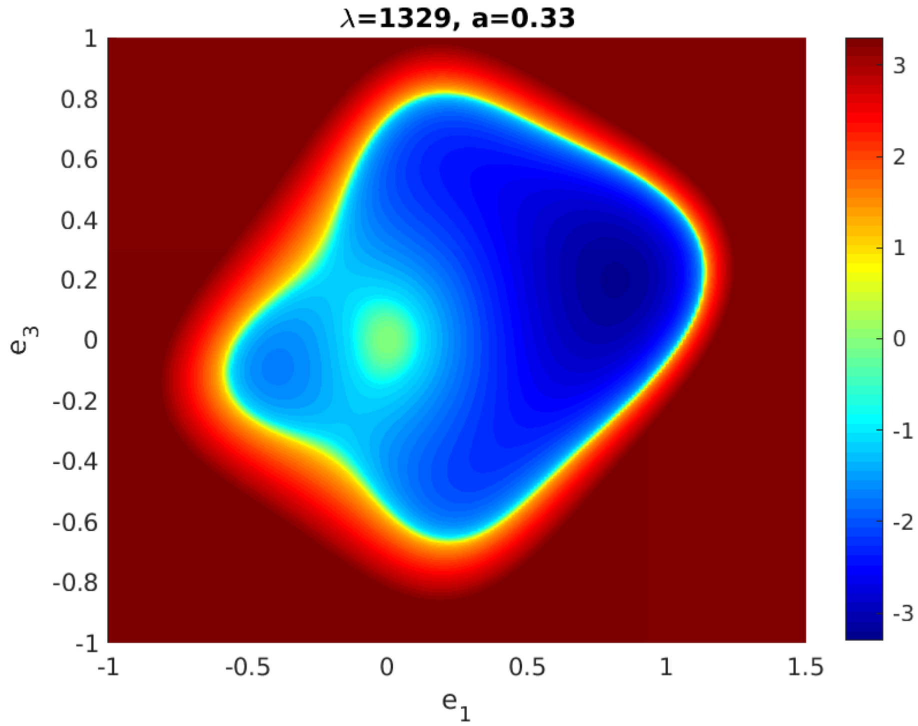

FIGURE 19. Energy functional $J_\lambda^{e_1,e_3}(\alpha,\beta,\theta)$. The values above 3 are not represented in the picture.

In Figure 20 a numerical minimization with IpOpt [96] of the following discretized functional has been performed:

$$\begin{cases} \min\limits_{v\in H_0^1(0,1)\cap C} \displaystyle\int_0^1 \frac{1}{2}v_x^2 - \lambda(F(v+a)-F(a))dx \\ C := \{v\in L^\infty(0,1) \quad \text{s.t.} \quad -b_1 < v(x) < b_2, \quad b_1,b_2\geq 0\} \end{cases}$$

In Figure 20, one can observe different solutions with different boundary conditions. The line in green is the section of the nontrivial solution in the whole domain $\mathbb{R}$.

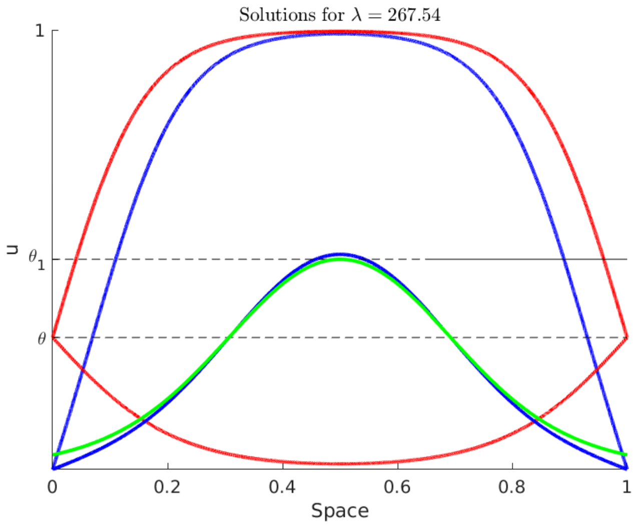

FIGURE 20. The blue (red) lines are for nontrivial solutions with Dirichlet condition 0 (respectively θ). In green, a section of the solution whole space $\mathbb{R}$.

6. The construction of paths of steady-states and the control strategy

In the previous sections, we have understood when barriers can appear. Whenever we are in such a setting, we cannot expect that for any initial data, we can reach steady-states such as 0 or θ.

The steady-states 0 and 1 are stable; this means that if we manage to be close to them, we will converge there naturally. However, since the steady-state 0 or 1 are respectively a subsolution and a supersolution, we cannot attain these steady-states in finite time.

On the other hand, the steady-state θ might be unstable or might not be the unique elliptic solution with boundary value θ. For these reasons, to understand how we can attain θ without violating the physical constraints is not obvious.

In this section, we are going to understand how we can reach the steady-state θ, avoiding both the unstability and the multiplicity issues.

The staircase method (Theorem 3.4 from [69]) ensures that if we have a connected path of admissible steady-states in the interior of the admissibility region, we are able to control from one steady-state to another. This section is highly devoted to the understanding of the construction of admissible paths of steady-states for the one-dimensional problem.

In general, the procedure to prove controllability is based on: first *constructing an admissible path of steady-states* connecting to θ. Then, once this path is known, the control strategy consists of two phases:

(1) *Dynamic phase.* For any initial data, find a control function a such that is able to steer system to an element of the path.
(2) *Quasistatic phase* Application of the staircase method in the constructed path to reach the target in large time.

6.1. Case $F(1) > 0$.

Recall that, we understand by bistable the following: $f < 0$ on $(0, \theta)$ and $f > 0$ on $(\theta, 1)$ assuming that $f'(0) < 0$, $f'(1) < 0$, and that $f'(\theta) > 0$.

In this subsection, we assume furthermore that the primitive $F(u) = \int_0^u f(s)ds$ evaluated at $F(1) > 0$. We denote by θ_1 the value different from 0 such that $F(\theta_1) = 0$. In the prototypical example of bistable nonlinearity, $f(u) = u(1-u)(u-\theta)$, one has that $F(1) = 0$ when $\theta = 1/2$.

Hereafter we present the strategy in [74] for finding the connected path of steady-states. The authors make use of the phase portrait to find a path of steady-states for ensuring under certain conditions the constrained controllability of the 1-D problem:

$$\begin{cases} u_t - \partial_{xx} u = f(y) & (x,t) \in (0,L) \times (0,T) \\ u(0,t) = a_1(t) & t \in (0,T) \\ u(L,t) = a_2(t) & t \in (0,T) \\ 0 \le u(x,0) \le 1 \end{cases}$$

We are looking for a path of steady-states with fixed length (note that, in the common argot of dynamical systems, one understands the spatial variable x as time) that converges to the stationary solution θ

$$\begin{cases} -\partial_{xx} u = f(u) \\ u(0) = a_1 \\ u(L) = a_2 \end{cases} \tag{6.1}$$

The following figures illustrate the control strategy taken after having steered the system in a steady-state with a small maximum value for different lengths. We emphasize that it might not always be possible to do this for any initial data. Indeed, if it appears a stationary solution

$$\begin{cases} -\partial_{xx}u = f(u) \\ u(0) = 0 \\ u(L) = 0 \\ 0 < u(x) < 1 \end{cases}$$

then, by the parabolic comparison principle, we will not be able to steer any initial data to a steady-state with a small maximum. In particular, when this solution appears, its maximum is above θ.

6.1.1. *Control strategy in [74].* For the first point, the authors set L so that 0 is the only steady-state of the problem. In this way, setting the control to 0 one approaches asymptotically this steady-state.

Afterward, whenever we are close to 0 the phase of attachment to the path starts.

More specifically one proceeds in the following way:

(1) Set both controls to 0. For $L < L^*$ one will approach the stationary solution 0 in the L^∞ norm.
(2) Whenever the maximum is below θ, say ϵ, set ϵ as a boundary value. $v \equiv \epsilon$ is a parabolic supersolution, for this reason, the solution is going to converge to a steady-state with boundary value ϵ which is below ϵ.

$$\begin{cases} -\partial_{xx}v = f(v) \\ v(0) = \epsilon = v(L) \\ 0 < v < \epsilon \end{cases}$$

(3) Wait a long time and apply local controllability to the steady-state at which we were converging. In this process we can guarantee that we do not violate the constraints provided that we wait enough time (see [69, Lemma 8.3]).
(4) Now we find the connected path of steady-states that bring us to the stationary solution θ for the Dirichlet condition being equal to θ. Take the Neumann trace of v at 0 and consider the following family of boundary conditions for the second order ODE of the steady-states:

$$\begin{cases} v^s(0) = (1-s)\epsilon + s\theta \\ \partial_x v^s(0) = \partial_x(1-s)v_\epsilon \end{cases}$$

solve from $x = 0$ to $x = L$:

$$\begin{cases} -\partial_{xx}v^s = f(v) \\ v^s(0) = (1-s)\epsilon + s\theta \\ \partial_x v^s(0) = \partial_x(1-s)v_\epsilon \end{cases}$$

The set of boundary conditions that we are looking for is

$$\begin{cases} u_1^s = v^s(0) \\ u_2^s = v^s(L) \end{cases}$$

(5) This path ends at θ by uniqueness of the solution of the ODE system:

$$\begin{cases} -\partial_{xx}v = f(v) \\ v(0) = \theta \\ \dfrac{\partial}{\partial x}v(0) = 0 \end{cases}$$

(6) Application of the staircase method with the family generated before.

The set of solutions is connected due to continuous dependence of the initial data.

The last property to verify is that the boundary condition $u_2^s = v^s(L)$ and the solution v^s is between 0 and 1. This is a consequence of these two facts:

- There is an invariant region Γ such that $(0,0) \in \partial\Gamma$ and $(\theta, 0) \in \Gamma$. Moreover, Γ is included in the admissible set of states and, by simplicity, assume that it is convex.
- In step (2), we have reached a stationary curve that lies inside Γ. (Proof in Proposition 6.1)
- The control strategy in [74] is based on tracing a line between the Neumann and Dirichlet traces in one extreme of the curve obtained in step 2 and the Dirichlet and Neumann trace of the target (the point $(\theta, 0)$). This line lies inside the invariant region Γ if Γ is convex; hence, by solving the ODE problem until $x = L$, we obtain a set of stationary solutions such that all of them lie in Γ and hence are admissible.

Proposition 6.1 (Invariant region Γ)**.** *Γ is an invariant region.*

Proof. Γ is enclosed by a homoclinic curve; by the uniqueness of the ODE, the result follows. □

Remark 6.2. Note that the convexity assumption on Γ is not needed. Indeed we need to require that there can exist a continuous curve $l : [0,1] \to \mathbb{R}^2$ such that $l(s) \in \Gamma$ for all $s \in [0,1]$ connecting the point $l(0) = (v(0), \partial_x v(0))$ to $l(1) = (\theta, 0)$.

Figure 21 is an illustration of the procedure described before.

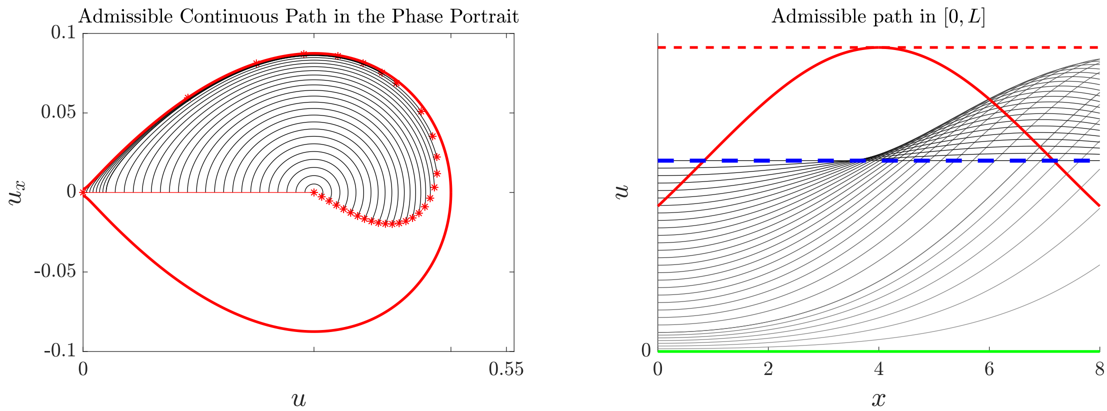

FIGURE 21. Strategy of [74] in the phase portrait of system (4.9). In black the representation of the steady-states of (6.1) that are part of a connected path that connects to the stationary solution θ. In red the values of the control that can be taken. $L = 8 > L_\theta$, $\theta = 0.33$. In the right, the stationary path plotted in the space domain, the red curve is the curve of maximum value in the invariant region Γ, in green the initial condition.

6.1.2. *Connected symmetric path.* A connected path of steady-states can be constructed following another strategy than the one used in [74] that provides controls that take the same value on both sides. The improvement is that we only require a function instead of two and enables to control the problem with the control in one side and Neumann boundary conditions in the other boundary. Moreover, we will see that looking at symmetric paths leads to an easier extension to several dimensions making use of radial steady-states.

The idea follows from the symmetry of the phase plane (and the symmetry of radial solutions of the Poisson equation).

- Instead of considering the boundary conditions at one extreme $x = 0$ for instance:
$$\begin{cases} v^s(0) = (1-s)\epsilon + s\theta \\ \partial_x v^s(0) = \partial_x (1-s) v_\epsilon \end{cases}$$
we consider the condition at the middle point $x = L/2$, which we know is a critical point of the solution of ODE (by symmetry), and hence it lies in the horizontal axis of the

phase portrait. We now solve the ODE from $x = L/2$ to $x = L$ (or backward to $x = 0$) for obtaining the necessary conditions in the boundary. The set of steady-states is then

$$v^s(L) = v^s(0) = a^s \quad s \in [0, 1]$$

where a^s is the projection in the first component of the solution at $x = L$ of the Cauchy problem. In other words, let $\Phi\left(x; \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)$ be the flow associated to $\frac{d}{dx}\begin{pmatrix} v \\ v_x \end{pmatrix} = \begin{pmatrix} v_x \\ -f(v) \end{pmatrix}$ and consider:

$$a^s := P_1\Phi\left(L/2; \begin{pmatrix} (1-s)v_\epsilon(L/2) + s\theta \\ 0 \end{pmatrix}\right),$$

where P_1 is the projection on the first component, and v_ϵ is the even stationary solution in which we arrived after step 2 in the procedure of [74]. Note that in the flow we set $L/2$ instead of L because $\Phi\left(0; \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ by definition and we are initializing the process at $L/2$.

