CONTROLLABILITY OF FRACTIONAL HEAT EQUATIONS UNDER POSITIVITY CONSTRAINTS

Umberto Biccari
umberto.biccari@deusto.es u.biccari@gmail.com
cmc.deusto.es/umberto-biccari
joint works with Harbir Antil, Víctor Hernández-Santamaría, Rodrigo Ponce, Mahamadi Warma, Sebastián Zamorano and Enrique Zuazua

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INTRODUCTION

Controllability of the fractional heat equation

Fractional heat equation

$$\begin{cases} z_t + (-d_x^2)^s z = f & (x,t) \in (-1,1) \times (0,T) =: Q \\ z \equiv g & (x,t) \in (-1,1)^c \times (0,T) =: Q^c \\ z(\cdot,0) = z_0 & x \in (-1,1). \end{cases}$$

FRACTIONAL LAPLACIAN:

$$(-d_x^2)^s z(x) = c_s \, \text{P.V.} \int_{\mathbb{R}} \frac{z(x) - z(y)}{|x-y|^{1+2s}} \, dy, \quad x \in \mathbb{R}.$$

We are interested in **controllability properties** under positivity constraints.

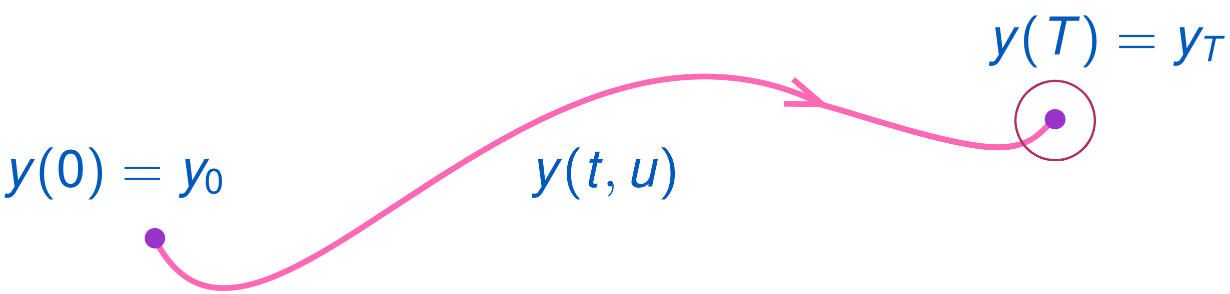
Linear control problem

$$\begin{cases} \dfrac{dy}{dt} = Ay + Bu, \quad t \in [0, T] \\ y(0) = y_0. \end{cases}$$

- $A : \mathcal{D}(A) \to H$
- $B \in \mathcal{L}(U; \mathcal{D}(A)^*)$
- u: **control**

Controllability

Starting from a given initial state at time $t = 0$ we want to act on the trajectories through a suitable control in order to match (or get close to) a desired final state in time $T > 0$.



Definitions of controllability

The system is **approximately controllable** at time T if, for any $y_0, y_T \in H$ and any $\varepsilon > 0$, there exists $u \in L^2(0, T; U)$ such that the solution y fulfills $\|y(T) - y_T\|_H < \varepsilon$.

The system is **null-controllable** at time T if, for any $y_0 \in H$, there exists $u \in L^2(0, T; U)$ such that the solution y fulfills $y(T) = 0$.

The system is **exactly controllable to trajectories** at time T if, for any $y_0 \in H$ and any $\widehat{y}$ solution with $\widehat{y}(0) = \widehat{y}_0 \in H$ and some given $\widehat{u}$, there exists $u \in L^2(0, T; U)$ such that the solution y fulfills $y(T) = \widehat{y}(T)$.

Hilbert Uniqueness Method (HUM)

The property of controllability of a system is equivalent to certain measurements (**observability**) of its adjoint

$$-\frac{dp}{dt} = A^* p, \quad t \in [0, T]; \qquad p(T) = p^T \in H.$$

Observability inequality

A system is null-controllable at time T if and only if there exists a constant $\mathcal{C} > 0$ such that

$$\|p(0)\|_H^2 \leq \mathcal{C} \int_0^T \|B^* p(t)\|_U^2 \, dt, \quad \textit{for all } p^T \in H.$$

Lions, SIAM Rev., 1988.

