EMERGENT COLLECTIVE BEHAVIORS OF STOCHASTIC KURAMOTO OSCILLATORS

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Abstract. We study the collective dynamics of Kuramoto ensemble under uncertain coupling strength. For a finite ensemble, we can model the dynamics of the Kuramoto ensemble by the stochastic Kuramoto system with multiplicative noise. In contrast, for an infinite ensemble, the dynamics is effectively described by the Kuramoto-Sakaguchi-Fokker-Planck(KS-FP) equation with state dependent degenerate diffusion. We present emergent synchronization estimates for the stochastic and kinetic models, which yield the stability of the phase-locked state for identical Kuramoto ensemble with the same natural frequencies. We also provide a brief explanation on the mean-field limit between two models.

1. Introduction

Collective behaviors of complex systems have been widely investigated in literature, and one of such collective phenomenon, *synchronization* [2, 4, 19] represents "*adjustment of rhythms in a weakly coupled oscillators*". Among phenomenological synchronization models, our main interest lies on the Kuramoto model in [20]. The Kuramoto model has been extensively analyzed to find out the synchronization mechanism [17, 29, 30], effect of network structures [18, 29], and the coupling

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strength which induces the synchronization [6, 9, 13, 14]. In this paper, we are interested in the dynamics of the Kuramoto ensemble in a mesoscopic regime, i.e., we may assume that the number of Kuramoto oscillators is sufficiently large so that one-oscillator distribution function can be described effectively for the Kuramoto-Sakaguchi (KS) equation [1, 8, 18, 23, 27]. More precisely, let $F = F(t, \theta, \nu)$ be a one-oscillator distribution function of the Kuramoto ensemble at phase $\theta \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and natural frequency $\nu \in \mathbb{R}$ at time t. In the absence of noise, the dynamics of F is governed by the KS equation:

$$(1) \qquad \begin{cases} \partial_t F + \partial_\theta(V[F]F) = 0, \quad (\theta, \nu) \in \mathbb{T} \times \mathbb{R}, \ t > 0, \\ V[F](t, \theta, \nu) := \nu + \kappa \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta_* - \theta) F(t, \theta_*, \nu_*) d\theta_* d\nu_*, \\ F(0, \cdot, \cdot) = F_0. \end{cases}$$

In this paper, we are interested in the emergent dynamics of (1) under the effect of random coupling strength. When the system is stochastically perturbed, a diffusive term will be added to the equation (1), and it becomes the KS-FP equation.

In literature [2, 10], additive white noise has been widely used to perturb the Kuramoto model to see the noise effect on the synchronization phenomenon. In this case, while the Kuramoto interactions gather the ensemble in some area, diffusive effect of white noise spreads solutions to have smoother shape.

On the other hand, multiplicative noise is known to have less dissipation of solution, and generates collective behaviors. For the Cucker-Smale flocking model, the studies on the noise effects have shown that multiplicative noise can enhance the gathering of ensembles [15]. It is conceivable for multiplicative noise to make ensembles closer, since the states stay much more time in a small noise area. The staying time can be significant for the whole space, where unbounded noise on the unbounded region captures individuals into a bounded area [7]. In the Kuramoto model, it is well known [1, 9, 12] that the synchronization occurs when the coupling strength is large, and then the solution tends to a phase-locked state asymptotically. When the noise intensity vanish near stable states, it is expected that these states are stable under the effect of multiplicative noise.

Let $\theta_t^i \in \mathbb{T}$ be the phase process of the i-th oscillator at time t in the Kuramoto model, whose dynamics is governed by the following stochastic differential equation (SDE) with multiplicative noise:

$$(2) \quad d\theta_t^i = \Big[\nu^i + \frac{\kappa}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^i)\Big]dt + \frac{\sqrt{2\sigma}}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^i) dB_t^i, \quad 1 \le i \le N, \ t > 0.$$

Here, κ and $\sqrt{2\sigma}$ are nonnegative coupling strength and noise intensity, respectively, and dB_t^i is a one-dimensional white noise acting on i-th oscillator. On the other hand, we may consider a distribution function of the oscillators as in (1), through the mean-field limit $N \to \infty$. Then, the one-oscillator distribution function F is governed by the Kuramoto-Sakaguchi-Fokker-Planck (KS-FP) equation: for $(\theta, \nu) \in \mathbb{T} \times \mathbb{R}, \ t > 0$,

$$(3) \qquad \begin{cases} \partial_t F + \partial_\theta(V[F]F) = \sigma \partial_\theta^2 \Big[F \Big(\displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta - \theta_*) F(t, \theta_*, \nu_*) d\theta_* d\nu_* \Big)^2 \Big], \\ V[F](t, \theta, \nu) := \nu + \kappa \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta_* - \theta) F(t, \theta_*, \nu_*) d\theta_* d\nu_*. \end{cases}$$

Since oscillators move in a compact domain $\mathbb{T}$, the noise affects the collective behavior in a slight different way compared to the Cucker-Smale model. While the deterministic drift distinguish stable and unstable points, noise does not have any directional forces. From the previous studies [25, 26] on the Kuramoto model with different dynamics, it is shown that multiplicative noise can generate bifurcation by changing stabilities of points, and hence it weakens the synchronization effect.