- a^s is continuous with respect to s due to continuous dependence of the initial data. There is a continuously connected set of solutions to the boundary value problem associated with a^s by construction.
- Notice that our trajectories lie in Γ since the path $(v_\epsilon(L/2)(1-s)+s\theta)$ is in the horizontal axis inside the invariant region Γ[5]

Remark 6.3. Notice that this procedure does not depend on L; the only requirement is to be able to reach an even stationary state inside the invariant region Γ.

Figures 22 24 27 are the representations of the proof for different values of L, for $L < L^*_\theta$, for $L^*_\theta < L < L^*$ and $L^* < L$.[6]

Figures 23 25 and 28 represent the value at the boundary of the continuous path of steady-states for different values of L.

Finally Figures 26 and 29 are bifurcation diagrams depending on a, i.e. as we rise the Dirichlet condition bifurcations or unifications of solutions might appear. Indeed the connected path uses this feature to reach θ from 0.

[5] We this curve will be inside Γ always, so we do not have to take care of building l such that is contained in Γ. This is due to the fact that $F(u)$ is negative until we reach $u = \theta_1$, this means that the separatrix is defined until θ_1 with a value strictly above zero (the upper side) and strictly below zero (the down part) therefore our initial conditions lie inside the invariant region.

[6] L^*_θ is the critical length for which a nontrivial solution around the condition θ appears.

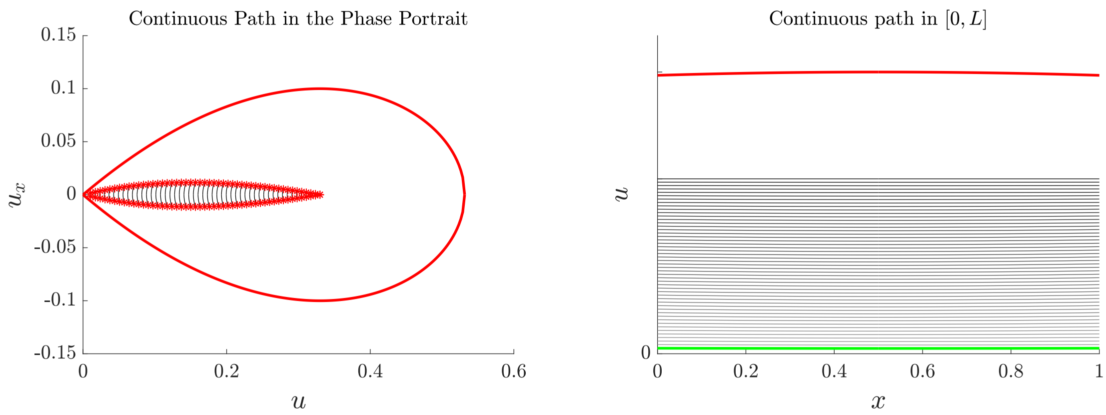

FIGURE 22. The even strategy is represented in the phase portrait of system (4.9). In black, the representation of the steady-states of (6.1) that are part of a connected path that connects to the stationary solution θ. In red, the values of the control that can be taken. Red points represent the value that has to be taken in the other extreme for the Dirichlet control. $L = 1$, $\theta = 0.33$. In the right, the stationary path plotted in the space domain, the red curve is the curve of the maximum value in the invariant region Γ, in green the initial condition.

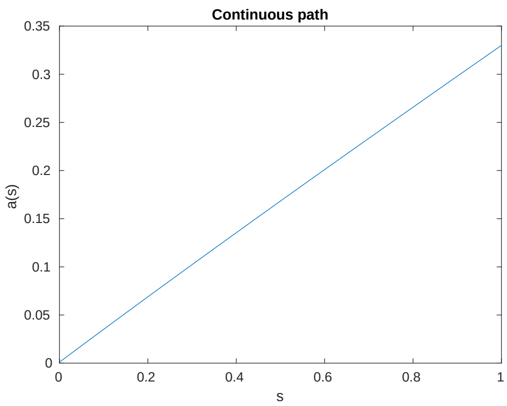

FIGURE 23. Connected path of steady-states. The value in the boundary in the vertical axis, in the horizontal axis the parameter $s \in [0, 1]$ in the vertical axis the value of the boundary

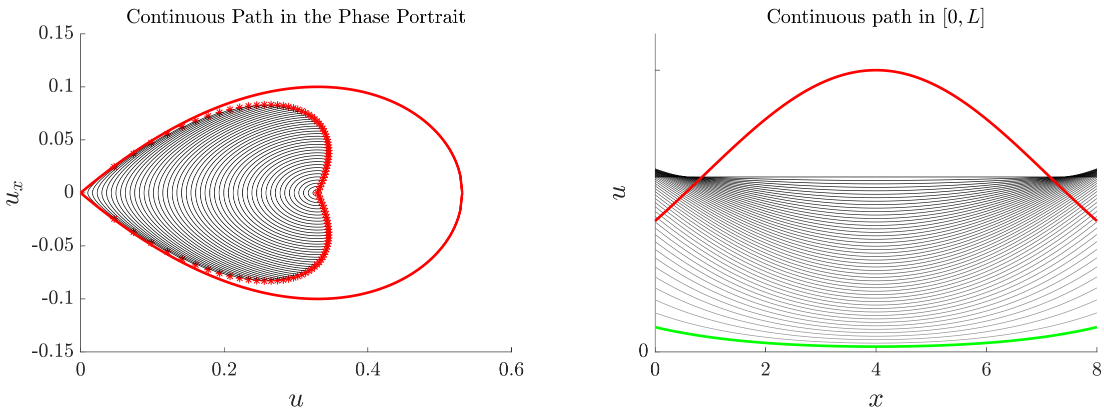

FIGURE 24. The even strategy is represented in the phase portrait of system (4.9). In black, the representation of the steady-states of (6.1) that are part of a connected path that connects to the stationary solution θ. In red, the values of the control that can be taken. Red points represent the value that has to be taken in the other extreme for the Dirichlet control. $L = 8$, $\theta = 0.33$. In the right, the stationary path plotted in the space domain, the red curve is the curve of the maximum value in the invariant region Γ, in green the initial condition.

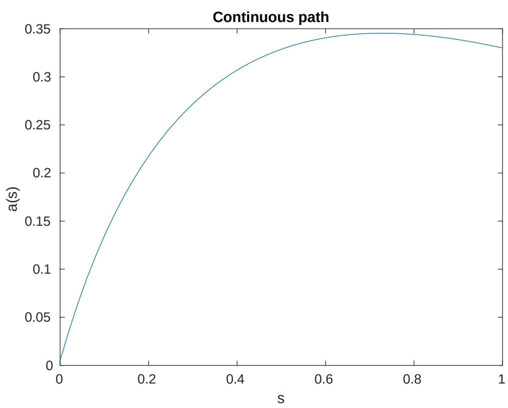

FIGURE 25. Connected path of steady-states. The value in the boundary in the vertical axis, in the horizontal axis the parameter $s \in [0, 1]$ in the vertical axis the value of the boundary

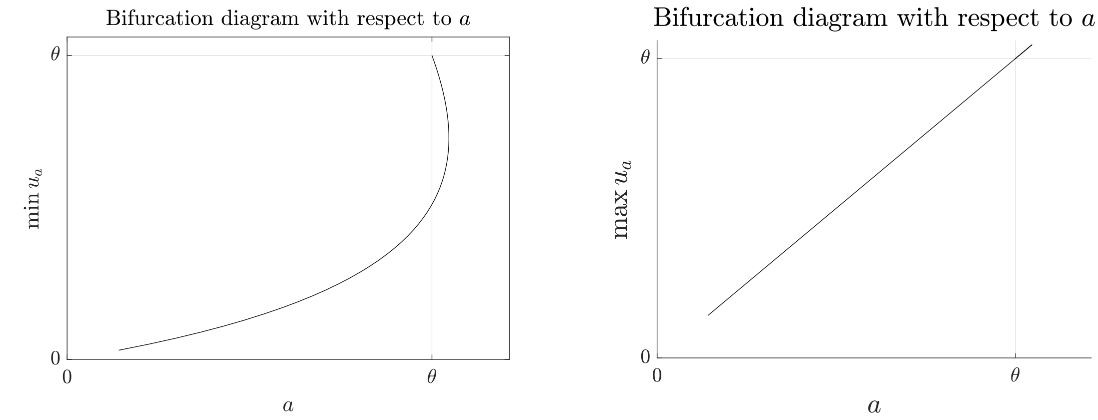

FIGURE 26. In the left, the minimum value of $u_a(x)$, $\min_{x\in[0,L]} u_a(x)$ is represented against a, for the connected path of steady-states. At the right, the maximum value $\max_{x\in[0,L]} u(x)$ against a

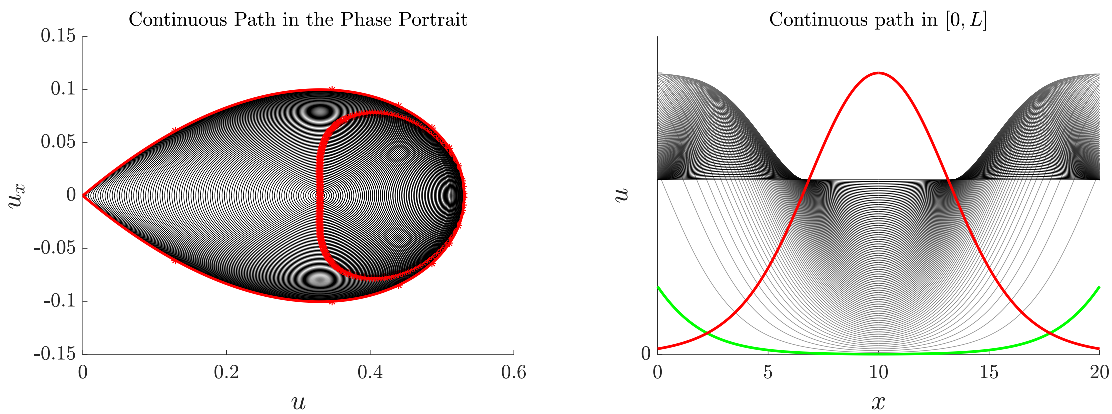

FIGURE 27. The even strategy is represented in the phase portrait of system (4.9). In black, the representation of the steady-states of (6.1) that are part of a connected path that connects to the stationary solution θ. In red, the values of the control that can be taken. Red points represent the value that has to be taken in the other extreme for the Dirichlet control. $L = 20$, $\theta = 0.33$. In the right, the stationary path plotted in the space domain, the red curve is the curve of the maximum value in the invariant region Γ, in green the initial condition.

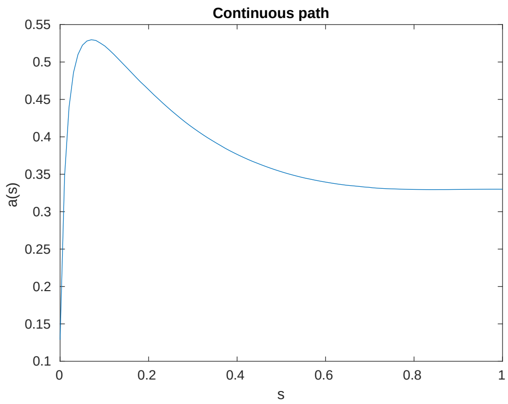

FIGURE 28. Connected path of steady-states. The value in the boundary in the vertical axis, in the horizontal axis the parameter $s \in [0, 1]$ in the vertical axis the value of the boundary

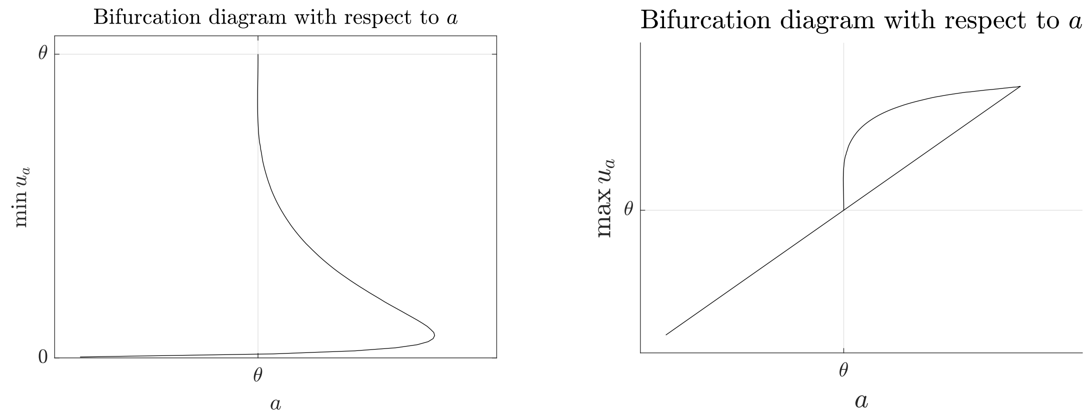

FIGURE 29. In the left, the minimum value of $u_a(x)$, $\min_{x\in[0,L]} u_a(x)$ is represented against a, for the connected path of steady-states. At the right, the maximum value $\max_{x\in[0,L]} u(x)$ against a

The key question is if we are able to reach Γ for certain initial conditions of the reaction-diffusion system even though the barrier has already appeared.

Do they exist connected paths of steady-states with even controls that connect certain steady-states to Γ ? For which initial conditions of the reaction-diffusion system, can we reach one of these connected components? Which action has to take the control to bring us there?