Computation of the HUM controls

The HUM controls are computed via the minimization of a given functional

$$u = B^*\widehat{p}$$

$$\widehat{p}: \textit{ solution of the adjoint equation with initial datum } \widehat{p}(T) = \widehat{p}^T$$

$$\widehat{p}^T = \min_{p^T \in H} J(p_T) := \frac{1}{2}\int_0^T |B^* p|^2 \, dt + \langle y_0, p(0)\rangle.$$

Direct Method of the Calculus of Variations

Let H be a Hilbert space with norm $\|\cdot\|_H$ and let $J : H \to \mathbb{R}$ be a continuous, convex and coercive functional. Then, J attains its minimum in H. Moreover, if J is strictly convex, this minimum is unique.

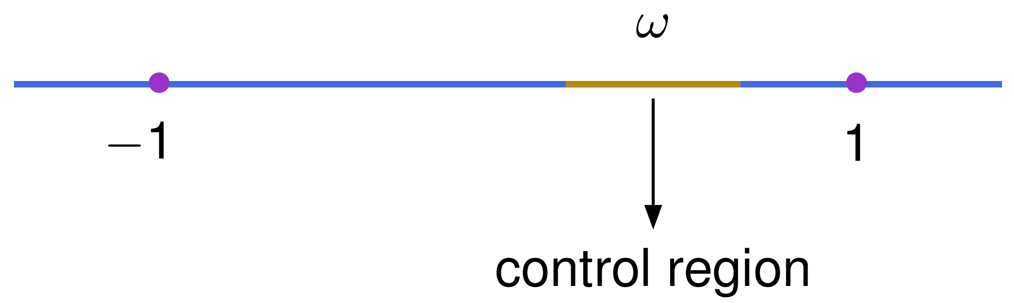
Observability inequality $\Longrightarrow$ Coercivity of J.

CONTROLLABILITY OF THE FRACTIONAL HEAT EQUATION

Controllability of the fractional heat equation

Controlled fractional heat equation

$$\begin{cases} z_t + (-d_x^2)^s z = g\chi_\omega & (x,t) \in Q \\ z \equiv 0 & (x,t) \in Q^c \\ z(\cdot,0) = z_0 & x \in (-1,1) \end{cases} \qquad (\mathcal{FH}1)$$



Controllability results

Theorem (B. and Hernández-Santamaría, IMA J. Math. Control Inf., 2019)

The fractional heat equation $(\mathcal{FH}1)$ *is*

- ***null-controllable*** *at time* $T > 0$ *if and only if* $s > 1/2$.
- ***approximately controllable*** *at time* $T > 0$ *for all* $s \in (0, 1)$.

Proof of the null controllability (sketch)

Moment method (Fattorini and Russell, Quart. Appl. Math., 1974)

$$g(x,t) := -\sum_{\ell\geq 1} \frac{z_\ell^0}{\|\varrho_\ell\|^2_{L^2(\omega)}} q_\ell(t)\varrho_\ell(x), \quad \int_0^T q_\ell(t)e^{-\lambda_k t}\,dt = \delta_{k,\ell}.$$

The control function g is well defined if and only if

1. $\lambda_{k+1} - \lambda_k \geq \gamma > 0, \quad \forall\, k \in \mathbb{N}$
2. $\sum_{k\geq 1} \lambda_k^{-1} < +\infty$
3. $\|\varrho_k\|_{L^2(\omega)} \geq C > 0$

Eigenvalues (Kwaśnicki, J. Funct. Anal., 2012)

$$\lambda_k = \left(\frac{k\pi}{2} - \frac{(1-s)\pi}{4}\right)^{2s} + O\left(\frac{1}{k}\right).$$

The conditions 1 and 2 above are both satisfied if and only if $s > 1/2$.

Proof of the approximate controllability (sketch)

The result follows from the following property.

Parabolic unique continuation

Given $s \in (0,1)$ and $p^T \in L^2(-1,1)$, let p be the unique solution to the adjoint equation

$$\begin{cases} -p_t + (-d_x^2)^s p = 0, & (x,t) \in Q \\ p = 0, & (x,t) \in Q^c \\ p(\cdot, T) = p^T(x), & x \in (-1,1). \end{cases}$$

Let $\omega \subset (-1,1)$ be an arbitrary open set. If $p = 0$ on $\omega \times (0,T)$, then $p = 0$ on $(-1,1) \times (0,T)$.

This, in turn, is a consequence of a **unique continuation** property for the fractional Laplacian.

