Control under constraints of reaction-diffusion equations

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Introduction

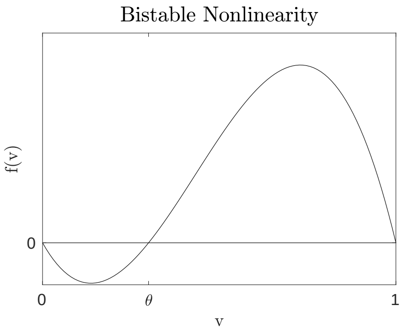
The models

Homogeneous bistable reaction-diffusion:

$$\begin{cases} u_t - \Delta u = f(u) & (x,t) \in \Omega \times (0,T) \\ u = a(x,t) & (x,t) \in \partial\Omega \times (0,T) \\ 0 \leq u(x,0) \leq 1 \end{cases}$$

Bistable reaction-diffusion with heterogeneous drift:

$$\begin{cases} u_t - \Delta u + \nabla b(x) \nabla u = f(u) & (x,t) \in \Omega \times (0,T) \\ u = a(x,t) & (x,t) \in \partial\Omega \times (0,T) \\ 0 \leq u(x,0) \leq 1 \end{cases}$$



The typical example is:

$$f(s) = s(1 - s)(s - \theta)$$

Applications

Reaction-diffusion equations typically model the evolution of quantities that are nonnegative or bounded by above and below, for example:

- Temperature
- Concentrations of chemicals
- Proportions

Control problem

$$\begin{cases} u_t - \Delta u = f(u) & (x,t) \in \Omega \times (0,T) \\ u = a(x,t) & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0 \end{cases} \tag{1}$$

Main Goal

Let f be bistable, and consider $0 \leq u_0 \leq 1$ there exists a $T > 0$ such that equation (1) is controllable towards the constant steady states 0,θ and 1 in a way that the trajectory fulfills for all $(x,t) \in \Omega \times [0,T]$ that $0 \leq u(x,t) \leq 1$?

Negative result: Barriers

Barriers

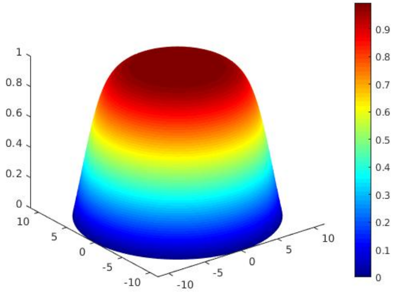
The comparison principle ensures that if a solution to the problem

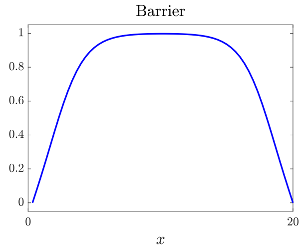
$$\begin{cases} -\Delta v = f(v) & x \in \Omega \\ v = 0 & x \in \partial\Omega \\ 1 > v > 0 & x \in \Omega \end{cases} \tag{2}$$

exists[1] and our initial data u_0 is above v then for any $a \in L^\infty(\Omega, [0,1])$ the solution of the parabolic problem will stay above v.

[1] P.L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Rev. 24 (1982), no. 4, 441–467

Barriers

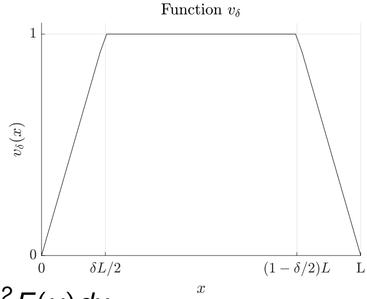

1
0.8
0.6
0.4
0.2
0
10
5
0
-5
-10
-10
-5
0
5
10
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0


Barrier
1
0.8
0.6
0.4
0.2
0
0
20
x

Existence of barriers

After a space rescaling one can rewrite the elliptic equation

$$\begin{cases} -v_{xx} = L^2 f(v) & x \in (0,1) \\ v(0) = v(1) = 0 \\ 1 > v > 0 & x \in (0,1) \end{cases}$$



The associated functional is:

$$J : H_0^1((0,1)) \longrightarrow \mathbb{R}$$

$$J(v) = \int_0^1 \frac{1}{2} v_x^2 - L^2 F(v) dx$$

Nonexistence of barriers

If λ is small, a barrier cannot exist:

$$\lambda_1 \int_0^1 v^2 \leq \int_0^1 v_x^2 = \lambda \int_0^1 v f(v) \leq \lambda \int_0^1 v^2 \|g\|_\infty$$