6.2. **A non returning path.**

The following result basically is a characterization of which are the admissible steady-states from which a connected path of steady-states can be constructed towards θ and 1.

Proposition 6.4. *Let w be an admissible steady-state fulfilling any admissible boundary conditions. Let $\underline{u}_{L^*}$ be the minimum solution of the problem:*

$$\begin{cases} -(\underline{u}_{L^*})_{xx} = f(\underline{u}_{L^*}) \\ \underline{u}_{L^*}(0) = \underline{u}_{L^*}(L^*) = 0 \\ \underline{u}_{L^*} > 0 \end{cases}$$

- *If* $\max w(x) \leq \max_{x\in[0,L]} \underline{u}_{L^*}$ *and*
- $$\frac{1}{2}w_x(0)^2 + F(w(0)) \leq F(\max_{x\in[0,L]} \underline{u}_{L^*})$$

Then there is a connected path of steady-states from w to θ and to 0*. Moreover, if w is symmetric, the second condition is not needed, and the path will maintain even boundary conditions.*

Proof. (Sketch) Let Λ denote the region between $\underline{u}_{L^*}$ and the vertical axis in the phase plane. Given any point, x in a solution arc inside Λ fulfills the two conditions stated in the proposition. The idea of the proof is to move first to a symmetric steady-state $w^*(x)$ following the trajectory from where the initial steady-state belongs to in the phase portrait (note that the first integral $\frac{1}{2}w_x^2 + F(w)$ is preserved along the trajectory). Then since $\frac{1}{2}w_x(0)^2 + F(w(0)) \leq F(\max_{x\in[0,L]} \underline{u}_{L^*})$ we know that $F(\max_{x\in[0,L]} w^*(x)) \leq F(\max_{x\in[0,L]} \underline{u}_{L^*})$. Then we use the same argument as in the previous section (taking the Dirichlet and Neumann conditions in $L/2$) to build the connected path of steady-states. The admissibility of them follows by the fact that $\underline{u}_{L^*}$ is the minimum solution of the elliptic problem and that the boundary of Γ is a homoclinic orbit (it takes infinite time $L = +\infty$). Indeed, any symmetric admissible trajectory in the phase plane in $\Lambda - \Gamma$ will not touch 0; otherwise, $\underline{u}_{L^*}$ will not be the minimum solution. □

Here we show that the assumption of having a connected path of admissible steady-states such that the control is strictly bigger than a constant bigger than zero is necessary.

Indeed we prove the following propositions.

Proposition 6.5. *When $L > L^*$ there is an admissible connected path of steady-states continuous with respect to the L^∞ topology that connects the stationary solution $u = 0$ with the minimum solution of the following problem:*

$$\begin{cases} -\partial_{xx}u = f(u) & x \in (0, L) \\ u(0) = u(L) = 0 \\ 1 > u > 0 & x \in (0, L) \end{cases}$$

Proof. Consider the following ODE system depending on the parameter $s \in [0, 1]$

$$\begin{cases} y_x = v \\ y_{xx} = v_x = -f(y) \\ y(L/2) = s \\ y_x(L/2) = 0 \end{cases} \tag{6.2}$$

As before, we will use this system to construct a path of stationary steady-states. We start for $s = 0$. By the uniqueness of the ODE, we have that $y(x) = 0$ is the only solution. Then, consider the family of solutions of (6.2) depending upon the parameter s that are defined on $x \in [0, L]$ (solving the ODE forward in time and backward). Let $s^* \in [0, 1]$ be such:

$$s^* = \min_{s\in[0,1]} \{\|y^s\|_{L^\infty([0,L])} \text{ such that } y^s \text{ is a solution of (6.2) and } y^s(L) = y^s(0) = 0 \text{ and } \|y^s\|_{L^\infty([0,L])} \leq 1\}$$

This $s^* \in [0,1]$ exists because $L > L^*$. The path of steady-states is:

$$\begin{aligned} \gamma : [0, s^*] &\longrightarrow L^\infty([0,L]) \\ s &\longrightarrow y^s \end{aligned}$$

By construction, this path is admissible. Moreover, by continuous dependence on the initial data, we have that the path is continuous: Let

$$\begin{aligned} y' &= F(y) \\ y(0) &= y_0 \end{aligned}$$

and let

$$\begin{aligned} z' &= F(z) \\ z(0) &= z_0 \end{aligned}$$

we have that:

$$\begin{aligned} \|y(x) - z(x)\|_\infty &= \left\| y_0 - z_0 + \int_0^x F(y(r)) - F(z(r)) dr \right\|_\infty \\ &\le \|y_0 - z_0\| + \int_0^x \|F(y(r)) - F(z(r))\|_\infty dr \\ &\le \|y_0 - z_0\| + \int_0^x \sup_{x \in \mathbb{R}^2 : \|x\| \le \max\{\|y\|_{L^\infty}, \|z\|_{L^\infty}\}} \|\nabla F\|_\infty \|y(r) - z(r)\| dr \\ &\le C(L, \|y\|_\infty, \|z\|_\infty) \|y_0 - z_0\| \end{aligned}$$

□

Proposition 6.6. *It does not exist any admissible control function such that can bring* u^* *to* 0

Proof. Consider the stationary solution y^{s^*}:

$$\begin{cases} -y^{s^*}_{xx} = f(y^{s^*}) \\ y^{s^*}(0) = y^{s^*}(L) = 0 \end{cases}$$

and consider the parabolic problem under discussion:

$$\begin{cases} y_t - y_{xx} = f(y) \\ y(0) = y(L) = a(t) \\ y(x,0) = y^{s^*}(x) \end{cases}$$

with $a \in L^\infty([0,T];[0,1])$. By the comparison principle 2.4 we have that

$$y(t) \ge y^{s^*} \qquad \forall a(t) \in L^\infty([0,T];[0,1])$$

□

Remark 6.7. As it was anticipated, we note that the assumption (3.12) of Theorem 3.4 is needed.

Remark 6.8. There is another procedure to follow a path of steady-states that does not rely on Theorem 3.4. Indeed, in [17], the authors construct a feedback law that stabilizes the quasi-static deformation while moving from a path of steady-states. This feedback law, in our case, can violate the constraints, and we know by the comparison principle and Proposition 6.6 that it is not going to work for this case. However, it can be a useful alternative to follow a path of steady-states that needs to be mentioned.

Remark 6.9. In this section, we proved that we can build a path of admissible steady-states outside Γ that connect to θ and 0.

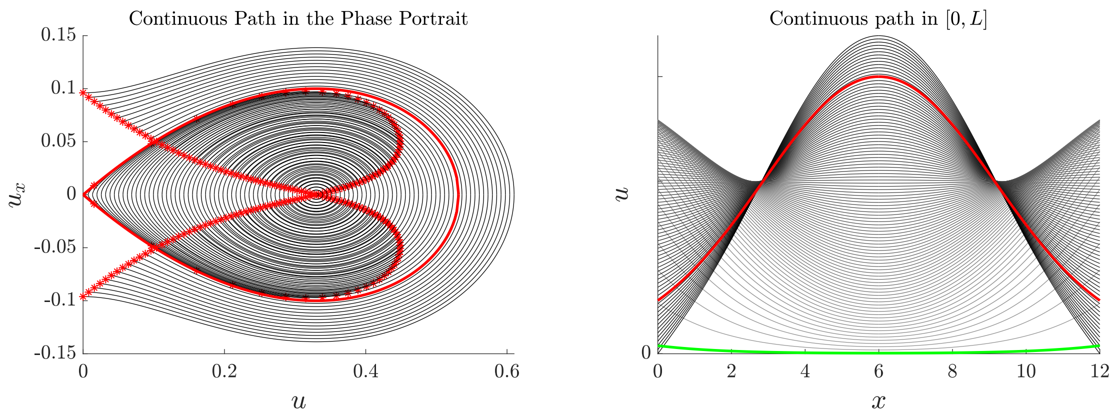

FIGURE 30. Connected path of steady-states connecting 0 and $\underline{u}_L$

6.3. Case $F(1) = 0$, even and not even controls.

In the case in which $F(1) = 0$ (for the prototypical example corresponds to $\theta = 1/2$), we have a positive result for the admissible paths of steady-states for any length L. For any even initial data of the reaction-diffusion system, we are able to bring it to a connected path of steady-states that connect 0, θ, and 1. Moreover, if we allow controls that are not constant in the boundary, we can connect 0 and 1 in any dimension. The last result follows from the fact that for $F(1) = 0$, the traveling wave that connects 0 to 1 is a stationary solution.

Hereafter we summarize the strategy in 1-d.

(1) Given any initial admissible data, we set any even admissible boundary condition:

$$\begin{cases} u_t - \partial_{xx} u = f(u) \\ u(0) = u(L) = a(t) \in (0,1) \\ u(x,0) = u_0(x) \end{cases}$$

(2) by Matano's theorem, we know that we are going to converge to a steady-state. Moreover, we know that this state will be even.
(3) Besides, by the parabolic comparison principle, this steady-state will be between 0 and 1.
(4) In case the initial datum was not even, we steer the system in finite time to a symmetric stationary state. This forces us to use different controls on both sides. Though, if the initial datum was already even, we do not need the previous steps.
(5) Noting that the (stationary) traveling waves now define the only invariant region that can have inside all symmetric solutions (See the discussion of the phase portrait above Proposition 4.26).
(6) We can find a connected path of steady-states following the same argument than in the case of $F(1) > 0$. Moreover, now 1 is in the boundary of the invariant region and 0 as well, which means that the path towards 0 and 1 exists.

Remark 6.10 (A simpler dynamic strategy). The path of admissible steady-states that connect any even solution to 1 or 0 is not needed. Indeed we can directly use a dynamic strategy for the parabolic problem. Note that in this case, by Matano, we already have the convergence to 0 and to 1 in infinite time since they are the only stationary solutions that take boundary values 0 and 1, respectively. From the phase portrait in the $F(1) = 0$ case, it is clear that the only solutions that take the same values in both boundaries should be inside the invariant region defined by the stationary traveling waves or separatrix that connect 0 to 1 and vice versa.

Theorem 6.11. *Let $[0, L] \subset \mathbb{R}$, and consider that $F(1) = 0$. Then there exists a connected path of steady-states $u^s(x)$ such that connects 0 and 1.*

Proof. For the case in which $F(1) = 0$ we know that the traveling wave v is a stationary solution for the problem in $\mathbb{R}$:

$$\begin{cases} -\partial_{xx} v = f(v) \\ v(+\infty) = 0, \quad v(-\infty) = 1 \\ v(0) = 1/2 \end{cases}$$

(the heteroclinic orbits in the phase portrait connecting 0 and 1). The restriction of V to $[0, L]$ is a stationary solution. Note that $V(x + c)$ is also a stationary solution for any $c \in \mathbb{R}$. Choose a direction in $c \in \mathbb{R}$ and by considering the restriction of $V(x + cs)$ on $[0, L]$ we obtain a connected path of steady-states.

□

The following Figures 31 34 and 37 are illustrations of the paths connecting a steady-state to the stationary solution θ to 1 and to 0. Figures 32 35 and 38 show its respective values at the boundary of the continuous paths of steady-states while Figures 33 36 and 39 represent their bifurcation diagrams.

One can observe that the path that brings any steady-state towards 1 or 0 for large L has Dirichlet and Neumann conditions in the border that are close to the stationary traveling waves.

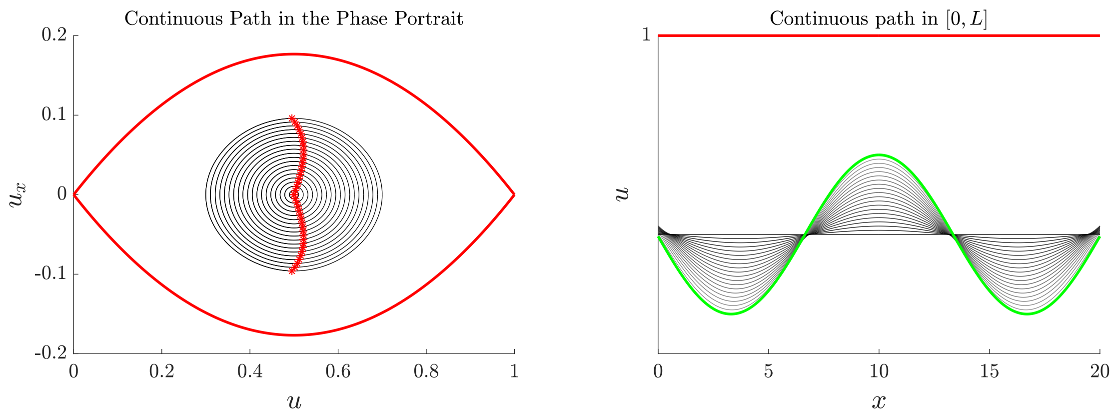

FIGURE 31. The even strategy is represented in the phase portrait of system (4.9). In black, the representation of the steady-states of (6.1) that are part of a connected path that connects to the stationary solution θ. In red, the values of the control that can be taken. Red points represent the value that has to be taken in the other extreme for the Dirichlet control. $L = 20$, $\theta = 0.5$. In the right, the stationary path plotted in the space domain, the red curve is the curve of the maximum value in the invariant region Γ, in green the initial condition.