The purpose of this paper is to study stochastic synchronization phenomena for the stochastic system (2) and kinetic model (3). We will briefly describe the relationship between (2) and (3), and then study the stability estimates on both models, under an assumption that all the oscillators have the same natural frequencies.

We first study the stochastic stability of the stochastic system (2) under the assumption of the common noise for the whole ensemble,

$$B_t^i = B_t^j =: B_t \quad \text{for all } i, j = 1, \cdots, N,$$

where an analogous idea was used for the Cucker-Smale model [3]. Under this condition, we proved a weak concept of stochastic stability: if phases are concentrated in a small interval, then with a high probability, phase configuration stays in a bounded region and asymptotically converges to a point (see Theorem 2.6).

Our second result shows that if the coupling strength κ is large enough compared to the noise scaling factor σ and the order parameter is initially nonzero, then one has

$$\lim_{t\to\infty} \int_{\mathbb{T}\times\mathbb{R}} F(t,\theta,\nu)\sin^2(\theta - \psi^\infty(t))d\theta d\nu = 0,$$

where ψ^∞ is the phase of the order parameter defined in (14) (see Theorem 2.7). Clearly, the above estimates imply that the phase density aggregates either ψ or $\psi + \pi$.

The rest of the paper is organized as follows. In Section 2, we briefly discuss two models (2) and (3) and present the main results on their emergent behavior. We also mention the regularity of (3) and the mean-field limit process from (2) to (3). In Section 3, we study the synchronization of the stochastic system (2) with the common noise condition. In Section 4, we present a synchronization estimate for the KS-FP equation. Finally, Section 5 is devoted to a brief summary and discussions for open problems.

2. Preliminaries and main results

In this section, we briefly discuss how the stochastic system (2) and its kinetic equation (3) can arise from the study of weakly coupled oscillators under uncertain coupling strength. The regularity of (3) and the mean-field limit process will be stated following the concepts of [21, 28]. After that, we will state main results of this paper.

2.1. A stochastic Kuramoto model.

Let θ_t^i be the phase of the i-th Kuramoto oscillators at time t. Then, the classical Kuramoto model is described by the following first-order consensus model:

$$\frac{d\theta_t^i}{dt} = \nu^i + \kappa \sum_{k=1}^N \sin(\theta_t^k - \theta_t^i), \quad t > 0, \quad i = 1, \cdots, N, \tag{4}$$

where ν^i and κ are constant. There are various ways to address randomness in the phase dynamics. Among them, we want to change the coupling strength κ into a time-dependent random variable $\kappa_i(t)$ in the equation of θ_t^i:

$$\kappa_i(t) = \frac{\kappa}{N} + \frac{\sqrt{2\sigma}}{N}\dot{B}_t^i, \quad t > 0, \quad i = 1, \cdots, N. \tag{5}$$

From (5), it formally follows from (4) that the phase process θ_t^i satisfies the stochastic Kuramoto model with a multiplicative noise:

$$d\theta_t^i = \Big[\nu^i + \frac{\kappa}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^i)\Big]dt + \frac{\sqrt{2\sigma}}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^i)dB_t^i, \quad t > 0, \quad 1 \le i \le N. \tag{6}$$

As in [3], in order to keep dynamical properties of (2), we assume the common noise $B_t = B_t^i$ for all i. Under this restricted condition, we can find a conserved quantity and rotational symmetry as in the following proposition. For phase and natural frequency vectors $\Theta_t := (\theta_t^1, \cdots, \theta_t^N)$ and $(\nu^1, \cdots, \nu^N)$, we set

$$\mathcal{C}_t := \sum_{j=1}^N \theta_t^j - t\sum_{j=1}^N \nu^j. \tag{7}$$

Proposition 1. *Let $\Theta_t := (\theta_t^1, \cdots, \theta_t^N)$ be a solution process of (6) with an extra identical assumption on B_t^i:*

$$B_t^i = B_t, \quad t > 0, \quad i,\ k = 1, \cdots, N.$$

Then, we have the following assertions:

