Touchdown localization for the MEMS problem with variable dielectric permittivity

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joint work with Philippe Souplet

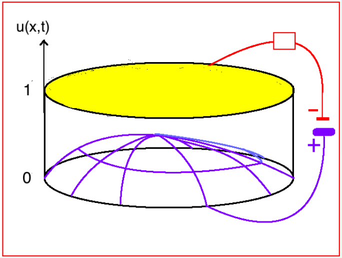
Micro-electromechanical systems (MEMS)

We consider the problem

$$
\left.\begin{array}{rll}
u_t - \Delta u = f(x)(1-u)^{-p}, & x \in \Omega, & t > 0, \\
u = 0, & x \in \partial\Omega, & t > 0, \\
u(0,x) = 0, & x \in \Omega, &
\end{array}\right\} \qquad (1)
$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, $p > 0$ and

$$f \geq 0 \text{ is a Hölder function.} \qquad (2)$$



Modèle de MEMS.

$p = 2$, $n = 2$ et $f(x) = \lambda g(x)$.

- λ is proportional to the applied voltage.
- $g(x)$ represents the dielectric permittivity of the material.

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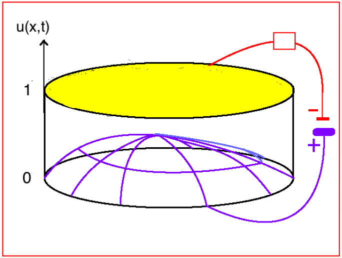

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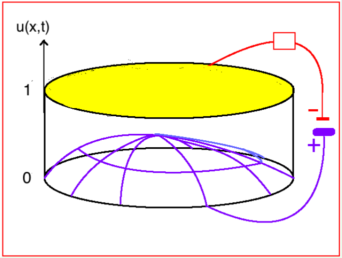
Full model.

$$\epsilon u_{tt} + u_t - \Delta u = \frac{\lambda g(x)}{(1-u)^2 \left(1 + \mu \int_\Omega \frac{1}{1-u} dx\right)^2}, \qquad x \in \Omega, \quad t > 0.$$

We consider $\epsilon = \mu = 0$.



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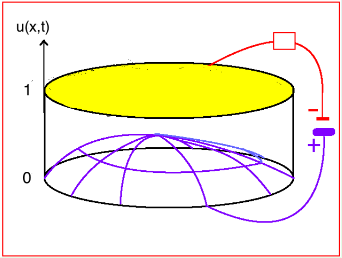

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Definition

We denote the maximal existence time of a classical solution by $T = T_f \in (0, \infty]$. $\|u(t)\|_{L^\infty} \longrightarrow 1$, as $t \nearrow T$.

Definition

A point $x_0 \in \overline{\Omega}$ is called a *touchdown point* if there exists a sequence $\{(t_n, x_n)\}$ in $(0, T) \times \overline{\Omega}$ such that

$$x_n \to x_0, \quad t_n \nearrow T, \quad \text{et} \quad u(t_n, x_n) \to 1 \text{ quand } n \to \infty.$$

The set of all such points is called the **touchdown set**, and is denoted by $\mathcal{T}_f$.

Motivation:

Can there be touchdown at a point where $f = 0$?

No for interior points of Ω (J.S. Guo and P. Souplet (2014-15)).

J.S. Guo and P. Souplet. *No touchdown at zero points of the permittivity profile for the MEMS problem.* SIAM J. Math. Analysis 47(2015), 614-625.

Quenching, or touchdown, phenomenon.

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No touchdown at points of small permittivity.

Natural question: Can we rule out touchdown at points of positive but small permittivity?

Theorem 1

Suppose

$$\begin{cases} T(f) \leq M, \quad \|f\|_\infty \leq M, \quad f \geq r\chi_B, \\ \text{where } M, r > 0 \text{ et } B \subset \Omega \text{ is a ball of radius } r. \end{cases} \tag{3}$$

There exists a constant $\gamma_0 > 0$ depending only on p, Ω, M, r such that, for all $x_0 \in \Omega$, if

$$f(x_0) < \gamma_0 \, dist(x_0, \partial\Omega)^{p+1}, \tag{4}$$

then x_0 is not a touchdown point.

Drawback: The threshold depends on x, and vanishes at the boundary.

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Theorem 2

Under the hypothesis of the previous theorem, there exists a constant $\gamma_0 > 0$ *depending only on* p, Ω, M, r *such that, for any* $\omega \subset\subset \Omega$*, if*

$$\sup_{x \in \overline{\Omega} \setminus \omega} f(x) < \gamma_0 \, dist(\omega, \partial\Omega)^{p+1}, \tag{5}$$

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- This is a partial contribution to the open problem whether or not touchdown can occur on the boundary, including at boundary points of zero permittivity. Previous results are given in this direction, where f is assumed either to vanish sufficiently fast or to be nonincreasing near the boundary.
- This result allows one to localize the touchdown in a small subdomain of Ω by choosing a suitable permittivity profile.

