The inverse problem for Hamilton-Jacobi equations and semiconcave envelopes

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joint work with Enrique Zuazua

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erc European Research Council Established by the European Commission

We consider the following initial-value problem

$$\begin{cases} \partial_t u + H(D_x u) = 0, & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{HJ}$$

where $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$ is the initial condition and

$$H : \mathbb{R}^n \longrightarrow \mathbb{R}$$

is a C^2 Hamiltonian satisfying

$$D^2 H(x) > 0, \ \forall x \in \mathbb{R}^n \text{ and } \frac{H(|x|)}{|x|} \underset{|x|\to\infty}{\longrightarrow} +\infty. \tag{H}$$

A problem in calculus of variations

We are given $T > 0$ and two cost functions:

Running cost: $L : \mathbb{R}^n \longrightarrow \mathbb{R}$ **Initial cost:** $u_0 : \mathbb{R}^n \longrightarrow \mathbb{R}$

For any $(t, x) \in]0, T[\times\mathbb{R}^n$, we introduce the set of **admissible arcs**

$$\mathcal{A}(t,x) := \{\alpha \in C^1([0,t];\mathbb{R}^n)\,;\ \alpha(t) = x\},$$

and consider the following **minimization problem**:

$$\text{minimize} \int_0^t L(\alpha'(s))ds + u_0(\alpha(0)) \text{ over all arcs } \alpha \in \mathcal{A}(t,x).$$

We define the **value function**:

$$u(t,x) = \inf_{\alpha(\cdot)\in\mathcal{A}(t,x)} \left\{ \int_0^t L(\alpha'(s))ds + u_0(\alpha(0)) \right\}.$$

This function satisfies the equation

$$\partial_t u + H(D_x u) = 0$$

at all points of differentiability of u. Here, H is given by

$$H(v) = \sup_{z\in\mathbb{R}^n} \{z \cdot v - L(z)\}.$$

We remark that u is Lipschitz in $[0,T] \times \mathbb{R}$ provided $u_0 \in \text{Lip}(\mathbb{R})$ and H satisfies (H).

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We remark that u is Lipschitz in $[0, T] \times \mathbb{R}$ provided $u_0 \in \mathrm{Lip}(\mathbb{R})$ and H satisfies (H).

$$\begin{cases} \partial_t u + H(D_x u) = 0, & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{HJ}$$

In general we cannot expect to have C^1 solutions. Therefore, we need to consider **generalized solutions**:

$$u \in W^{1,\infty}_{\text{loc}}, \text{ satisfying (HJ) a.e.}$$

We have no uniqueness of generalized solutions.

$$\begin{cases} \partial_t u + H(D_x u) = 0, & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{HJ}$$

Definition

We say $u \in C([0,T] \times \mathbb{R}^n)$ is a viscosity solution if

$$u(0,x) = u_0(x)$$

and for any $(t,x) \in (0,T) \times \mathbb{R}^n$ we have

$$\begin{array}{ll} p_t + H(p_x) \leq 0 & \text{for all } (p_t, p_x) \in D^+ u(t,x) \\ p_t + H(p_x) \geq 0 & \text{for all } (p_t, p_x) \in D^- u(t,x) \end{array}$$

where the super- and sub-differentials are defined by

$$\begin{aligned} D^+ u(t,x) = \{(p_t, p_x) \ : \ p_t = \varphi_t(t,x),\ p_x = D\varphi(t,x), \quad \exists \varphi \in C^1, \\ u - \varphi \leq 0,\ (u-\varphi)(t,x) = 0\}, \\ D^- u(t,x) = \{(p_t, p_x) \ : \ p_t = \varphi_t(t,x),\ p_x = D\varphi(t,x), \quad \exists \varphi \in C^1, \\ u - \varphi \geq 0,\ (u-\varphi)(t,x) = 0\}. \end{aligned}$$

Viscosity solutions

$$\begin{cases} \partial_t u + H(D_x u) = 0, & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{HJ}$$

Theorem: Crandall-P.L. Lions, 1980's

Let $T > 0$, $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$ and H satisfy (H). The problem (HJ) admits a unique viscosity solution and coincides with the value function of the problem in calculus of variations.