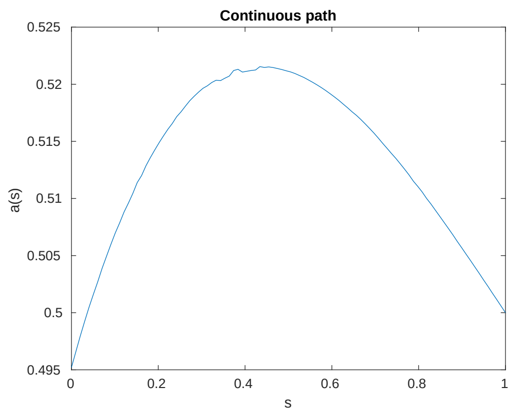

FIGURE 32. Connected path of steady-states, value in the boundary, in the horizontal axis the parameter $s \in [0,1]$ in the vertical axis the value of the boundary

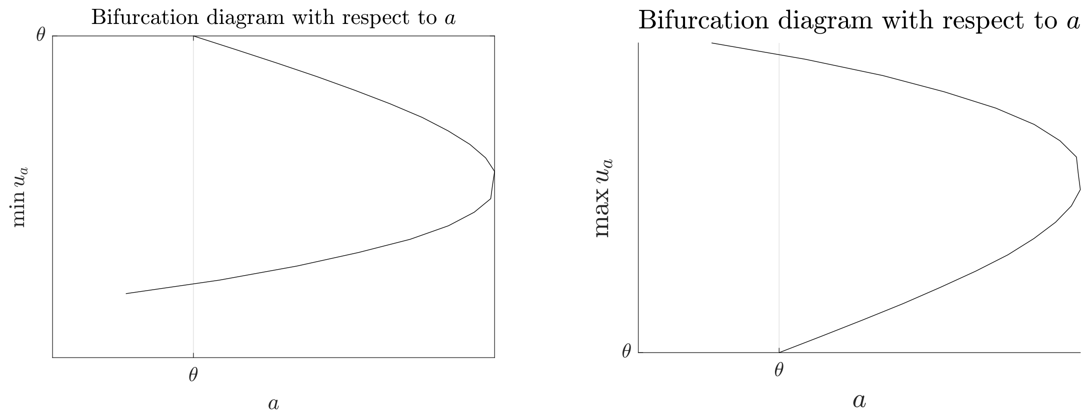

FIGURE 33. In the left, the minimum value of $u_a(x)$, $\min_{x\in[0,L]} u_a(x)$ is represented against a, for the connected path of steady-states. At the right, the maximum value $\max_{x\in[0,L]} u(x)$ against a

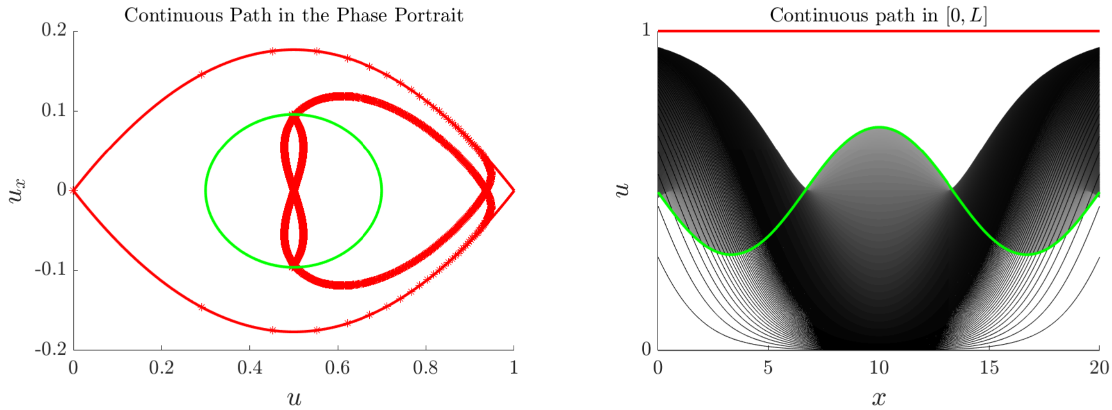

FIGURE 34. The even strategy is represented in the phase portrait of system (4.9). In black, the representation of the steady-states of (6.1) that are part of a connected path that connects to the stationary solution 0. In red, the values of the control that can be taken. Red points represent the value that has to be taken in the other extreme for the Dirichlet control. $L = 20$, $\theta = 0.5$. In the right, the stationary path plotted in the space domain, the red curve is the curve of the maximum value in the invariant region Γ, in green the initial condition.

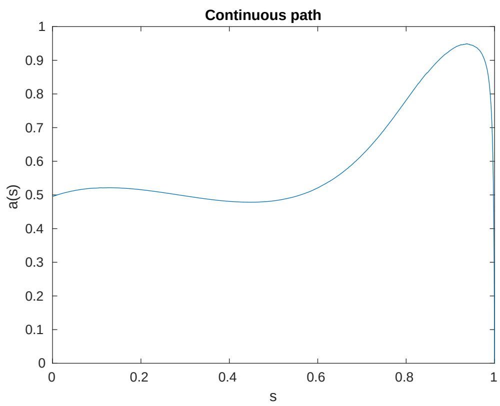

FIGURE 35. Connected path of steady-states. The value in the boundary in the vertical axis, in the horizontal axis the parameter $s \in [0, 1]$ in the vertical axis the value of the boundary

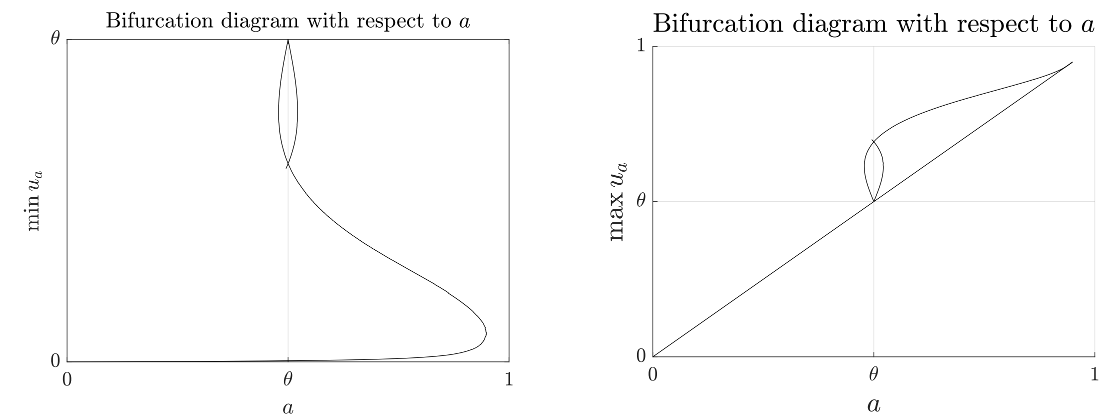

FIGURE 36. In the left, the minimum value of $u_a(x)$, $\min_{x\in[0,L]} u_a(x)$ is represented against a, for the connected path of steady-states. At the right, the maximum value $\max_{x\in[0,L]} u(x)$ against a

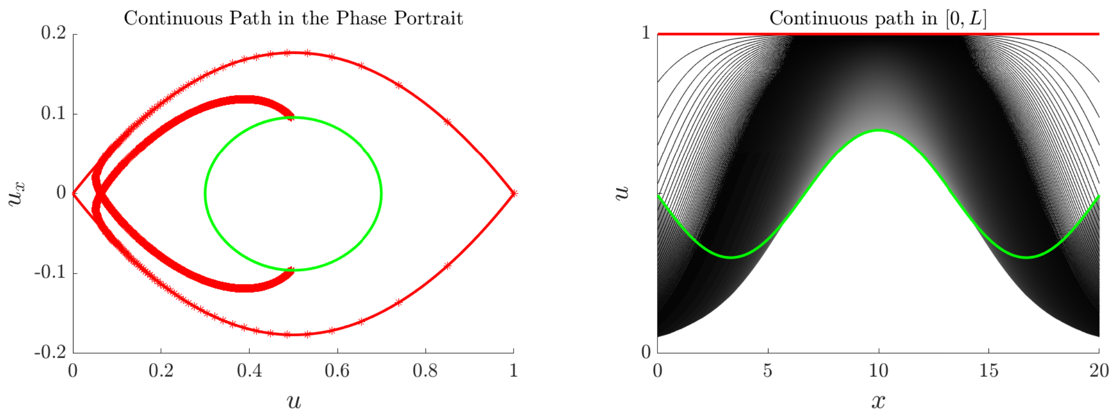

FIGURE 37. Even strategy represented in the phase portrait of system (4.9). In black the representation of the steady-states of (6.1) that are part of a connected path that connects to the stationary solution θ. In red the values of the control that can be taken. Red points represent the value that has to be taken in the other extreme for the Dirichlet control. $L = 20$, $\theta = 0.5$. In the right, the stationary path plotted in the space domain, the red curve is the curve of maximum value in the invariant region Γ, in green the initial condition.

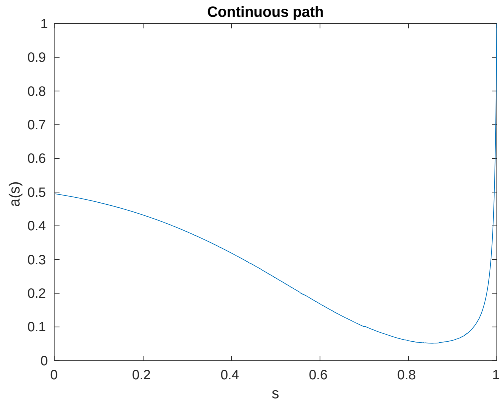

Figure 38. Connected path of steady-states, value in the boundary, in the horizontal axis the parameter $s \in [0, 1]$ in the vertical axis the value of the boundary

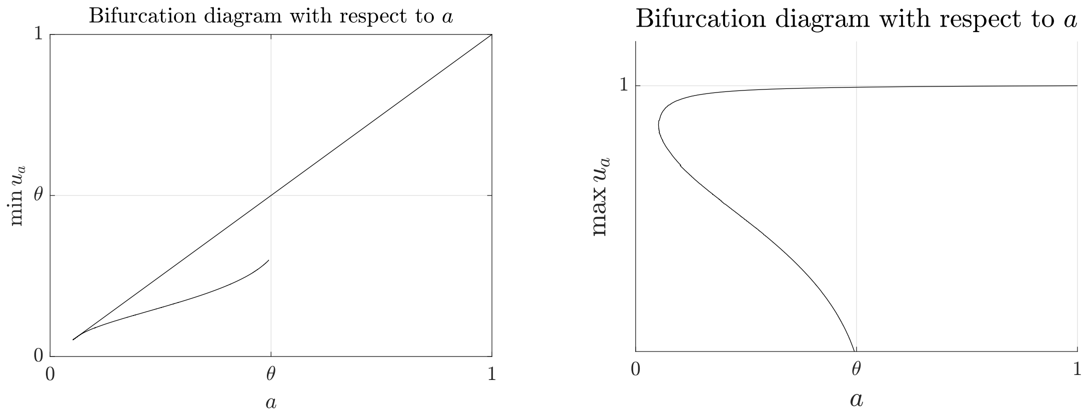

Figure 39. In the left, the minimum value of $u_a(x)$, $\min_{x\in[0,L]} u_a(x)$ is represented against a, for the connected path of steady-states. At the right, the maximum value $\max_{x\in[0,L]} u(x)$ against a

7. Results on controllability

The Figures 40 and 41 are a summary of the study. In them, the direction of the traveling waves between constant stationary solutions is shown together with connected paths of steady-states between those constant solutions in a domain $[0, L]$. In the case of $F(1) = 0$, we have that the traveling wave is stationary. Therefore, as discussed previously, its restrictions on domains of length L give a connected path of steady-states between 0 and 1 naturally. However, note that this path does not pass through the stationary solution θ.

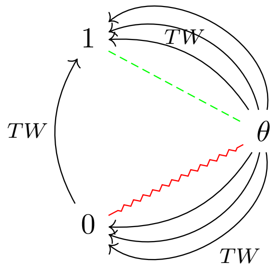

FIGURE 40. Connectivity map for $F(1) > 0$. In red, it is shown an admissible continuous path of steady-states (for any L) connecting stationary solutions. In green, it is an admissible and continuous path of steady-states connecting two stationary solutions, but in this case, its existence depends on L. In black, the existence of traveling waves for the Cauchy problem is shown. The Travelling wave from 0 to 1 is unique, while the traveling waves from θ to 1 or to 0 are infinitely many.

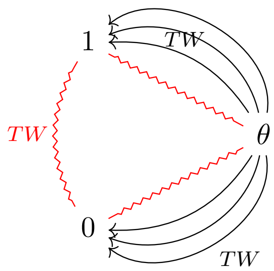

FIGURE 41. Connectivity map for $F(1) = 0$. In red, it is shown an admissible continuous path of steady-states (for any L) connecting stationary solutions. The Travelling wave from 0 to 1 is unique and stationary, giving a continuous path of admissible steady-states connecting 0 and 1. In black, the existence of non-stationary traveling waves for the Cauchy problem is shown. The traveling waves from θ to 1 or to 0 are infinitely many.

7.1. **Bistable Nonlinearities.**

Consider the following reaction-diffusion equation:

$$\begin{cases} u_t - \partial_{xx} u = f(u) & (x,t) \in (0,L) \times (0,T), \\ u(0) = a_1(t), \quad u(L) = a_2(t) & t \in (0,T), \\ u(x,0) = u_0(x) \in [0,1], \end{cases} \tag{7.1}$$

where f is a C^2 bistable function satisfying $f(0) = f(1) = f(\theta) = 0$ for a certain $\theta \in (0,1)$. Let F be defined as $F(t) = \int_0^t f(s)ds$ and $a_1(t), a_2(t) \in [0,1]$.

Condition 7.1. Let w be any stationary solution of (7.1) satisfying:

$$\begin{cases} \frac{1}{2}w_x(0)^2 + F(w(0)) \leq F\left(\max_{x\in[0,L]} \underline{u}_{L^*}\right), \\ \max w(x) < \max_{x\in[0,L]} \underline{u}_{L^*}, \end{cases}$$

where $\underline{u}_{L^*}$ is the minimum solution with respect to the infinity norm to the problem:

$$\begin{cases} -(\underline{u}_{L^*})_{xx} = f\left(\underline{u}_{L^*}\right) & x \in (0,L), \\ \underline{u}_{L^*}(0) = \underline{u}_{L^*}(L^*) = 0, & \\ 1 > \underline{u}_{L^*} > 0 & x \in (0,L), \end{cases} \tag{7.2}$$

where $L^* \in \mathbb{R}^+$ is the minimum value for which the problem (7.2) has a solution.