Fall and Felli, Comm. Partial Differential Equations, 2014.

Control computation

Penalized Hilbert Uniqueness Method: we have to solve the following minimization problem

$$\min_{p^T \in L^2(-1,1)} J(p^T) := \frac{1}{2}\int_{\omega\times(0,T)} |p|^2 dxdt + \frac{\varepsilon}{2}\left\|p^T\right\|^2_{L^2(-1,1)} + \int_{-1}^{1} z_0 p(0),$$

where p is the solution of the adjoint problem.

J is **continuous**, **coercive** and **strictly convex**, thus the existence and uniqueness of a minimizer p^T_{min} is guaranteed.

The control g is chosen as

$$g = p_{min}|_{\omega}$$

where p_{min} is the solution to the adjoint with initial datum p^T_{min}.

Control computation (cont.)

- We use Conjugate Gradient to minimize the functional J.
- The fractional Laplacian is approximated in Finite Element.

B. and Hernández-Santamaría, IMA J. Math. Control. Inf., 2019.

Analyzing the behavior of the penalized problem with respect to the parameter ε, we can deduce controllability properties for our system.

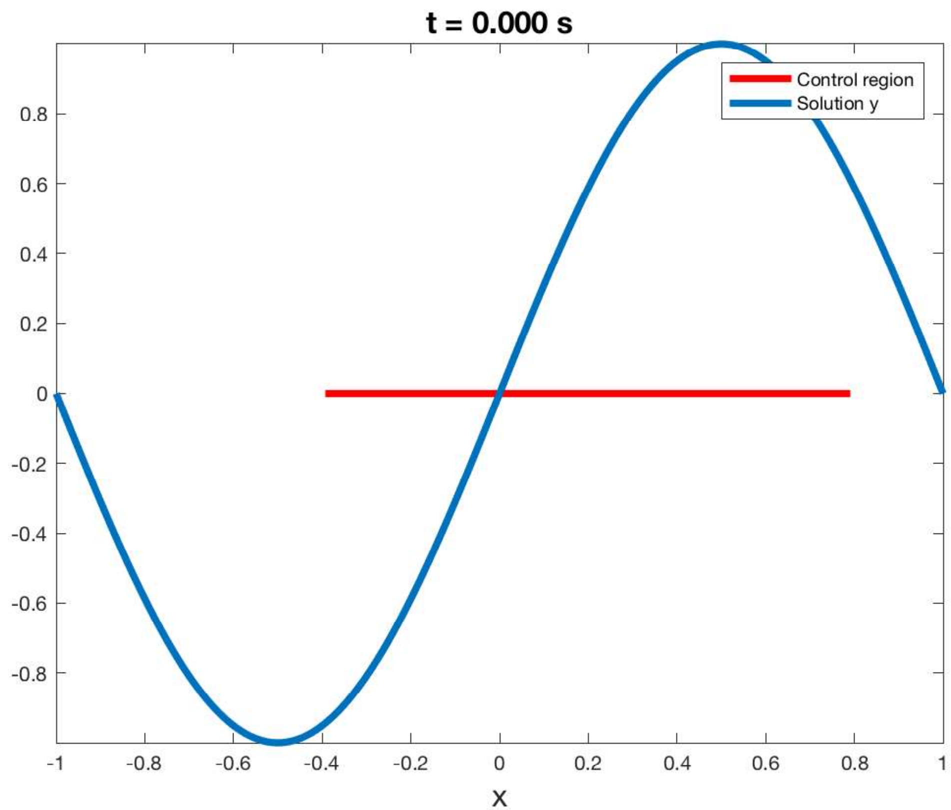
Theorem (Boyer, ESAIM Proc., 2013)

Let p_ε^T be the unique minimizer of J_ε. The system is

- ***NULL CONTROLLABLE*** *at time* $T \Leftrightarrow M_{z_0}^2 := 2\sup_{\varepsilon>0}(\inf J) < +\infty$.
  In this case:
  $$\|g\|_{L^2(\omega\times(0,T))} \le M_{z_0} \quad \text{and} \quad \|p_{min}^T\| \le M_{z_0}\sqrt{\varepsilon}$$
- ***APPROXIMATELY CONTROLLABLE*** *at time* $T \Leftrightarrow p_{min}^T \to 0$ *as* $\varepsilon \to 0$.