We define the following nonlinear operator:

$$\begin{array}{rccc} S_T^+ : & \mathrm{Lip}(\mathbb{R}) & \longrightarrow & \mathrm{Lip}(\mathbb{R}) \\ & u_0 & \longmapsto & S_T^+ u_0 := u(T,\cdot) \end{array}$$

Goals

Consider the nonlinear operator:

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where u is the viscosity solution of (HJ).

The inverse problem: for a given target $u_T \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$ fixed,

- Study the reachability of u_T, i.e. determine if the set

$$I_T(u_T) := \left\{ u_0 \in \text{Lip}(\mathbb{R}) \, ; \; S_T^+ u_0 = u_T \right\}$$

  is empty or not.
- If u_T is reachable, construct all the initial conditions in $I_T(u_T)$.
- If u_T is not reachable, define a projection of u_T on the set of reachable targets and study its geometrical properties.

Long-time behavior: for a given initial condition $u_0 \in \text{Lip}(\mathbb{R}^n)$, set the target $u_T := S_T^+ u_0$ and the set of initial conditions $I_T(u_T) \neq \emptyset$, for each $T > 0$.

- Describe the evolution of $I_T(u_T)$ as T increases.
- Study the behavior of $I_T(u_T)$ as T goes to infinity.

The backward operator

Definition

A uniformly continuous function $w : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is called a **backward viscosity solution** of (HJ) if the function $v(t, x) := w(T - t, x)$ is a viscosity solution of

$$\partial_t v - H(D_x v) = 0, \quad \text{in } [0, T] \times \mathbb{R}^n.$$

Lemma

We say $w \in C([0, T] \times \mathbb{R}^n)$ is a backward viscosity solution if and only if for any $(t, x) \in (0, T) \times \mathbb{R}^n$ we have

$$\begin{aligned} p_t + H(p_x) &\geq 0 \quad \text{for all } (p_t, p_x) \in D^+ w(t, x) \\ p_t + H(p_x) &\leq 0 \quad \text{for all } (p_t, p_x) \in D^- w(t, x) \end{aligned}$$

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Using similar arguments as for (forward) viscosity solutions, for any terminal condition $u_T \in \mathrm{Lip}(\mathbb{R}^n)$, the problem

$$\begin{cases} \partial_t w + H(\partial_x w) = 0, & \text{in } [0,T] \times \mathbb{R}^n, \\ w(T,x) = u_T(x), & \text{in } \mathbb{R} \end{cases} \tag{BHJ}$$

admits a unique backward viscosity solution.

We define the following nonlinear operator:

$$\begin{array}{cccc} S_T^- : & \mathrm{Lip}(\mathbb{R}) & \longrightarrow & \mathrm{Lip}(\mathbb{R}) \\ & u_T & \longmapsto & S_T^- u_T := w(0,\cdot) \end{array}$$

where w is the backward viscosity solution of (BHJ).

Hopf formula

$$S_T^+ u_0(x) = \min_{y\in\mathbb{R}^n} \left[u_0(y) + T\,H\left(\frac{x-y}{T}\right)\right]$$

$$S_T^- u_T(x) = \max_{y\in\mathbb{R}^n} \left[u_T(y) - T\,H\left(\frac{y-x}{T}\right)\right]$$

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Definition

1. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is semiconcave with linear modulus if it is continuous and there exists $C \geq 0$ such that
$$f(x+h) + f(x-h) - 2f(x) \leq C\,h^2, \qquad \text{for all } x, h \in \mathbb{R}^n.$$
The constant C above is called a semiconcavity constant of f.
2. We say that f is semiconvex with linear modulus and constant $C > 0$ if the function $g = -f$ is semiconcave with linear modulus and constant C.