In the case that the solution of (7.2) does not exist, it is assumed that, given a steady-state, the condition is always fulfilled.

Note that Condition 7.1, in particular, is asking to be below the minimum nontrivial solution. Together with the energy requirement, one can construct a parabolic supersolution with a steady-state that is below the nontrivial solution and then either approach to 0 or to use the staircase method with the path constructed in Subsection 6.2.

Theorem 7.2. *For f bistable.*

If $F(1) > 0$ there exist a time T (possibly $T = +\infty$) such that the solution of the system (7.1) *can be driven by means of two functions $a_1, a_2 \in L^\infty([0,T],[0,1])$ to:*

- 0 *asymptotically:*
  - *for any initial data u_0 iff $L < L^*$.*
  - *for any $L > 0$ if u_0 can be controlled to a steady-state fulfilling Condition 7.1.*
- θ *in finite time:*
  - *for any initial data u_0 iff $L < L^*$.*
  - *for any $L > 0$ if u_0 can be controlled to a steady-state fulfilling Condition 7.1.*
- 1 *asymptotically for any initial data u_0 and for any L.*

Furthermore one has the following estimate for L^*:

$$\frac{\pi}{\sqrt{\max_{s\in[0,1]} f'(s)}} \leq L^* \leq 2\sqrt{2}\sqrt{\frac{F(1)-F(\theta)}{F(1)^2}}.$$

If $F(1) = 0$ then: If the initial condition is symmetric with respect $L/2$ satisfying $v(0) = v(L)$, there exist a function $a(t) = a_1(t) = a_2(t)$ such that drives the system in finite time to θ and in infinite time to 0 and 1 for any L.

Remark 7.3. Moreover, if the initial data is symmetric with respect $L/2$ taking the same value in $u_0(0) = u_0(L)$ the path can be constructed in a symmetric way $a_1(t) = a_2(t) = a(t)$.

Remark 7.4. The construction of a symmetric path allows to control the bistable equation with one boundary control in one extreme and Neumann condition to the other extreme. To generate a path for several dimensions is done in [82]. However, how to proceed if we act only in a part of the boundary is an open question (see Section 11).

Remark 7.5. Note that the continuous path of steady-states between 0 and 1 can exist if we allow ourselves to break the constraints (see Figure 42).

In general, an important concern is if the ODE dynamics is going to blow up for a finite L. This would stop any continuous path to exist regardless of the constraints.

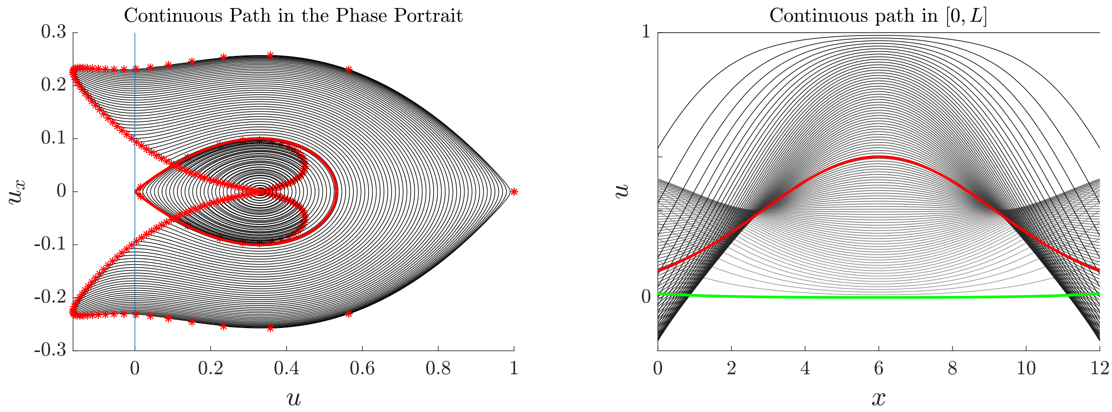

FIGURE 42. Non admissible continuous path from 0 to 1

7.2. **Monostable Nonlinearities.**

Previously we gave precise thresholds and conditions for when the nontrivial positive solutions of the semilinear elliptic problem appear. We have the following result:

Theorem 7.6. *For f monostable.*
Let $[0, L]$ be a domain, the system (7.1) can be asymptotically driven to:

- 1 *for any initial condition and any measure of the domain.*
- 0 *for any initial condition if $L^2 < \lambda^*$.*

where:

$$\frac{\lambda_1([0,1])}{\max_{s\in[0,1]} f'(s)} \leq \lambda^* \leq \overline{\lambda} < +\infty$$

where $\lambda_1([0, 1])$ is the first eigenvalue of the Dirichlet Laplacian in $[0, 1]$.

Remark 7.7. Note that for $\lambda > \lambda^*$ one can still have that 0 is stable.

Following the computations in 1-D for the bistable case, it can be shown that for the monostable case in dimension 1 one has $\overline{\lambda} = \frac{8}{F(1)}$.

8. Numerical Simulations

This section is devoted to seeing numerical implementations of the control. We want to observe the results showed above, together with some qualitative experiments. The numerical implementations performed are optimization problems of the form:

$$\min_{a\in\mathcal{A}} J(u_a; u_0) \tag{8.1}$$

for different $\mathcal{A}$ and where u_a solves:

$$\begin{cases} u_t - \partial_{xx}u = u(1-u)(u-1/3) & (x,t)\in(0,L)\times(0,T),\\ u(0,t) = a_1(t), \quad u(L,t) = a_2(t), & t\in(0,T),\\ u(x,0) = u_0(x) \end{cases}$$

We have used IpOpt [96] to do the optimization procedures. The order will be the following:

(1) The control strategy for $L_\theta < L < L^*$. Here we take:

$$J = \|u(T;a)-\theta\|^2,$$
$$\mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1])\Big\},$$

and we observe in Figure 44 how the control goes below $\theta = 1/3$. This is natural since for $L > L_\theta$ there is a nontrivial solution above θ that has to be overpassed. Furthermore, it can be observed that the state can be controlled, Figure 43.

(2) The lack of controllability towards 0 for $L > L^*$ for $u_0 = 1$ in Figure 45. One can observe the emergence of a barrier. The functional employed in the figure is:

$$J = \|u(T;a)\|^2,$$
$$\mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1])\Big\}.$$

However, we are able to reach θ when starting from 0. In Figure 47 the functional:

$$J = \|u(T;a)-\theta\|^2,$$
$$\mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1])\Big\},$$

is being minimized.

(3) The existence of a minimal controllability time and the control in minimal time. First, we observe the lack of controllability when the time horizon is too short (Figure 49) for

$$J = \|u(T;a)-\theta\|^2,$$
$$\mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1])\Big\},$$

In Figure 51 the functional minimized is the time horizon

$$J = T,$$
$$\mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1]) : \text{ the trajectory fulfills } \theta-\epsilon \le u_a(T;u_0) \le \theta+\epsilon\Big\}.$$

Numerically this can be achieved by first generating an initial guess minimizing the L^2 norm of the difference of the final datum with the target. Then, minimizing δt in the discretization. The control showed in Figure 52 points out that the control strategy associated to the control in minimal time is a bang-bang function.

(4) A quasistatic control that approximately follows the path of steady-states constructed above. We set T large, and we minimize

$$J = \int_0^T \|a_t\|^2 dt,$$
$$\mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1]) : \text{ the trajectory fulfills} \theta - \epsilon \le u_a(T;u_0) \le \theta + \epsilon\Big\}.$$

Moreover, in Figure 54, one can see snapshots of the parabolic controlled state being close in the phase plane to the elliptic ones.

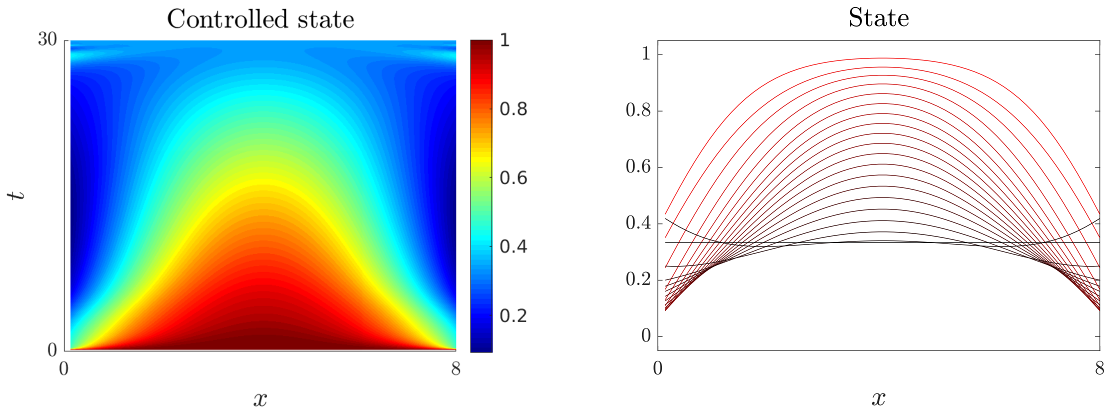

FIGURE 43. Controlled state for $L = 8$, $T = 30$ and initial datum $u_0(x) = 1$.

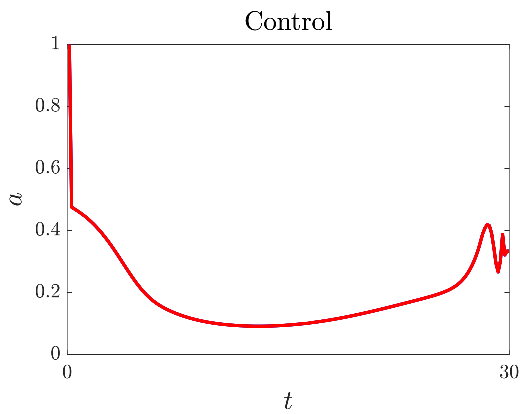

FIGURE 44. Optimal constrained control associated with Figure 43.

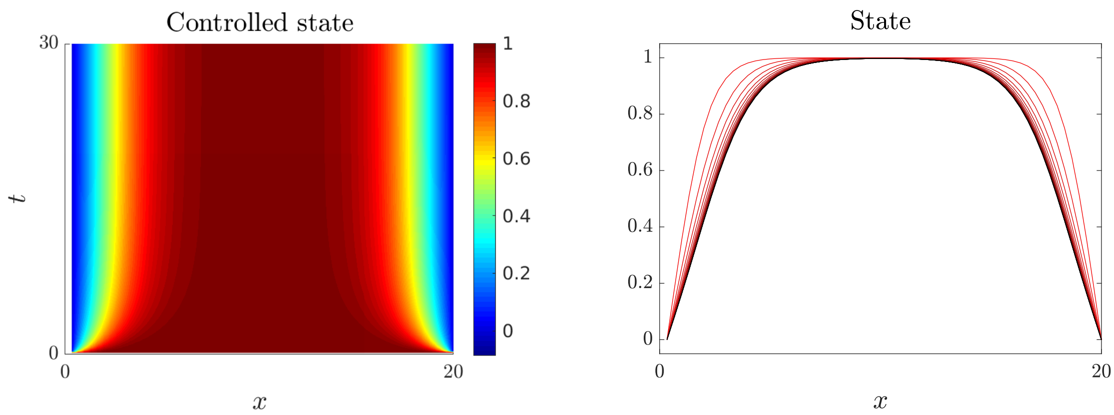

FIGURE 45. Controlled state for $L = 20$, $T = 30$ and initial datum $u_0(x) = 1$.

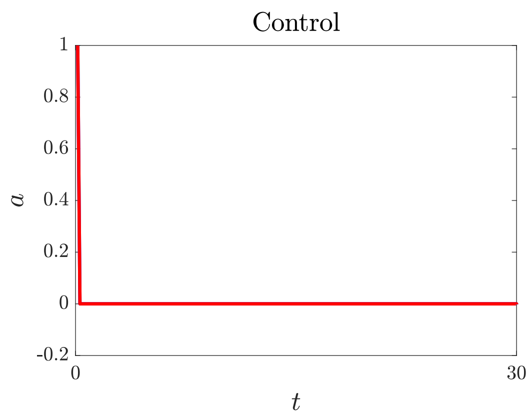

FIGURE 46. Optimal constrained control associated with Figure 45.

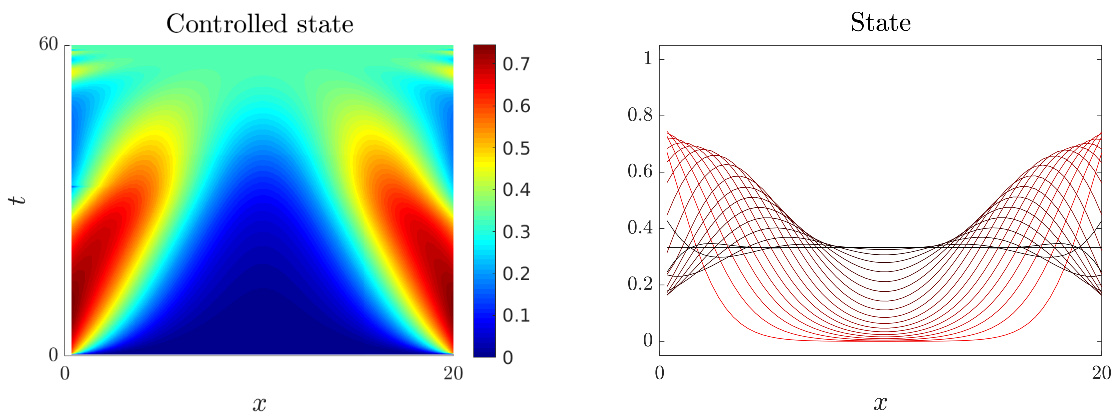

FIGURE 47. Controlled state for $L = 20$, $T = 60$ and initial datum $u_0(x) = 0$.