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Proposition: type I estimate

Under the assumptions of the previous theorems, the solution u of problem (1) satisfies

$$u(t,x) \leq 1 - \gamma \mathrm{dist}(x, \partial\Omega)(T-t)^{\frac{1}{p+1}}, \quad \text{for all } t \in [0,T) \text{ and } x \in \Omega, \tag{6}$$

where γ depends only on p, Ω, M, r.

Proof of Theorems 1 and 2

Once we have proven the type I estimate for interior points, the proof of Theorems 1 and 2 consists in constructing a comparison function of the form

$$w(t,x) = y(t)\psi(x), \quad \text{in}[0,T) \times D,$$

where $D = B(x_0, b)$, $b > 0$ small enough for the Theorem 1 and $D = \Omega \setminus \omega$ for Theorem 2.

Proof of Type I estimate

The proof is a modification of the Friedman-McLeod method (1985). The proposition is proved by means of the maximum principle applied to the auxiliary function

$$J(t,x) = u_t - \varepsilon a(x) h(u),$$

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In one space dimension we can make a much more accurate study of the techniques used in the proof of the type one estimate.

Let $\tilde{\Omega} \subset\subset \Omega$. We now define the auxiliary function

$$J(t,x) := u_t - \varepsilon a(x) h(u),$$

in $[0,T) \times \tilde{\Omega}$.
Now the function h is given by

$$h(u) = (1-u)^{-p} + K(1-u)^q, \quad \text{with } K > 0,\ 0 \le q \le 1.$$

We choose $a(x)$ as a solution of the following ODE

$$a''(x) \ge a(x)\, F\Big(x, \frac{a'(x)}{a(x)}\Big), \qquad 0 \le x < 1+\beta,$$

where

$$F(x,\xi) = \begin{cases} \sup\limits_{u\in(0,1)} \left[\dfrac{h'^2(u)}{hh''(u)}\right] \xi^2, & x > 1, \\ \sup\limits_{u\in(1-\eta,1)} \left[\dfrac{h'^2(u)}{hh''(u)}\xi^2 - \dfrac{(p+q)K\mu}{(1-u)^{p+1-q}h(u)}\right], & x < 1. \end{cases}$$

Quantitative results in one space dimension.

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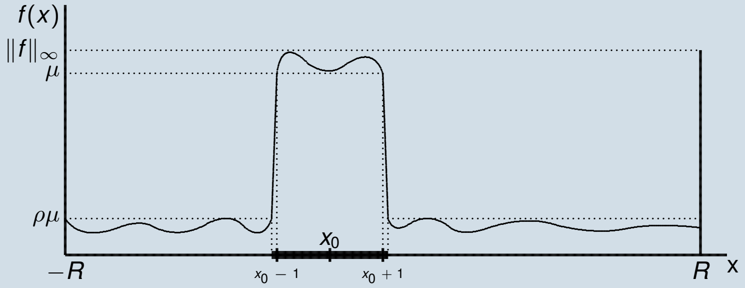
In order to give good estimates of the threshold, we shall consider two typical situations, which roughly correspond to a "one-bump" or a "two-bump" shape for the profile f.

Permittivity profile with one bump



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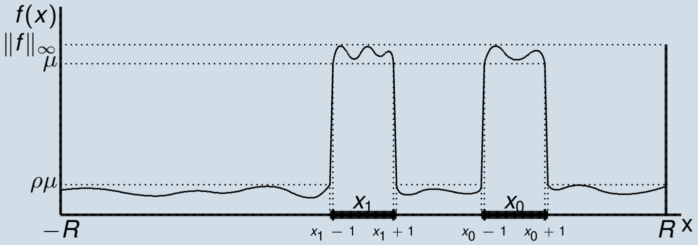
Permittivity profile with two bumps



Numerical estimates for the threshold ratio ρ

With the help of computational tools such as *Matlab* with very good accuracy, we can compute a lower estimate $\overline{\rho}$ of the ratio ρ, which is the **ratio between the height of the bump and the threshold.**

μ	$\|f\|_\infty$	d	d_0	$\overline{\rho}$
1	1.1	0.1	5	**0.2237**
2	2.25	0.1	3	**0.2146**
3	3.5	0.01	5	**0.2725**
6	6.2	0.01	10	**0.2858**
10	10	0.005	10	**0.2917**

- μ is the height of the bump.
- $\|f\|_\infty$ is the maximum of the permittivity profile.
- d is the distance to the bump.
- d_0 is the distance between the bump and the boundary.