Lemma

Let $T > 0$ and $u_0, u_T \in \mathrm{Lip}(\mathbb{R})$. Then,

1. the function $S_T^+ u_0$ is semiconcave with linear modulus;
2. the function $S_T^- u_T$ is semiconvex with linear modulus.

Semiconcavity and semiconvexity

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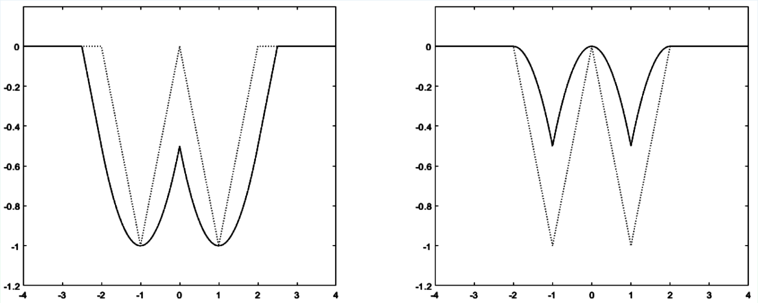
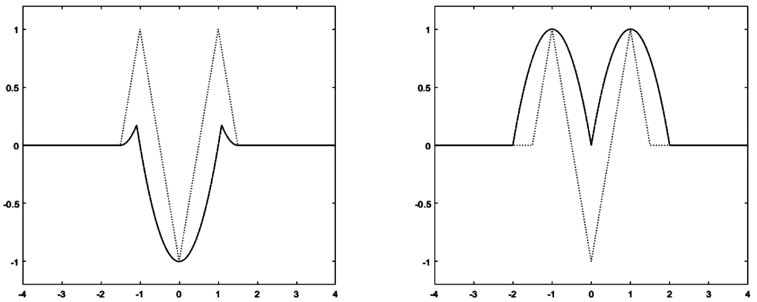
$$u_1(x) := \begin{cases} |x+1|-1 & \text{if } -2 < x \leq 0 \\ |x-1|-1 & \text{if } 0 < x < 2 \\ 0 & \text{else.} \end{cases}$$



For $T = 1$, the function $S_T^+ u_1$ at the left and the function $S_T^- u_1$ at the right.

$$u_2(x) := \begin{cases} 1-2|x+1| & \text{if } -1{,}5 < x \le 0 \\ 1-2|x-1| & \text{if } 0 < x < 1{,}5 \\ 0 & \text{else.} \end{cases}$$



For $T = 0{,}5$, the function $S_T^+ u_2$ at the left and the function $S_T^- u_2$ at the right.

Lemma

Let $T > 0$ and $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$. Set the function

$$\tilde{u}_0(x) := S_T^-(S_T^+ u_0)(x), \qquad \text{for all } x \in \mathbb{R}^n.$$

Then it holds

$$S_T^+ u_0 = S_T^+ \tilde{u}_0, \qquad \text{and} \qquad u_0(x) \geq \tilde{u}_0(x), \quad \text{for all } x \in \mathbb{R}^n.$$

Reachability condition

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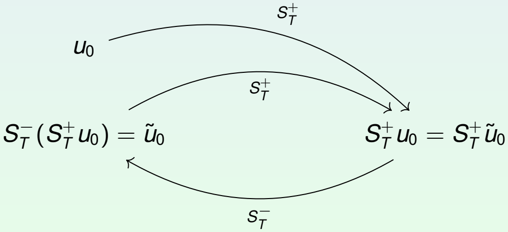

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Theorem (reachability condition)

Let $u_T \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$. Then, the set $I_T(u_T)$ is nonempty if and only if $S_T^+ \left(S_T^- u_T \right) = u_T$.