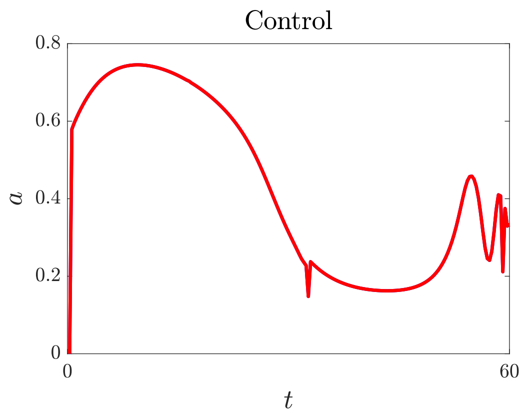

FIGURE 48. Optimal constrained control associated with Figure 47.

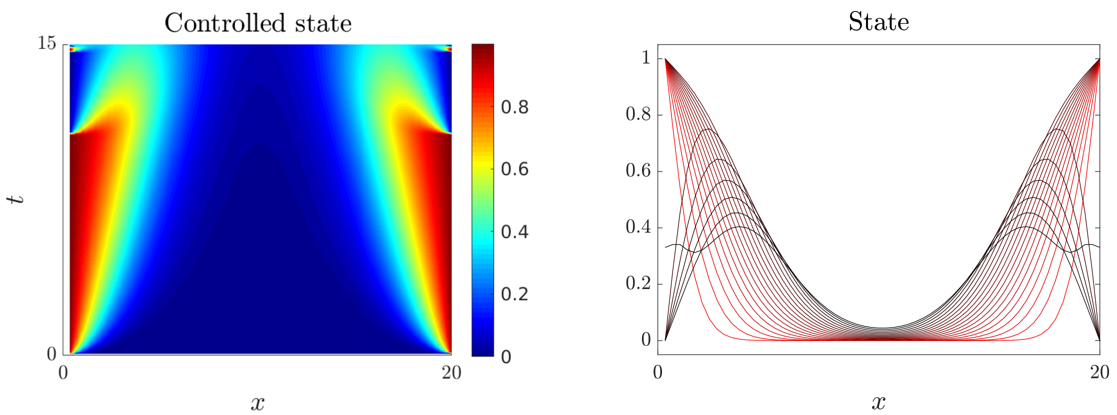

FIGURE 49. Controlled state for $L = 20$, $T = 15$ and initial datum $u_0(x) = 0$.

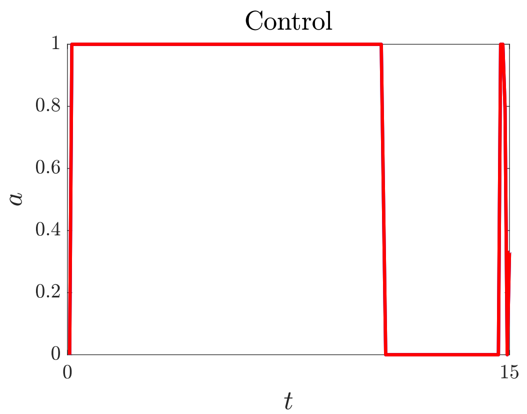

FIGURE 50. Optimal constrained control associated with Figure 49.

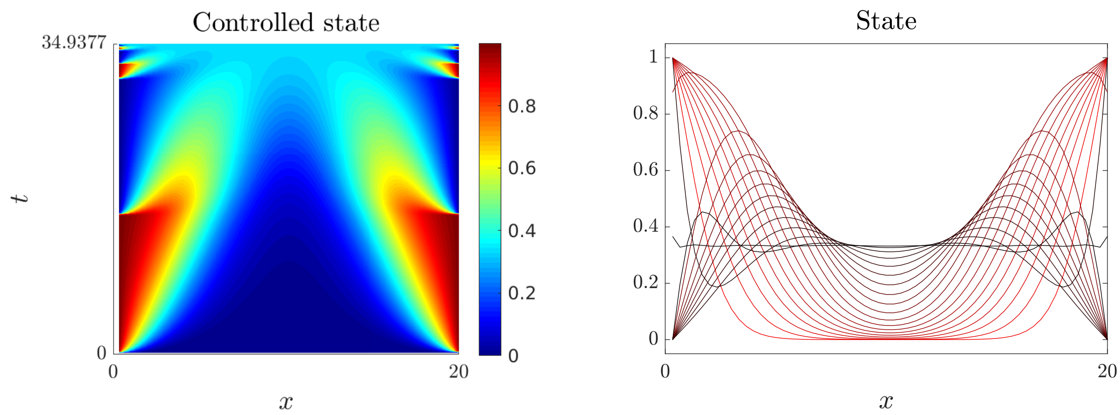

FIGURE 51. Controlled state in minimal time for $L = 20$ and initial datum $u_0(x) = 1$.

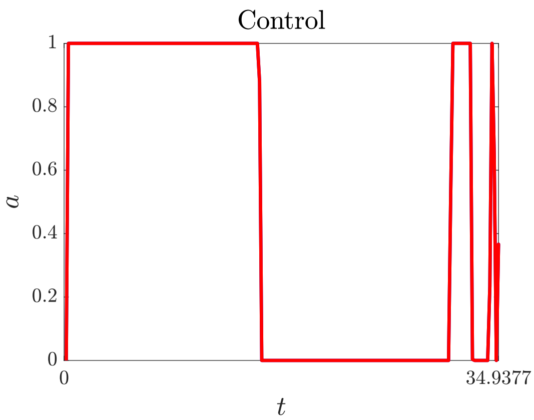

FIGURE 52. Optimal constrained control associated with Figure 51.

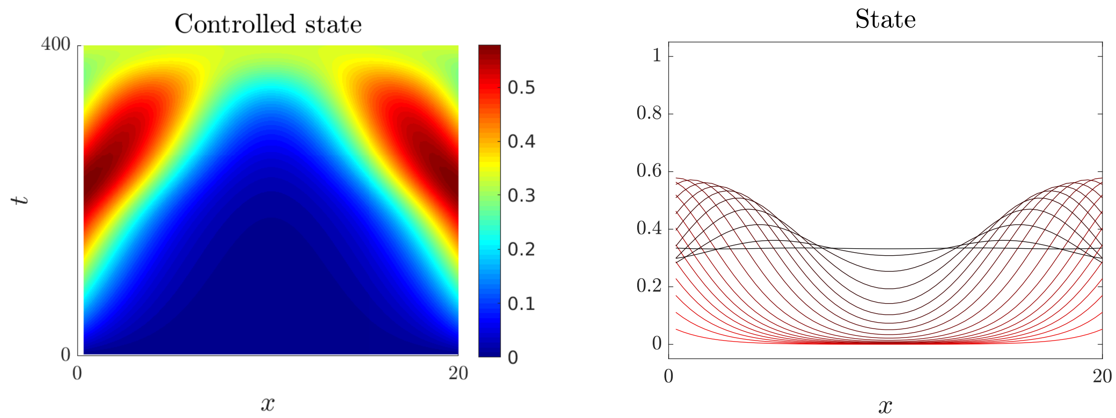

FIGURE 53. Controlled state for $L = 20$, $T = 400$ and initial datum $u_0(x) = 0$.

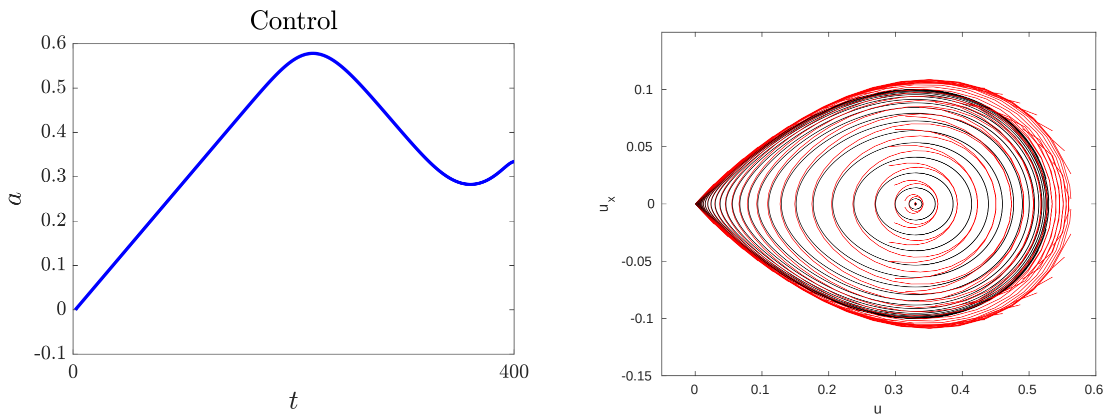

FIGURE 54. (Left) Optimal constrained control associated with Figure 53. (Right) in black steady-states in the phase plane, in red snapshots of the parabolic state in the phase plane

9. Extensions and other problems

9.1. **Combining Allee Control and Boundary Control.**

9.1.1. *Preliminaries.* The goal of this subsection is to present the control strategy in [90] schematically. In the reference above, the authors present the modeling of the evolution of a population of mosquitoes when sterile mosquitoes are released. This way of acting to the system leads to an interaction to the nonlinearity, more precisely to the Allee parameter θ:

$$\begin{cases} v_t - \partial_{xx} v = v(1-v)(v-\theta(t)) \quad x \in \mathbb{R}, \\ 0 \leq \theta(t) \leq 1, \\ 0 \leq v(x,0) \leq 1. \end{cases}$$

The goal of the authors is to control approximately to a traveling wave in large time. The strategy of the proof relies on the following arguments:

- Fix $\theta \in (0,1)$ and let the system converge exponentially to a traveling wave profile. For this step to work, one needs to require that the limits at $+\infty$ and $-\infty$ exists and are different at some point.
- Using that for any θ, the profile of the traveling wave for the cubic nonlinearity is the same one can move θ to move the center of the traveling wave. Note that if $\theta < 1/2$, the traveling wave moves in the direction such that will make 1 invade and if $\theta > 1/2$ the opposite.

9.1.2. *Combined Strategy.* Combining the two interactions, the boundary control and the Allee interaction of [90], we can obtain controllability to any steady (despite 1 and 0 whose convergence is only asymptotic) state and to sections of traveling waves. We consider the following problem:

$$\begin{cases} v_t - \partial_{xx} v = v(1-v)(v-\theta(t)), \\ v(0,t) = a_1(t) \quad v(L,t) = a_2(t), \\ 0 \leq v(x,0) \leq 1. \end{cases}$$

The strategy is presented afterwards:

- Take $a_1 = a_2 = 1/2$ for long time. Matano's Theorem ensures the convergence to a steady-state.
- Local controllability to a stationary point.

At this point in the phase portrait, we have the traveling waves being stationary and connecting 0 and 1. In particular, looking at how the flow in the phase portrait goes, one can see that there is no obstruction if we start inside Γ, i.e. there is no curve that starts at the vertical axis $\{u = 0\}$ or $\{u = 1\}$ and ends at the same. This allows reaching any steady-state inside Γ.

Moreover, we can have exact controllability to the traveling waves. Using the staircase method, we can approach a traveling wave arc and reach it in finite time. Then changing θ, we can move the traveling wave.

9.2. **Multi dimensional case.**

The results and techniques explained before can be extended in several dimensions [82]. Let us consider a bounded domain $\Omega \subset \mathbb{R}^d$ with C^2 boundary of $|\Omega| = 1$ where the following dynamics is taking place:

$$\begin{cases} u_t - \Delta u = \lambda f(u) & (x,t) \in \Omega \times (0,T], \\ u(x,t) = a(x,t) & (x,t) \in \partial\Omega \times (0,T], \\ 0 \leq u(x,t) \leq 1 & (x,t) \in \Omega \times [0,T], \end{cases} \tag{9.1}$$

where $\lambda > 0$.

Note that in this context, the bounds derived in section 4 also work by adapting the proofs properly.

One should note that the first Dirichlet eigenvalue depends on the shape of the domain. This feature has more relevance than the measure of the domain since we can consider a set with very large measure but with a very big first eigenvalue. Proof of Theorem 4.15 is adapted by finding the biggest ball inside the domain and making the computations for finding a subsolution. While the proof of Theorem 4.14 holds the same, and the result will depend on the first eigenvalue that, in turn, depend on the domain geometry. Therefore, the existence of barriers is a general feature of these problems see Figure 55.

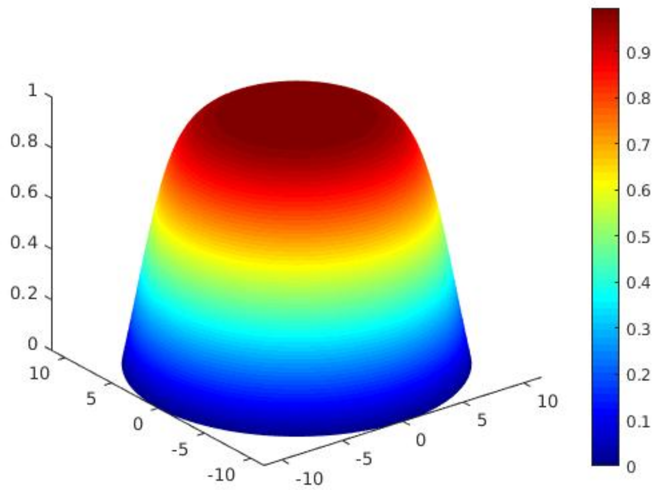

FIGURE 55. Simulation of the semilinear heat equation with nonlinearity $f(s) = s(1-s)(1-\theta)$ with boundary value $a(x,t) = 0$ finding a barrier.

The proof of convergence to the constant steady-state 1 in case that $F(1) > 0$ follows by the traveling wave solutions as in the one-dimensional case Proposition 4.25. Note that by extending the one-dimensional traveling wave to be constant in all other $d-1$ remaining dimensions, one obtains a traveling wave profile for the multi-dimensional case.