Theorem 3 (Continuity of the touchdown time and upper semi-continuity of the touchdown set)

Let $p > 0$ and $\Omega \subset \mathbb{R}^n$ a smooth bounded domain. Let $1 \leq q \leq \infty$ with $q > \frac{n}{2}$, $B \subset \Omega$ a ball of radius $r > 0$, $M \geq \mu > \mu_0(p,n)r^{-2}$ and set

$$\tilde{E} = \left\{ f \in E;\ M \geq f \geq \mu\chi_B \right\}. \tag{7}$$

For all $f \in \tilde{E}$ with $\mathcal{T}_f \subset\subset \Omega$ and all $\sigma > 0$, there exists $\varepsilon > 0$ such that,

$$\text{if } g \in \tilde{E} \text{ and } \|g - f\|_q \leq \varepsilon, \text{ then } |T_g - T_f| \leq \sigma \text{ and } \mathcal{T}_g \subset \mathcal{T}_f + B(0,\sigma).$$

This result allows one to construct profiles for which the touchdown set is located far away from the maximum points of f.

This implies that some kind of **smallness condition as the ones presented above cannot be avoided in order to rule out touchdown at a point.**

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Nontrivial touchdown sets.

Theorem 4 (Stability of single point touchdown under perturbation)

Let $p > 0$, $\Omega = B_R \subset \mathbb{R}^n$, $1 \leq q \leq \infty$ with $q > \frac{n}{2}$, $M > 0$, $\rho \in (0, R)$. Let $f \in E \cap C^1(\overline{B}_\rho)$ be radially symmetric nonincreasing, with $f(r) > \mu_0(p, n)\rho^{-2}$ on $\overline{B}_\rho$. There exists $\varepsilon > 0$ such that, if $g \in E \cap C^1(\overline{B}_\rho)$ is radially symmetric and satisfies

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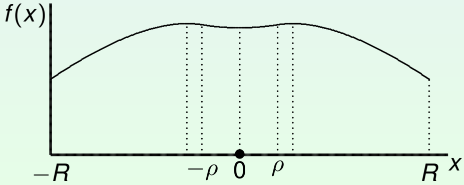

An illustration of Theorem 4 in one space dimension for an "M"-shaped profile.

We see that the single touchdown point is located far away from the maximum points of f.

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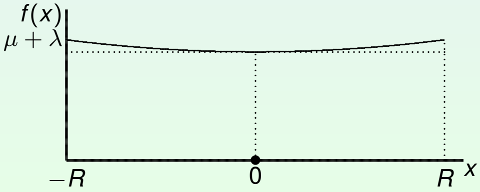

An illustration of Theorem 4 for a strictly convex profile. For example

$$f_\lambda(x) := \mu + \lambda \frac{|x|^2}{R^2},$$

with $\mu > \mu_0(p, n)\rho^{-2}$ and $\lambda \geq 0$.

Nontrivial touchdown sets.

Theorem 5

Let $p > 0$. Let $\Omega \subset \mathbb{R}^n$ a smooth bounded domain.
(i) (Touchdown set concentrated near two arbitrary points.) *For any $x_1, x_2 \in \Omega$ and any $\rho > 0$, there exist positive profiles $f \in E$ such that*

$$\mathcal{T}_f \subset B(x_1, \rho) \cup B(x_2, \rho), \qquad \mathcal{T}_f \cap B(x_1, \rho) \neq \emptyset, \qquad \mathcal{T}_f \cap B(x_2, \rho) \neq \emptyset.$$

(ii) (Touchdown set concentrated near two arbitrary spheres.) *Let $\Omega = B_R \subset \mathbb{R}^n$, $0 < r_1 < r_2 < R$, $\rho > 0$ and set $A_i = \{x \in \mathbb{R}^n;\ |x| \in (r_i - \rho, r_i + \rho)\}$. There exist positive, radially symmetric profiles $f \in E$ such that*

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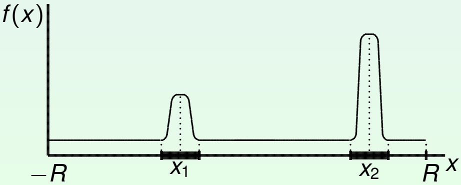

Illustration of Theorem 5(i) for $n = 1$.

The touchdown set contains, at least, two connected components close to both bumps.

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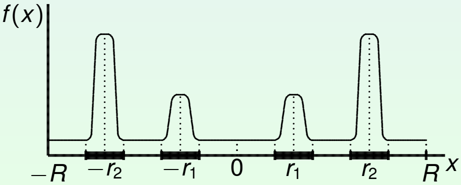

Illustration of Theorem 5(ii) for $n = 1$.

The touchdown set contains, at least, two $n-1$ dimensional spheres.

Thanks for the attention!!