Theorem (initial data construction)

Let $T > 0$ and consider $u_T \in \mathrm{Lip}(\mathbb{R}^n)$ such that $I_T(u_T) \neq \emptyset$. Define the function

$$\tilde{u}_0 := S_T^- u_T.$$

For any $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$, the two following statements are equivalent:

1. $u_0 \in I_T(u_T)$;
2. $u_0(x) \geq \tilde{u}_0(x), \ \forall x \in \mathbb{R}^n \quad \text{and} \quad u_0(x) = \tilde{u}_0(x), \ \forall x \in X_T(u_T)$,

where $X_T(u_T)$ is the subset of $\mathbb{R}$ defined by

$$X_T(u_T) := \left\{ z - T\,\nabla_x u_T(z); \ \forall z \in \mathbb{R}^n \text{ such that } u_T(\cdot) \text{ is differentiable at } z \right\}.$$

Remark: Observe that, by the reachability condition,

$$I_T(u_T) \neq \emptyset, \quad \text{implies} \quad \tilde{u}_0 \in I_T(u_T).$$

In view of this theorem, we can write

$$I_T(u_T) = \left\{ \tilde{u}_0 + \varphi; \ \varphi \in \mathrm{Lip}(\mathbb{R}) \text{ such that } \varphi \geq 0 \text{ and } \mathrm{supp}(\varphi) \subset \mathbb{R} \setminus X_T(u_T) \right\}.$$

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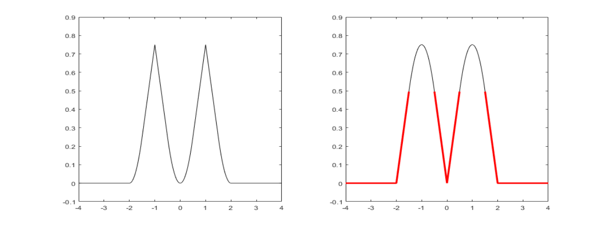
Consider $T = 0{,}5$ and the target function

$$u_T(x) := S_T^+ u_3(x), \quad \text{where} \quad u_3(x) := \begin{cases} 1 - |x+1| & \text{if } -2 < x \leq 0 \\ 1 - |x-1| & \text{if } 0 < x < 2 \\ 0 & \text{else.} \end{cases}$$

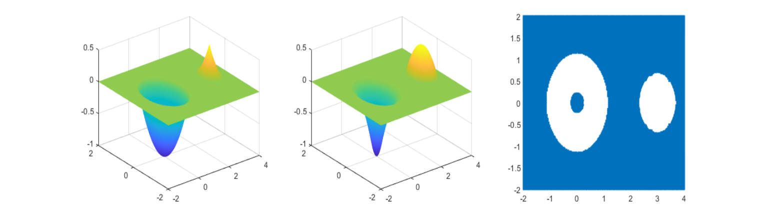
$$X_T(u_T) = \mathbb{R} \setminus ([-1{,}5, -0{,}5] \cup [0{,}5, 1{,}5]) .$$



The function u_T is represented at the left. The function $\tilde{u}_0$ is represented at the right. The restriction of $\tilde{u}_0$ to the set $X_T(u_T)$ is marked by a red line.

Consider $T = 0{,}5$ and the target function

$$u_T(x) := S_T^+ u_4(x), \quad \text{where} \quad u_4(x) := \begin{cases} -(1 - 4|x|^2) & \text{if } |x| < \dfrac{1}{2} \\ 1 - 4|x - (3,0)|^2 & \text{if } |x - (3,0)| < \dfrac{1}{2} \\ 0 & \text{else.} \end{cases}$$



From left to right we have: the function u_T, the function $\tilde{u}_0$ and the set $X_T(u_T)$ in blue.

The concave envelope

For a given function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, the **concave envelope** f^* is the smallest concave function which stays above f.

$$f^*(x) := \inf\{v(x)\,;\ v \text{ is concave and } v(x) \geq f(x),\ \forall x \in \mathbb{R}^n\}.$$

Theorem: (Oberman in 2007) Let $f \in \text{Lip}(\mathbb{R}^n)$, then f^* is the viscosity solution of the following fully nonlinear obstacle problem:

$$\min\{v(x) - f(x),\ -\lambda_n[D^2 v(x)]\} = 0.$$

Here, $\lambda_n[D^2 v(x)]$ denotes the biggest eigenvalue of the Hessian matrix $D^2 v(x)$.