The construction of the continuous path of steady-states relies now on extending the domain to a bigger ball, Figure 56,

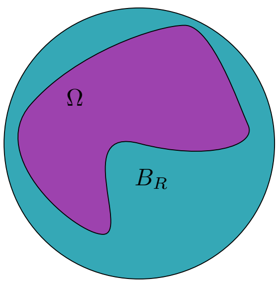

FIGURE 56. Ball containing the original domain.

and write the problem (9.2)

$$\begin{cases} -\Delta u = \lambda f(u) & x \in B_r \subset \mathbb{R}^d, \\ u(0) = a, \\ Du(0) = 0, \end{cases} \tag{9.2}$$

as an ODE problem:

$$
\begin{cases} u_{rr}(r) + \dfrac{d-1}{r} u_r(r) = -f(u(r)) & r \in [0, R_{B,\lambda}), \\ u(0) = a, \\ u_r(0) = 0. \end{cases} \tag{9.3}
$$

where $R_{B,\lambda}$ is the radius of the ball after rescaling for absorbing λ. The solutions of a semilinear elliptic equation in a ball are radially symmetric [23]. This is why the transformation is allowed.

The local wellposedness of the ODE (9.3) comes from Banach contraction and Gromwall inequality.

The key points are the following:

(1) Radial solutions of the problem (9.3) dissipate:
Assume that $F(1) \geq 0$ and consider the energy

$$
E(u, v) = \frac{1}{2} v^2 + F(u)
$$

where $F(u) = \int_0^u f(s) ds$. Now we define the region:

$$
M := \{(u, v) \in \mathbb{R}^2 \quad \text{such that} \quad E(u, v) \leq 0\}
$$

Note that the region defined by

$$
\Gamma := \{(u, v) \in [0, \theta_1] \times \mathbb{R} \quad \text{such that} \quad |v| \leq \sqrt{-2F(u)}\}
$$

satisfies $\Gamma \subset M$. In fact, this region is the same than in the one dimensional case.

Now one considers an initial datum of the form $(u_0, 0) \in \Gamma$, then the solution of (9.3) with initial datum $(u_0, 0)$ satisfies:

$$
\frac{d}{dr} E(u, v) = v v_r + f(u) v = -\frac{d-1}{r} v^2 < 0.
$$

As a consequence $(u, v) \in \Gamma$ for all $r > 0$ and the path is admissible and globally defined.
In Figure 57 one can see the construction of the construction of the path.
Then, restricting to our original domain Ω, one obtains the desired path.

(2) Another important feature is how to control to the path. For this, one realizes that the minimum solution of

$$
\begin{cases} -\Delta u = \lambda f(u) & x \in \Omega, \\ u = a(x) & x \in \partial\Omega, \\ 0 < u < 1 & x \in \Omega, \end{cases}
$$

is radial with respect to some ball, and therefore it belongs to a path of steady-states. The argument follows by contradiction using a comparison argument [82].

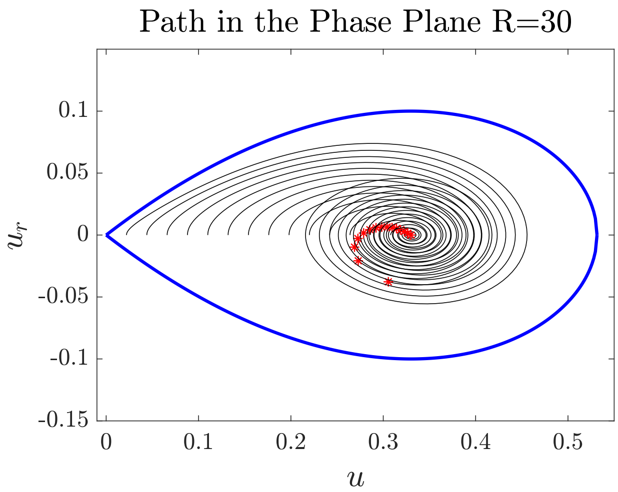

FIGURE 57. In blue the invariant region in the phase space, in black the radial trajectories forming the continuous path of steady-states where the red stars indicate the condition in the boundary. $\theta = 1/3$, $R = 30$ and $d = 2$.

9.3. **Spatially Heterogeneous case.**

In this section sketch a more general model studied in [61]:

$$\begin{cases} u_t - \Delta u + \left\langle \dfrac{2\nabla N}{N}, \nabla u \right\rangle = f(u) & x \in \Omega \times (0,T), \\ u(x,t) = a(x,t) & x \in \partial\Omega \times (0,T), \\ 0 \leq u(x,t) \leq 1, \end{cases}$$

where $N : \mathbb{R}^d \to (0, +\infty)$ be a C^∞ function.

9.3.1. *Modeling.* Taking care of the modeling done in Section 1, one noted that we have assumed that the whole population N was behaving as a homogeneous heat equation. If we consider, instead, that N follows:

$$\begin{cases} N_t - \Delta N = N(\kappa(x) - N), \\ \dfrac{\partial N}{\partial \nu} = 0, \\ N(x,0) = N_0(x) \geq 0, \end{cases} \tag{9.4}$$

where κ is a C^∞ function, then, it is known that thanks to the gradient structure of the semilinear heat equation and together with the fact that there is only one steady-state of (9.4), the solution of (9.4) converges to (9.5),

$$\begin{cases} -\Delta N = N(\kappa(x) - N), \\ \dfrac{\partial N}{\partial \nu} = 0. \end{cases} \tag{9.5}$$

Taking N from (9.5) and employing the same approximation done in Section 1 we arrive at the following heterogeneous bistable equation

$$\begin{cases} u_t - \Delta u + \left\langle \dfrac{2\nabla N}{N}, \nabla u \right\rangle = f(u) & (x,t) \in \Omega \times (0,T), \\ u(x,t) = a(x,t) & (x,t) \in \partial\Omega \times (0,T), \\ 0 \leq u(x,t) \leq 1 & (x,t) \in \Omega \times [0,T], \end{cases}$$

where u is a proportion.

The effect of the mother population N on the state depends on its shape. Figure 58 shows the effect of the drift produced by N. In the right hand side of Figure 58, one can see that the drift pushes from the boundary to the interior. For this reason, one intuitively expects that the controllability is easier in such setting. In the left hand side of Figure 58, on can see that the effect of the boundary control is diminished by the drift. This will lead to the existence of new nontrivial solutions as it is shown in the subsequent subsections.

The new barriers block controllability from 0 to θ in Figure 59 (left) one can see how the minimal controllability time blows up when varying a parameter.

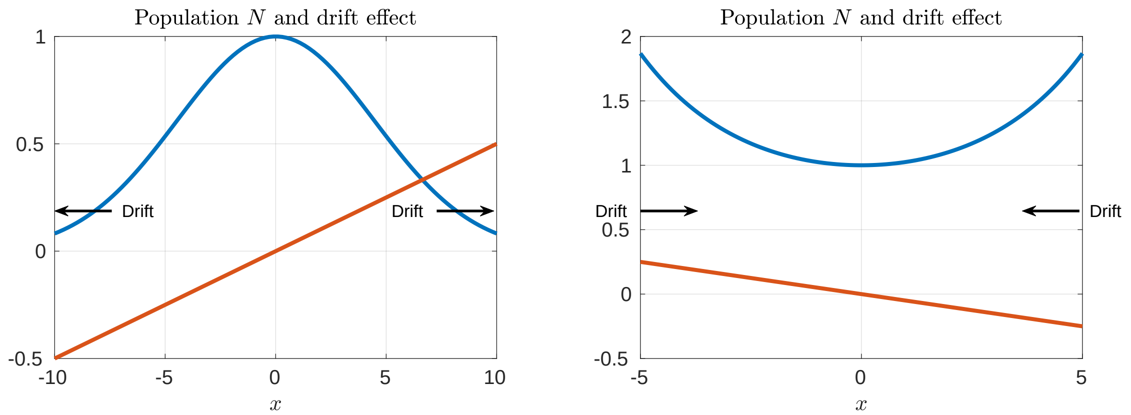

FIGURE 58. In blue the curve $N(x)$, in orange the quotient $-\frac{N_x(x)}{N(x)}$ responsible of the drift effect. Left $N(x) = e^{-\frac{x^2}{\sigma}}$, right $N(x) = e^{\frac{x^2}{\sigma}}$.

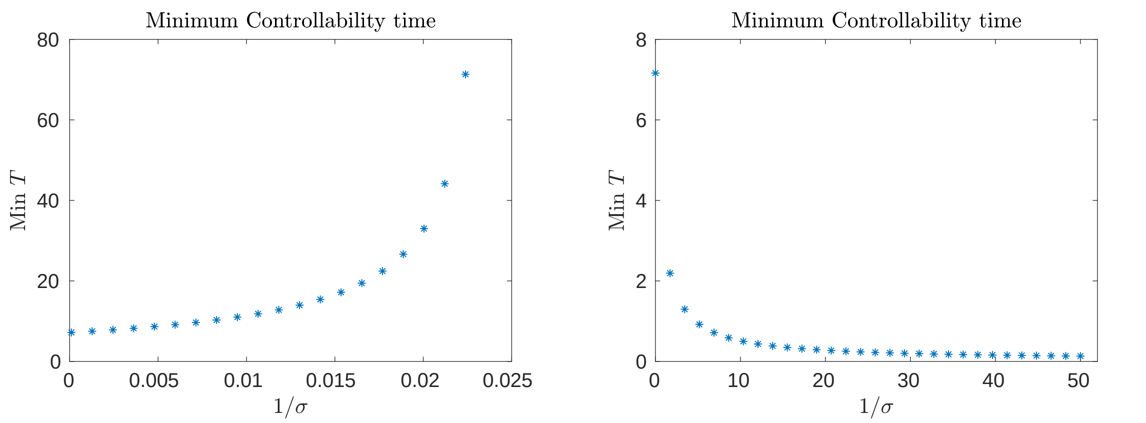

FIGURE 59. Minimal controllability time from 0 to $\theta = \frac{1}{3}$ versus $\epsilon = \frac{1}{\sigma}$. In the right the minimal controllability time for the Gaussian case $N(x) = e^{-\frac{x^2}{\sigma}}$, in the right $N(x) = e^{\frac{x^2}{\sigma}}$.

9.3.2. *Small drifts.* In [61], a method concerning perturbations of the homogeneous path is developed to guarantee the controllability for small drifts. Take finite sequence on the path of steady-states for the homogeneous equation and consider small drift equation:

$$\begin{cases} u_t - \Delta u + \epsilon \langle \nabla p(x), \nabla u \rangle = f(u) & x \in \Omega \times (0,T), \\ u(x,t) = a(x,t) & x \in \partial\Omega \times (0,T), \\ 0 \leq u(x,t) \leq 1 & (x,t) \in \Omega \times [0,T]. \end{cases} \tag{9.6}$$

The implicit function theorem can be applied to all the elements selected from the homogeneous path to obtain a sequence of steady states for (9.6) in a way that remain close enough to apply the staircase method (see Figure 60), see [61, Theorem 2] for details. It has to be remarked that we do not know if a continuous path for the perturbed problem exists, the only requirement to apply the staircase method is to have a sequence whose elements are successively close, see Figure 60.

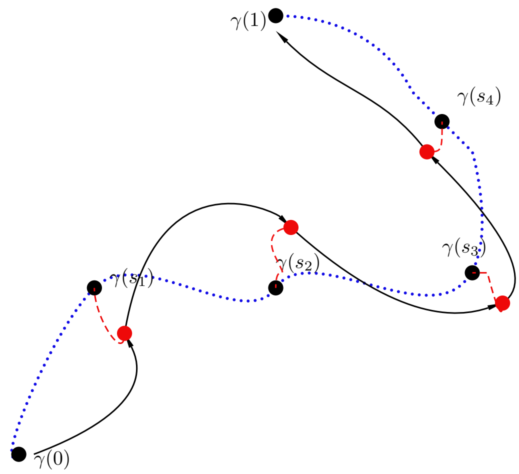

FIGURE 60. The blue dotted line represents the continuous path of steady states for the homogeneous equation. In red, the perturbed steady states, linked to the unperturbed steady states (black) that belong to a continuous path for the homogeneous equation.

This perturbative argument is an improvement of the Staircase method [69].

Figure 59 shows that we cannot expect this method to work for big ϵ.

9.3.3. *Big drifts and new barriers.* In the case of radial drifts in [61], one can conclude that if the following differential inequality is fulfilled, then one also has an invariant region in the phase portrait where the stationary states 0 and θ belong to. In case of multi-D one has that the energy $E(u,v) = \frac{1}{2}v^2 + F(u)$ follows:

$$\frac{d}{dr}E = -\frac{N'}{N}v^2 - \frac{d-1}{r}v^2$$

If

$$\frac{d}{dr}N(r) \geq -\frac{d-1}{r}N(r), \tag{9.7}$$

then the energy decreases and the set:

$$E(u,v) \leq 0$$

is positively invariant (see [61, Theorem 3]).

However, in the opposite case, new barriers can appear in the one-dimensional case. Moreover, a scheme of the proof of the upper barrier, Figure 61, is also shown.

Let us consider the a Gaussian drift $N(x) = e^{-x^2/\sigma}$. From the modeling perspective, this amounts to say that there is a big concentration of individuals around $x = 0$. Consider, in addition, the following boundary value problem:

$$
\begin{cases} -\partial_{xx}u + 4\left(\frac{x}{\sigma}\right)u_x = f(u) & x \in (-L, L), \\ u(-L) = u(L) = a \in \{0, 1\}. \end{cases} \tag{9.8}
$$

Theorem 9.1 (Theorem 4 [61])**.** *Let $F(1) > 0$. For every $\sigma > 0$, there exists a solution of* (9.8) *satisfying the state constraints $0 \le u \le 1$ with $a = 1$ for L big enough. Moreover, there exist another nontrivial solution for certain $L > 0$ with $a = 0$.*

To prove the existence of the barrier function to reach 0, one can proceed with the same arguments discussed previously. However, for proving the existence of the upper barrier, the same methods do not apply, since the stationary solution 1 is a global minimum of the energy functional.

For this reason, in [61] Theorem 9.1 is proved using phase plane techniques (shooting method), namely find an initial condition for the system:

$$
\begin{cases} \dfrac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -f(u) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{4}{\sigma}xv \end{pmatrix}, \\ \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \end{cases} \tag{9.9}
$$

for which at a certain L the trajectory of 9.9 reaches 1.