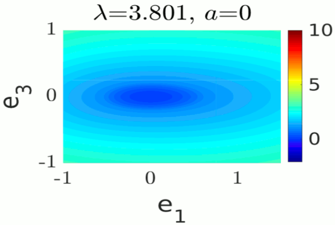
where $f(v) = vg(v)$. Choose λ small enough so that:

$$\lambda_1 \int_0^1 v^2 > \lambda \int_0^1 v^2 \|g\|_\infty$$

Visual representation

$$J : V_h \longrightarrow \mathbb{R}$$

$$J(v) = \int_0^1 \frac{1}{2} v_x^2 - \lambda F(v) dx$$



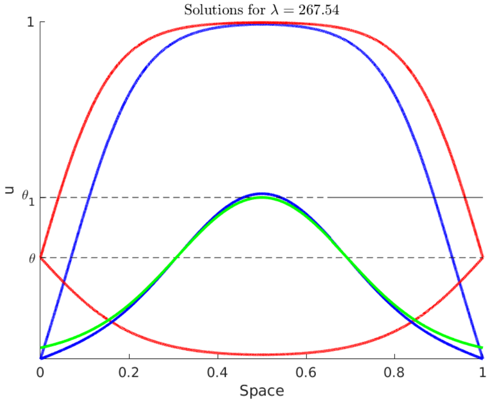
Nonexistence of barriers II

Nonexistence of barriers for reaching 1

Let $F(1) \geq 0$, for any $\lambda > 0$ there exists a unique solution ($v \equiv 1$) to the problem:

$$\begin{cases} -v_{xx} = \lambda f(v) & x \in (0,1) \\ 0 \leq v \leq 1 & x \in (0,1) \\ v(0) = v(1) = 1 \end{cases} \tag{3}$$

Visual representation

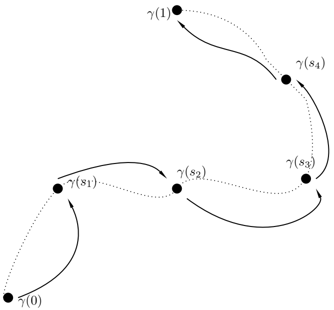


Control towards θ

Staircase method

Theorem

Let v_0 and v_1 be path connected. If T is large enough, $\exists a \in L^\infty$ such that the problem (1) with initial datum v_0 and control a admits unique solution verifying $v(T, \cdot) = v_1$ s.t. its trajectory is admissible.



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[2] D. Pighin and E. Zuazua, Controllability under positivity constraints of semilinear heat equations, Math. Control Relat. Fields 8 (2018), 935.

Continuous Path

$$\mathcal{S} := \left\{ v \in H^1(0,L) \text{ such that } v \text{ satisfies (4)} \right\}$$

$$\begin{cases} -u_{xx} = f(u) \\ u(0) = a_1, \quad u(L) = a_2 \\ 0 \leq u \leq 1 \end{cases} \tag{4}$$

The construction of the path involves to find a map

$$\gamma : [0,1] \to \mathcal{S}$$

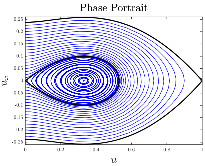
fulfilling

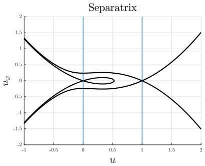
- $\gamma(0) = 0$
- $\gamma(1) = \theta$
- γ is continuous with respect to the L^∞ topology.

Phase portrait

Assume $F(1) > 0$. Observe that the one dimensional elliptic equation can be written as an ODE[3]

$$\frac{d}{dx}\begin{pmatrix} u \\ u_x \end{pmatrix} = \begin{pmatrix} u_x \\ -f(u) \end{pmatrix}$$

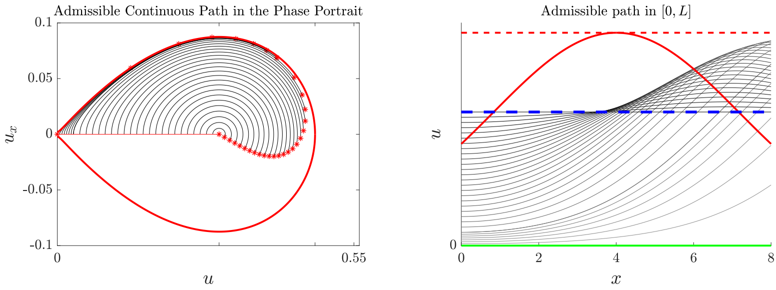




[3] C. Pouchol, E. Trélat, and E. Zuazua, Phase portrait control for 1d monostable and bistable reaction–diffusion equations, Nonlinearity 32 (2019), no. 3, 884–909

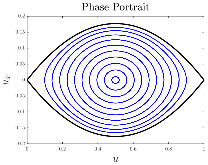
Invariant region

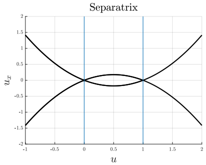
$$\frac{d}{dx}\begin{pmatrix} u \\ u_x \end{pmatrix} = \begin{pmatrix} u_x \\ -f(u) \end{pmatrix}, \quad \begin{pmatrix} u(0) = s\theta \\ u_x(0) = 0 \end{pmatrix}$$



Traveling waves

- When $F(1) = 0$ the traveling waves are stationary and they enclose an invariant region.