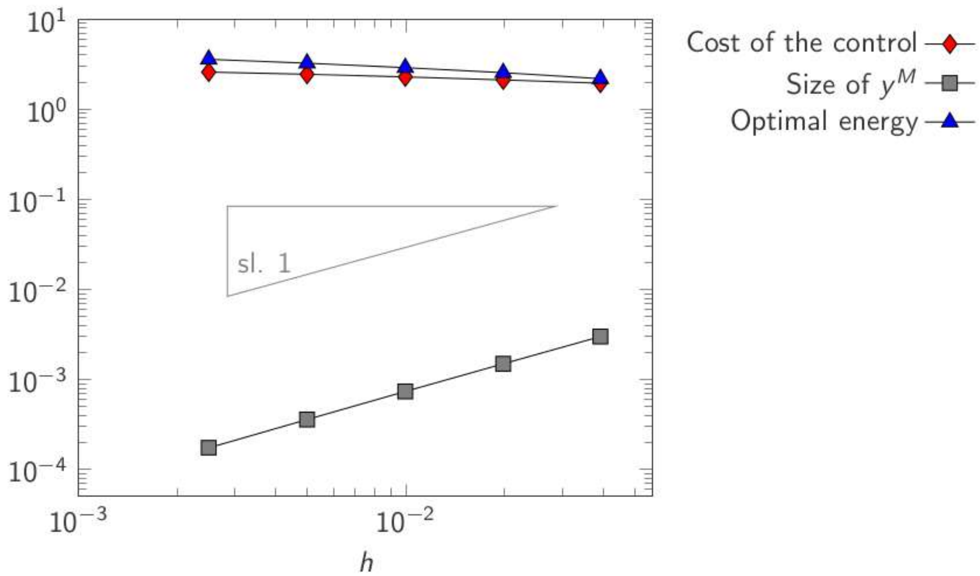
Controlled solution

- $T = 0.3s$
- $s = 0.8$
- $\omega = (-0.4, 0.8)$



Behavior of the penalized HUM

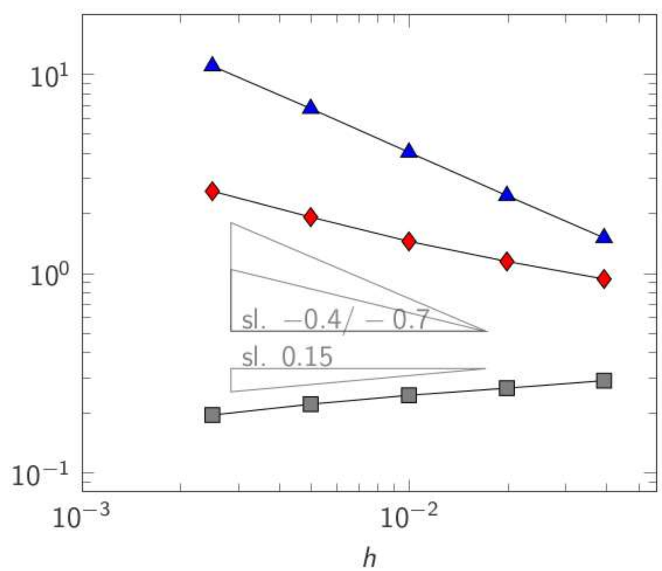
$s = 0.8$



- Cost of control and opt. energy are bounded as $h \to 0$.
- $|y(T)|_{L^2} \sim \sqrt{\varepsilon}$

NULL CONTROLLABLE

$s = 0.4$



- Cost of control and opt. energy are not bounded as $h \to 0$.
- $|y(T)|_{L^2} \sim Ch^{0.15}$

APPROXIMATELY CONTROLLABLE

Constrained controllability of the fractional heat equation

Constrained controllability

- The fractional heat equation $(\mathcal{FH}1)$ is null controllable in any time $T > 0$ by means of a control $g \in L^2(\omega \times (0, T))$, if and only if $s > 1/2$. Besides, the equation being linear, by translation the same result holds if the final target is a trajectory $\widehat{z}$.
- The fractional heat equation preserves positivity: if z_0 is a given non-negative initial datum in $L^2(-1, 1)$ and g is a non-negative function, then so it is for the solution z of $(\mathcal{FH}1)$.

Question

Can we control the fractional heat dynamics $(\mathcal{FH}1)$ from any initial datum $z_0 \in L^2(-1, 1)$ to any positive trajectory $\widehat{z}$, under positivity constraints on the control and/or the state?