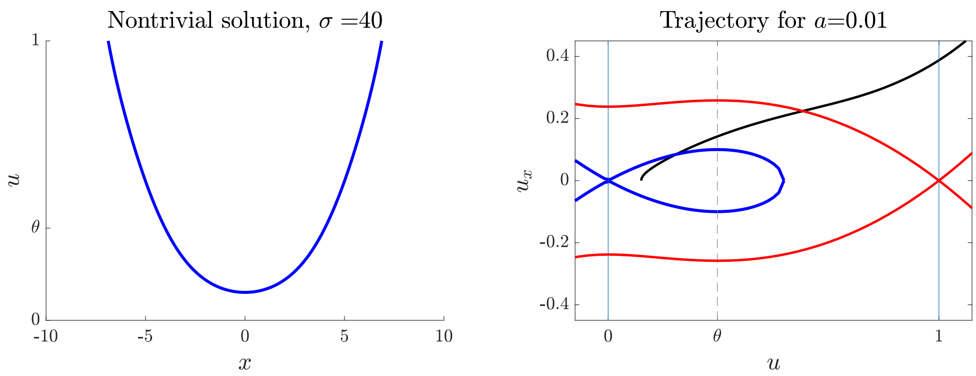

FIGURE 61. (Left) Upper barrier, solution of (9.8) for $\sigma = 40$. (Right) Sketch of the phase-plane analysis for the trajectory leading to a solution of (9.8)

Therefore, for big domains, one cannot guarantee the controllability to any of the constant steady-states.

10. Further comments

When dealing with higher dimensions and spatial heterogeneity, one should take care of new phenomena. For instance, it is known that in several dimensions, we do not necessarily converge o a steady-state. Indeed, in dimension greater than one, Matano's result is not true. The following Theorem is a counterexample[7]:

Theorem 10.1 (Theorem 1.2 in [72]). *Given any dimension $d \geq 2$ and any bounded domain $\Omega \subset \mathbb{R}^d$ with C^2 boundary, there exist a function $g \in C^\infty\left(\mathbb{R}^{d+1}\right)$ and an infinite dimensional manifold $\mathcal{F} \subset W_0^{1,p}(\Omega)$ with $p > d$ such that for every initial datum $v_0 \in \mathcal{F}$ the solution $v(\cdot, t; v_0)$ of*

$$\begin{cases} v_t - \Delta v = g(x, v) & (x,t) \in \Omega \times (0,T], \\ v(x,t) = 0 & (x,t) \in \partial\Omega \times [0,T], \\ v(x,0) = v_0 & x \in \Omega, \end{cases} \tag{10.1}$$

is bounded and its ω-limit set $\omega(v_0)$ is a continuum in $W_0^{1,p}(\Omega)$ homeomorphic to $\mathbb{S}^1$.

For coming up with this issue, one has two options. The first will be if one is able to prove that the set of steady-states is discrete for then employing the LaSalle's invariance principle to ensure its convergence. The second is to use a result of [87] [45] of the Łojasiewicz gradient inequality for analytic nonlinearities, as the prototypical examples $f(t) = t(1-t)(t-\theta)$, to ensure its convergence to a steady-state:

Recall that the semilinear heat equation (10.1) can be expressed as a gradient system in functional spaces, understanding u as a function of time giving values $u(t) \in H_0^1(\Omega)$:

$$\frac{d}{dt}u(t) = -\nabla_u I[u(t)],$$

where I is:

$$\begin{aligned} I : H_0^1(\Omega) &\longrightarrow \mathbb{R}, \\ v &\longrightarrow I[v] = \int_\Omega \frac{1}{2}|v_x(x)|^2 - \int_0^{v(x)} f(s)dsdx. \end{aligned}$$

For $d \geq 2$, for nonlinearities $f(x,v)$ that satisfy:

$$\left\{ \begin{array}{l} \exists C > 0, \alpha \geq 0 \quad \text{such that } (d-2)\alpha \leq 2, \\ \left|\dfrac{\partial f}{\partial s}(x,s)\right| \leq C\left(1 + |s|^\alpha\right) \quad \text{a.e. on } \Omega \times \mathbb{R}, \\ f \quad \text{is analytic in } s, \text{ uniformly with respect to } x \in \Omega, \end{array} \right. \tag{10.2}$$

we have the following Theorem based on gradient systems[8]:

Theorem 10.2 (Łojasiewicz convergence, Theorem 10.4.3 in [38]). *Let $v \in C^1\left(\mathbb{R}_+; H_0^1(\Omega)\right)$ be a bounded solution of (10.1). Assume that for all $\phi \in S := \{\phi \in H_0^1(\Omega) \quad \text{such that} \quad -\Delta\phi + f(x,\phi) = 0\}$ we have that $\phi \in L^\infty(\Omega; \mathbb{R})$ and assume that we have that f satisfies (10.2). Then, there exists $\phi \in S$ such that:*

$$\lim_{t\to\infty} \|v(t) - \phi\|_{H^1(\Omega)} = 0.$$

We have furthermore the following classical result.

[7] The reader can furthermore check [71] for a more particular example, Ω being the unit disk in $\mathbb{R}^2$ and the construction of a nonlinearity of the class C^m with m finite such that a bounded nonconvergent solution of (10.1) exists

[8] Check [38] for stronger and more general form of Theorem 10.2, check also [45]

Theorem 10.3 (LaSalle Theorem 9.2.7 from [14])**.** *Let V be a strict Lyapunov function for $\{Q_t\}_{t\geq 0}$ and Z a complete metric space, and let $z \in Z$ be such that $\cup_{t\geq 0}\{Q_t z\}$ is relatively compact in Z. Let S be the set of equilibrium points of $\{Q_t\}t0$. Then:*

- *S is a non-empty closed subset of Z.*
- *$d(Q_t z, S) \to 0$ as $t \to \infty$ (i.e. $\omega(z) \subset S$).*

Note that if S is discrete, we have the convergence to a steady-state. For the homogeneous heat equation, we have the results of [56], which guarantees this fact (see also [39] for a special case).

11. Perspectives

In this study, we have treated the 1-D constrained controllability of certain semilinear heat equations. The presence of natural state constraints has a fundamental importance in control theory. For this reason, further works are pointed out.

- *Spatial heterogeneities.* Heterogeneities are essential in practice since the environment is, in general diverse. In some places, the growth of the population can be bigger due to a higher capacity, or the diffusion be smaller because the terrain is abrupt. In [61], a particular type of heterogeneities are considered, to consider more general heterogeneities is still an open problem:

$$\begin{cases} u_t - \operatorname{div}(A(x)\nabla u) + \langle b(x), \nabla u\rangle = f(u,x) & (x,t) \in \Omega \times (0,T], \\ u = a(x,t) & (x,t) \in \partial\Omega \times (0,T], \\ 0 \le u(x,0) \le 1. \end{cases}$$

- *Other types of diffusions.* In this text, we have been discussing the case of the classical diffusion by means of the Laplacian. However, other types of diffusion also can model nonnegative quantities. This is the case, for instance, of the porous medium equation [16, 63, 94]:

$$\begin{cases} u_t - (u^m)_{xx} = f(u) & (x,t) \in [0,L] \times (0,T], \\ u(0,t) = a_1(t), \quad u(L,t) = a_2(t) & t \in (0,T], \\ 0 \le u(x,t) \le 1 & (x,t) \in \Omega \times [0,T]. \end{cases}$$

  A specific analysis also has to be done for dealing with other diffusions such as the fractional diffusion [9, 37] or more nonlinear problems [1, 2].
- *Systems.* For instance, in the context of evolutionary game theory [41–43], typically one deals with more than two strategies. In this text, we motivated the problem in this context with a two strategy game that was reduced to a single evolution equation. However, if we deal with more strategies, the reduction to a scalar equation is not possible:

$$\begin{cases} \partial_t u_1 - \mu_1 \Delta u_1 = f_1(u_1,u_2,u_3) & (x,t) \in \Omega \times (0,T], \\ \partial_t u_2 - \mu_2 \Delta u_2 = f_2(u_1,u_2,u_3) & (x,t) \in \Omega \times (0,T], \\ \partial_t u_3 - \mu_3 \Delta u_3 = f_3(u_1,u_2,u_3) & (x,t) \in \Omega \times (0,T], \\ u_1 = a(x,t) \in [0,1] & (x,t) \in \partial\Omega \times (0,T], \\ \dfrac{\partial}{\partial \nu} u_j = 0 & (x,t) \in \partial\Omega \times (0,T] \quad j = 2,3, \\ 0 \le u_i(x,0) \le 1 & i = 1,2,3. \end{cases}$$

  Furthermore, in the case of systems, more interesting phenomenology appears, such as Turing instabilities. The control of the system to those patterns can also be a relevant problem with real applications to morphogenesis.

  We refer to [49, 50, 83] for the controllability and optimal control of systems in the context of reaction-diffusion.
- *Hyperbolic models* Note that the semilinear wave equation:

$$\begin{cases} u_{tt} - u_{xx} = f(u) & (x,t) \in [0,L] \times (0,T], \\ u(0,t) = a_1(t), \quad u(L,t) = a_2(t) & t \in (0,T], \\ 0 \le u(x,t) \le 1 & (x,t) \in \Omega \times [0,T]. \end{cases}$$

  has the same steady-states than the semilinear heat equation discussed in this text. This means that the paths of steady-states and nontrivial solutions are the same. However, for the wave equation, we do not have a maximum principle, this means that barriers might be avoidable, see [70].

On the other hand, the damped semilinear wave equation (telegraph equation) might present under certain conditions a maximum principle (see [35]).

More variety of systems where positivity of the state has sense can also be considered, such as thermoelastic systems [99].

- *Partial boundary.* Due to technical reasons, in [61, 82], the authors considered a control acting in the whole boundary. How can the paths be constructed when one acts only on a portion of the boundary? How is the structure of the set of the steady-states? Let $\eta \subset \partial\Omega$ and consider:

$$\begin{cases} u_t - \Delta u = \lambda f(u) & (x,t) \in \Omega \times (0,T], \\ u(x,t) = a(x,t) & (x,t) \in \eta \times (0,T], \\ \dfrac{\partial}{\partial \nu} u(x,t) = 0 & (x,t) \in \partial\Omega \backslash \eta \times (0,T], \\ 0 \leq u(x,t) \leq 1 & (x,t) \in \Omega \times [0,T]. \end{cases}$$

- *Optimal placement of controls.* We have seen that the length of the domain has a crucial role in the context of constraints. For big domains, boundary control can not be effective. However, the situation might be different if we consider interior control. If we set Neumann boundary conditions and a pointwise control in the middle of the interval, by symmetrization, we observe the same problem with big domains. On the other hand, if we allow ourselves to choose in which region are we placing our control, one can clearly split the control region into small pieces distributed over the domain so that a barrier function cannot exist.

  For the 1d homogeneous reaction-diffusion equation, this placement is an easy problem. However, if we consider a multidimensional problem with spatial heterogeneities, to determine where is the "best" control region with a fixed measure is a nontrivial problem with high relevance in applications.

  For related literature on the placement of sensors and actuators we refer to [75–77].
- *Alternatives to the staircase method and control in minimal time.* The main general strategy to prove controllability is the staircase method [69]. However, we know that there are multiple ways to reach a specific target, for instance, we can set the boundary to be equal to one, or to follow a traveling wave, or when available using the staircase method. In numerical simulations, we have observed that in particular, the minimal time control is far away from being a quasistatic trajectory. Further research could be to develop other methods rather than the staircase method to be able to characterize other trajectories that, in certain situations, might be better than a quasistatic one.
- *Control in minimal time.* Simulations suggest that a bang-bang control might be possible for the minimal time control. How is the control in minimal time? Can we develop precise estimates or a formula for the minimal controllability time?

  One possible way to attack these questions would be to make use of the Pontryagin maximum principle (see [58] for similar questions).
- *Distinguish among control strategies.* Knowing that generally speaking, there is not a unique way to reach a specific configuration. How can we distinguish among different control possibilities? What is the best control (in terms of minimal L^2 norm or minimal flow, for example) for going from 0 to θ, for instance?
- *Periodic targets.* In these notes we have mainly considered steady-states as targets. However, there are dynamic trajectories that could be of interest such as periodic (in time) solutions. How can we control to these trajectories? In which part of the trajectory is faster to control to?
- *Modeling and other problems.* One important remark is that control theory should be intimately related to the modeling. The natural constraints on the modeling give rise to new mathematical challenges in control theory.

In the context of game theory, we modeled in Section 1 two strategies and an evolution of the proportion by the replicator dynamics. However, in practical situations makes sense to consider a continuum of strategies. For instance, in linguistics, most of the individuals are not perfectly bilingual, and their ability in the minority language can range in a continuum spectra. Interactions with other individuals can create an increase or decrease in this trait. For the modeling of this situation, we cite [5].

Moreover, diffusion models are more appropriate for simple living species or chemicals. When dealing with intelligent animals, the understanding of their way of moving in the environment is essential. In this situation also finite-dimensional models are appropriate, and the role of the network in which individuals move plays a crucial role in the dynamics [66, 89].

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Domènec Ruiz-Balet
Departamento de Matemáticas,
Universidad Autónoma de Madrid, 28049 Madrid, Spain
and
Chair of Computational Mathematics, Fundación Deusto
Av. de las Universidades, 24
48007 Bilbao, Basque Country, Spain
Email address: domenec.ruizi@uam.es

Enrique Zuazua
Chair in Applied Analysis,
Alexander von Humboldt-Professorship,
Department of Mathematics,
Friedrich-Alexander-Universität Erlangen-Nürnberg
91058 Erlangen, Germany

and
Chair of Computational Mathematics,
Fundación Deusto
Av. de las Universidades, 24,
48007 Bilbao, Basque Country, Spain

and
Departamento de Matemáticas,
Universidad Autónoma de Madrid, 28049 Madrid, Spain
Email address: enrique.zuazua@fau.de