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This kind of operators, and its connection with geometry and game theory, have been largely studied during the past 10 year by many authors: A.M Oberman, L. Silvestre, I. Birindelli, F.R. Harvey, H.B. Lawson, H. Ishii, M. Parviainen, P. Blanc, J.D. Rossi...

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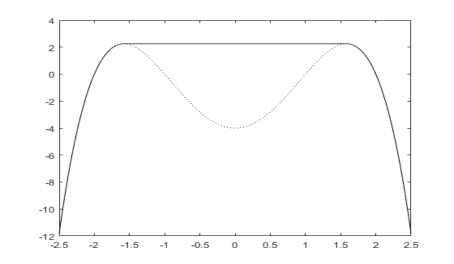

What if the target u_T is not reachable?

Consider the composition operator

$$S_T^+ \circ S_T^- : \quad \begin{array}{ccc} \mathrm{Lip}(\mathbb{R}^n) & \longrightarrow & \mathrm{Lip}(\mathbb{R}^n) \\ u_T & \longmapsto & S_T^+(S_T^- u_T) \end{array}$$

Note the the function $u_T^* := S_T^+(S_T^- u_T)$ satisfies $I_T(u_T) \neq \emptyset$.

The operator $S_T^+ \circ S_T^-$ can be viewed as a projection of $\mathrm{Lip}(\mathbb{R}^n)$ onto the set of reachable targets.

Theorem

Let $H(p) = \frac{|p|^2}{p}$ and $u_T \in \mathrm{Lip}(\mathbb{R}^n)$. Then, the function $u_T^* := S_T^+(S_T^- u_T)$ is the viscosity solution of the obstacle problem

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Observe that, the inequality $\lambda_n[D^2 u_T^*(x)] \leq \frac{1}{T}$ implies that the function u_T^* is semiconcave with linear modulus and constant $C = \frac{1}{T}$.

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In analogy with the concave envelope, we refer to the function u_T^* as the $\frac{1}{T}$−**semiconcave envelope** of u_T in $\mathbb{R}^n$.

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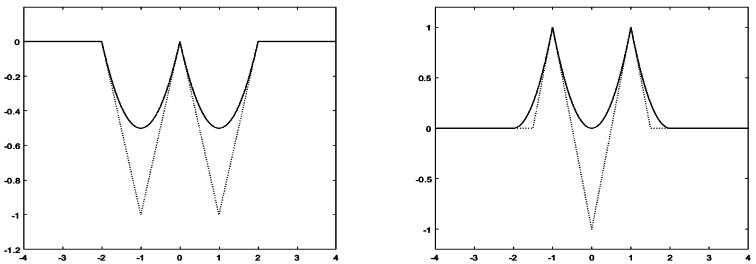
Corollary (reachability condition)

Let $u_T \in \mathrm{Lip}(\mathbb{R}^n)$ and $T > 0$, then the set $I_T(u_T)$ is nonempty if and only if u_T satisfies the inequality $\lambda_n[D^2 u_T(x)] \leq \frac{1}{T}$ in a viscosity sense.

Examples

$$u_1(x) := \begin{cases} |x+1|-1 & \text{if } -2 < x \leq 0 \\ |x-1|-1 & \text{if } 0 < x < 2 \\ 0 & \text{else.} \end{cases}$$

$$u_2(x) := \begin{cases} 1-2|x+1| & \text{if } -1{,}5 < x \leq 0 \\ 1-2|x-1| & \text{if } 0 < x < 1{,}5 \\ 0 & \text{else.} \end{cases}$$



Here can see the $\frac{1}{T}-$semiconcave envelopes of u_1 and u_2 respectively and the functions u_1 and u_2 represented by dotted lines.

Thanks for the attention!

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