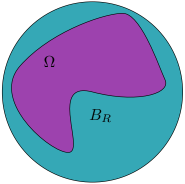


- For $F(1) > 0$ they give a natural control towards the stationary solution 1.

Multi-D

$$\begin{cases} u_t - \mu \Delta u = f(u) & (x,t) \in \Omega \times (0,T) \\ u = a(x,t) & (x,t) \in \partial\Omega \times (0,T) \\ 0 \leq u(x,0) \leq 1 \end{cases} \quad (5)$$

$$\begin{cases} -u_{rr} - \frac{N-1}{r} u_r = \frac{1}{\mu} f(u) & r \in (0,R) \\ u(0) = a \\ u_r(0) = 0 \end{cases}$$



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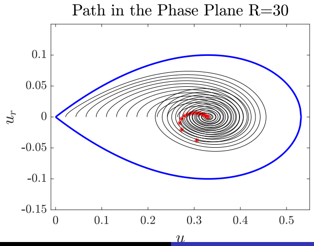
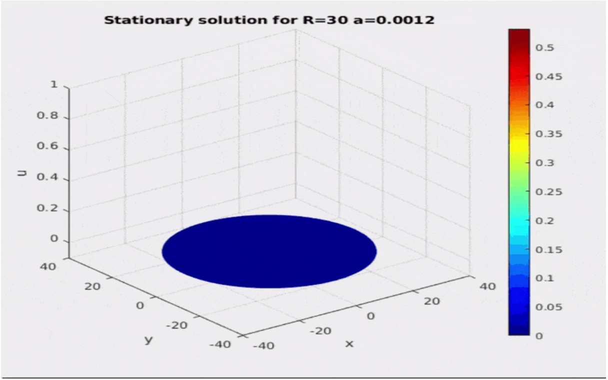
[4] D. Ruiz-Balet and E. Zuazua, Control under constraints for multi-dimensional reaction-diffusion monostable and bistable equations, Preprint (2019)

$$\frac{d}{dr}\begin{pmatrix} u \\ u_r \end{pmatrix} = \begin{pmatrix} u_r \\ -f(u) \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{N-1}{r} u_r \end{pmatrix}$$

$$E(u, u_r) = \frac{1}{2} u_r^2 + F(u)$$

$$\frac{d}{dr} E = -\frac{N-1}{r} u_r^2 < 0$$




Stationary solution for R=30 a=0.0012
u
1
0.8
0.6
0.4
0.2
0
40
20
0
-20
-40
y
-40
-20
0
20
40
x
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0

Theorem (Theorem 1.2 in (4))

Let f be bistable. Let $\Omega \subset \mathbb{R}^N$ be a C^2-regular domain of measure 1*. If* $F(1) > 0$*,* $\exists T \in (0, +\infty]$*, and* $\exists \mathcal{A} \subset L^\infty(\Omega; [0, 1])$ *s.t. the solution of the system* (5) *can be controlled by means of a function* $a \in L^\infty(\partial\Omega \times [0, T], [0, 1])$

- *in infinite time to* $w \equiv 0$*:*
    - *for any initial data* u_0 *iff* $\mu > \mu^*(\Omega, f)$*,*
    - *for any* $\mu > 0$ *if* $u_0 \in \mathcal{A}$*,*
- *in finite time to* $w \equiv \theta$*:*
    - *for any initial data* u_0 *iff* $\mu > \mu^*(\Omega, f)$*,*
    - *for any* $\mu > 0$ *if* $u_0 \in \mathcal{A}$*,*
- *in infinite time to* $w \equiv 1$ *for any admissible initial data* u_0 *and for any* $\mu > 0$.

Furthermore $\mu^*(\Omega, f) > 0$.