Constrained controllability

Theorem (B., Warma and Zuazua, Comm. Pure Appl. Anal., 2020)

1. *Let $s > 1/2$. There exists a minimal time $T_{min} > 0$ such that the fractional heat equation $(\mathcal{FH}1)$ is* ***controllable to positive trajectories*** *at time $T > T_{min}$ through the action of a non-negative control $g \in L^\infty(\omega \times (0,T))$. Moreover, if $z_0 \geq 0$, we also have $z(x,t) \geq 0$ a.e. in $(-1,1) \times (0,T)$.*

2. *The controllability time T_{min} is strictly positive.*

3. *For $T = T_{min}$, controllability to trajectories holds with a non-negative control $g \in \mathcal{M}(\omega \times (0, T_{min}))$, the space of Radon measures on $\omega \times (0, T_{min})$.*

Idea of the proof

The proof is based on two main ingredients:

1. Controllability through L^∞ **controls**, consequence of the following observability inequality

$L^2 - L^1$ observability

$$\|p(\cdot,0)\|^2_{L^2(-1,1)} \leq C\left(\int_0^T\int_\omega |p(x,t)|\,dxdt\right)^2$$

2. **Dissipativity** of the fractional heat semi-group.

Some preliminary results

To prove the observability in $L^1(-1,1)$ we use:

Theorem

Let $\{\mu_k\}_{k\geq 1}$ be a sequence of real numbers such that $\sum_{k\geq 1}\mu_k^{-1} < +\infty$. Assume also that $\mu_{k+1}-\mu_k \geq \gamma$ for all $k \geq 1$, with $\gamma > 0$. Then, for any $T > 0$, there is a positive constant $\mathcal{C} = \mathcal{C}(T) > 0$ such that, for any finite sequence $\{c_k\}_{k\geq 1}$ it holds the inequality

$$\sum_{k\geq 1} |c_k|^2 e^{-2\mu_k T} \leq \mathcal{C} \left\| \sum_{k\geq 1} c_k e^{-\mu_k t} \right\|^2_{L^1(0,T)}.$$

Lemma

For any open subset $\omega \subset (-1,1)$, there is a positive constant $\beta > 0$ such that each eigenfunction of the fractional Laplacian satisfies $\|\phi_k\|_{L^1(\omega)} \geq \beta > 0$.

Numerical simulations

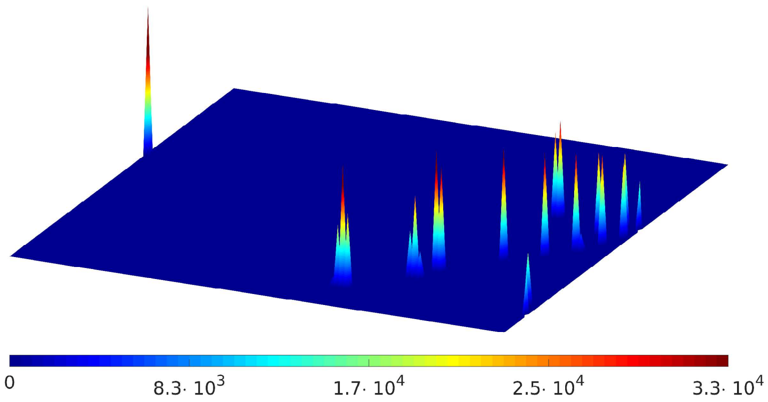
- We consider the problem of steering the initial datum $z_0(x) = \frac{1}{2}\cos\left(\frac{\pi}{2}x\right)$ to the target trajectory $\widehat{z}$ solution of $\mathcal{FH}1$ with initial datum $\widehat{z}_0(x) = 6\cos\left(\frac{\pi}{2}x\right)$ and right-hand side $\widehat{u} \equiv 1$.
- We choose $s = 0.8$ and $\omega = (-0.3, 0.8) \subset (-1, 1)$ as the control region.
- The approximation of the minimal controllability time is obtained by solving the following constrained minimization problem:

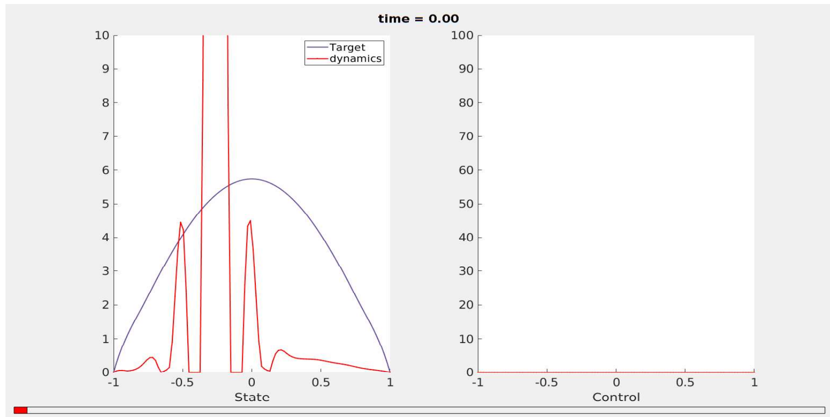
$$\text{minimize T}$$

$$\begin{cases} T > 0 \\ z_t + (-d_x^2)^s z = u\chi_\omega, & a.e.\ in\ (-1,1)\times(0,T) \\ z(\cdot,0) = z_0 \geq 0, & a.e.\ in\ (-1,1) \\ z \geq 0, & a.e.\ in\ (-1,1)\times(0,T) \\ u \geq 0, & a.e.\ in\ \omega\times(0,T). \end{cases}$$

Numerical simulations

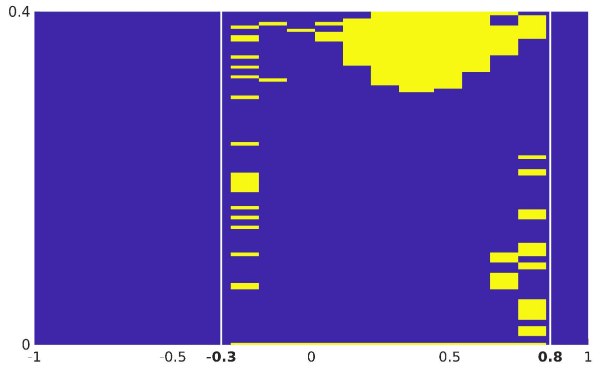
- We obtain the minimal time $T_{min} = 0,2101$
- In this time horizon, the equation is controllable from the initial datum z_0 to the desired trajectory $\widehat{z}(\cdot, T)$ by maintaining the positivity of the solution.

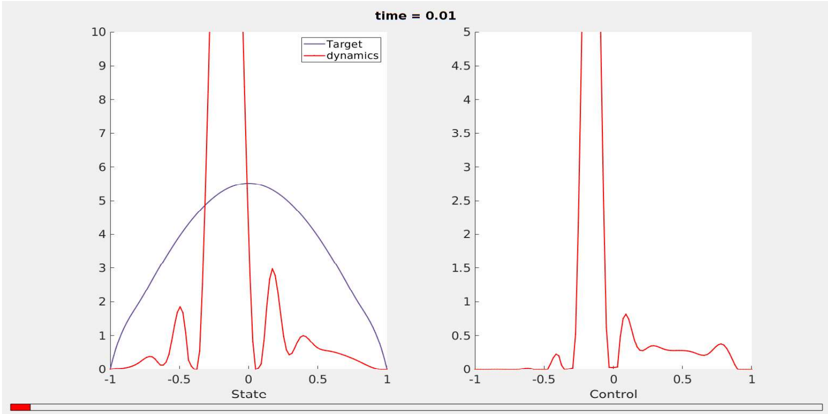




Numerical simulations

The impulsional behavior of the control is lost for a time horizon $T > T_{min}$.