Heterogeneous drifts

The model

Consider a distribution of population $N > 0$. Consider that the population is divided between two traits. We model the evolution of the proportion of one trait by[5]:

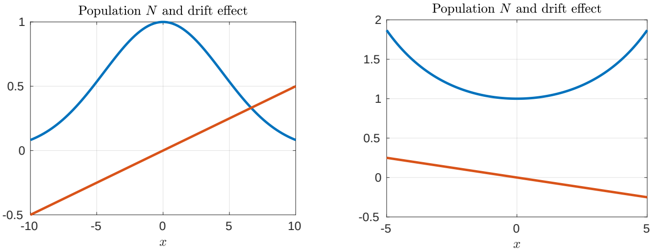
$$\begin{cases} u_t - \Delta u - \frac{\nabla N(x)}{N(x)} \nabla u = f(u) & (x,t) \in \Omega \times (0,T) \\ u = a(x,t) & (x,t) \in \partial\Omega \times (0,T) \\ 0 \leq u(x,0) \leq 1 \end{cases}$$

we will also use the notation $\nabla b(x) = \frac{\nabla N(x)}{N(x)}$

[5] I. Mazari, D. Ruiz-Balet, and E. Zuazua, Constrained control of bistable reaction-diffusion equations: Gene- flow and spatially heterogeneous models, Preprint (2019)

Two examples

- For $N(x) = e^{-\frac{x^2}{\sigma}}$, $\quad u_t - u_{xx} + \frac{2x}{\sigma} u_x = f(u)$
- For $N(x) = e^{\frac{x^2}{\sigma}}$, $\quad u_t - u_{xx} - \frac{2x}{\sigma} u_x = f(u)$



Firsts Barriers

Nontrivial solutions with 0 boundary also exist:

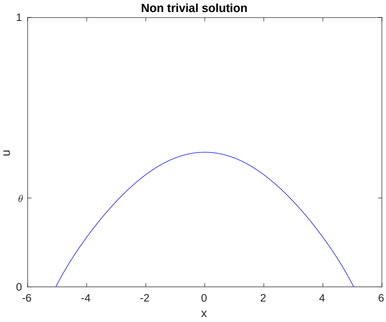


Figure: $\nabla b(x) = 2\frac{x}{\sigma}$.

Differential inequality

When can we ensure controllability?

- The multi-dimensional case shows us that the important fact is to have a conservative or dissivative ODE dynamics.
- If we set a radial drift that makes the ODE dynamics dissipative, would be enough to guarantee the construction of the path.
- Theorem 3 in (5) ensures the existence of a continuous path whenever the drift is radial and fulfills:

$$N'(r) \geq -\frac{N-1}{2r} N(r) \tag{6}$$

Upper barriers

Question

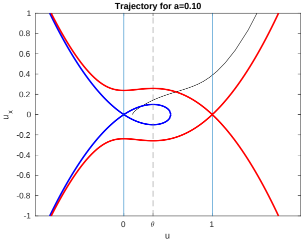
What happens if the differential inequality (6) is not satisfied?

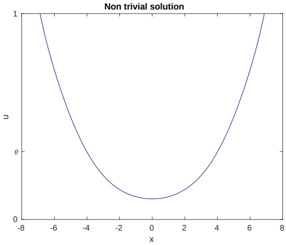
Take $N(x) = e^{-\frac{x^2}{\sigma}}$. We observed that an upper barrier can exist (Theorem 4 in (5)).

$$\begin{cases} -u_{xx} + 2\frac{x}{\sigma}u_x = f(u) \\ u(-L) = u(L) = 1 \end{cases} \tag{7}$$

Shooting method

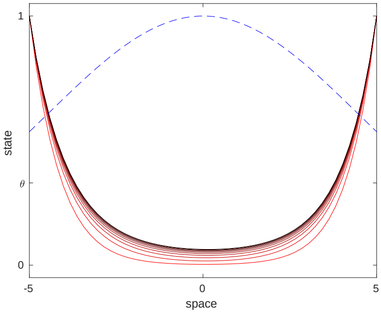
This steady state cannot correspond to a global energetic minima hence a shooting method is employed.





Emergence of a barrier

In the following numerical simulation one can observe how the parabolic trajectory finds the barrier found before.



References

1. P.L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Rev. 24 (1982), no. 4, 441–467
2. D. Pighin and E. Zuazua, Controllability under positivity constraints of semilinear heat equations, Math. Control Relat. Fields 8 (2018), 935.
3. C. Pouchol, E. Trélat, and E. Zuazua, Phase portrait control for 1d monostable and bistable reaction–diffusion equations, Nonlinearity 32 (2019), no. 3, 884–909
4. D. Ruiz-Balet and E. Zuazua, Control under constraints for multi-dimensional reaction-diffusion monostable and bistable equations, Preprint (2019)
5. I. Mazari, D. Ruiz-Balet, and E. Zuazua, Constrained control of bistable reaction-diffusion equations: Gene- flow and spatially heterogeneous models, Preprint (2019)

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