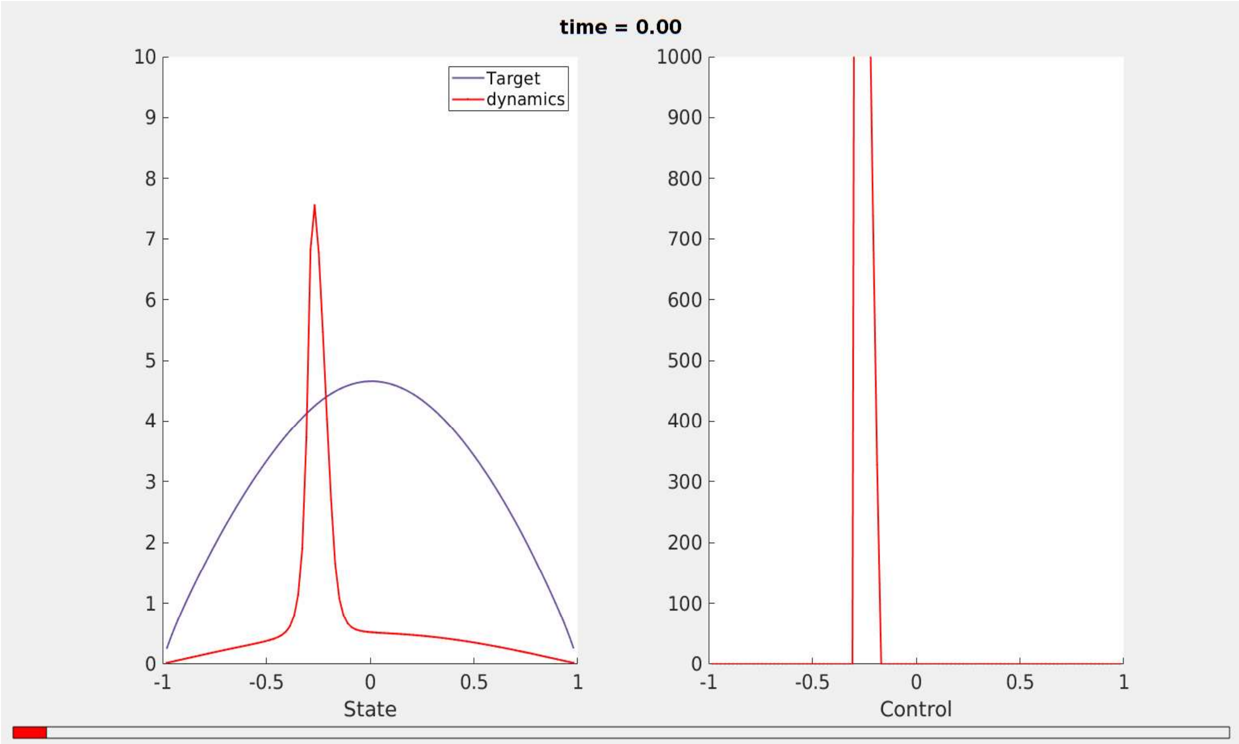
This control has been computed by solving the minimization problem:

$$\min \ \|z(\cdot, T) - \widehat{z}(\cdot, T)\|_{L^2(-1,1)}$$

$$\begin{cases} T > 0 \\ z_t + (-d_x^2)^s z = u\chi_\omega, & a.\,e.\ in\ (-1,1)\times(0,T) \\ z(\cdot,0) = z_0 \geq 0, & a.\,e.\ in\ (-1,1) \\ z \geq 0, & a.\,e.\ in\ (-1,1)\times(0,T) \\ u \geq 0, & a.\,e.\ in\ \omega\times(0,T). \end{cases}$$

Numerical simulations

When considering a time horizon $T < T_{min}$, constrained controllability fails.



Exterior controllability

Exterior controllability of the fractional heat equation

Fractional heat equation with exterior control

$$\begin{cases} z_t + (-d_x^2)^s z = 0 & (x,t) \in Q \\ z \equiv g\chi_\omega & (x,t) \in Q^c \\ z(\cdot, 0) = z_0 & x \in (-1,1) \end{cases} \qquad (\mathcal{FH}2)$$

Controllability results

Theorem (Warma and Zamorano, 2019)

The fractional heat equation $(\mathcal{FH}2)$ *is* ***null controllable*** *at time* $T > 0$ *if and only if* $s > 1/2$.

Theorem (Antil, B., Warma, Ponce and Zamorano, 2019)

1. *Let* $s > 1/2$. *There exists* $T_{min} > 0$ *such that the fractional heat equation* $(\mathcal{FH}2)$ *is* ***controllable to positive trajectories*** *at time* $T > T_{min}$ *through the action of a non-negative control* $g \in L^{\infty}(\omega \times (0,T)) \cap L^2(0,T; H_0^s(\omega))$. *Moreover, if* $z_0 \geq 0$, *we also have* $z(x,t) \geq 0$ *a.e. in* $(-1,1) \times (0,T)$.
2. *The controllability time* T_{min} *is strictly positive.*
3. *For* $T = T_{min}$, *controllability to trajectories holds with a non-negative control* $g \in \mathcal{M}(\omega \times (0,T_{min}))$, *the space of Radon measures on* $\omega \times (0,T_{min})$.

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