(1) The process $\mathcal{C}_t$ is conserved:

$$\mathcal{C}_t = \mathcal{C}_0, \quad t > 0.$$

(2) For any C^1-function $\alpha = \alpha(t)$, the processes $\widetilde{\theta}_t^i := \theta_t^i + \alpha$ satisfy (6) with $\widetilde{\nu}^i := \nu^i - \dot{\alpha}$:

$$d\widetilde{\theta}_t^i = \Big[\widetilde{\nu}^i + \frac{\kappa}{N}\sum_{k=1}^N \sin(\widetilde{\theta}_t^k - \widetilde{\theta}_t^i)\Big]dt + \frac{\sqrt{2\sigma}}{N}\sum_{k=1}^N \sin(\widetilde{\theta}_t^k - \widetilde{\theta}_t^i)dB_t.$$

Proof. (i) We sum (6) over all i to get

$$d\sum_{i=1}^N \theta_t^i = \sum_{i=1}^N \nu^i dt, \quad \text{i.e.,} \quad \frac{d}{dt}\Big(\sum_{i=1}^N \theta_t^i - t\sum_{i=1}^N \nu^i\Big) = 0.$$

This yields the first assertion.

(ii) The translational invariance follows directly from (6). □

Next, we introduce representative random variables of the system, R_t^N and ψ_t^N, called *order parameters*. For a given phase vector $\Theta_t := (\theta_t^1, \cdots, \theta_t^N)$, define R_t^N and ψ_t^N as the modulus and phase of the centroid of N-unit vectors $e^{\mathrm{i}\theta_t^j}$:

$$R_t^N e^{\mathrm{i}\psi_t^N} := \frac{1}{N}\sum_{k=1}^N e^{\mathrm{i}\theta_t^k}. \tag{8}$$

From the definition of order parameters, we can further simplify the model (2) using order parameters. We divide the both sides of (8) by $e^{\mathrm{i}\theta_t^i}$ and compare real and imaginary parts of the resulting relation to get

$$\begin{aligned} R_t^N \cos(\psi_t^N - \theta_t^i) &= \frac{1}{N}\sum_{k=1}^N \cos(\theta_t^k - \theta_t^i), \\ R_t^N \sin(\psi_t^N - \theta_t^i) &= \frac{1}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^i). \end{aligned} \tag{9}$$

We may use (9) to rewrite system (2) as follows:

$$d\theta_t^i = \Big[\nu^i - \kappa R_t^N \sin(\theta_t^i - \psi_t^N)\Big]dt - \sqrt{2\sigma} R_t^N \sin(\theta_t^i - \psi_t^N) dB_t^i, \quad 1 \le i \le N,\ t > 0. \tag{10}$$

Note that the above system looks like a decoupled system where θ_t^i is only affected by the mean-field quantities R_t^N and ψ_t^N. This is why the system (6) is called a *"mean-field model"*, where each oscillator is affected by an averaged interaction.

2.2. **The KS-FP equation.** Next, we discuss the KS-FP equation (3) arising from the mean-field limit from the stochastic system (2). In order to erase the dependence of (2) on the index i, we additionally consider the dynamics of the constants $\nu_t^i \equiv \nu^i$ to distinghuish oscillators with different natural frequencies:

$$\begin{aligned} & d\theta_t^i = \Big[\nu_t^i + \frac{\kappa}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^i)\Big]dt + \frac{\sqrt{2\sigma}}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^i) dB_t^i, \\ & \frac{d}{dt}\nu_t^i = 0, \quad t > 0, \quad 1 \le i \le N. \end{aligned} \tag{11}$$

Let $F = F(t, \theta, \nu)$ be the density function for (11) at phase θ with a natural frequency ν, time t. For example, the solution $\{(\theta_t^i, \nu_t^i)\}_{i=1,\cdots,N}$ at time t can be represented by the empirical measure

$$F(t, \theta, \nu) = \frac{1}{N}\sum_{i=1}^N \delta(\theta - \theta_t^i) \otimes \delta(\nu - \nu_t^i). \tag{12}$$

Then, the standard BBGKY hierarchy argument [28] yields a kinetic equation of F as follows. We first adopt a formal assumption called *propagation of chaos*:

$$law(\theta_t^1, \nu_t^1, \cdots, \theta_t^k, \nu_t^k) = \prod_{i=1}^k F^N(t, \theta_t^i, \nu_t^i),$$

which implies that small number (k) of particles among the system of N particles are independent and identical, so that the total distribution on the left-hand side is a multiple of a representative one-particle distributions $F^N = law(\theta_t^1, \nu_t^1)$. Then, the equation for F^N can be derived from (11). From the formal limit ($N \to \infty$ and $k \to \infty$) on the equations of F^N, we get the KS-FP equation:

$$\begin{cases} \partial_t F + \partial_\theta (V[F]F) = \partial_\theta^2 (\mu[F]F), \quad (\theta, \nu) \in \mathbb{T} \times \mathbb{R},\ t > 0, \\ F(0, \cdot, \cdot) = F_0, \quad \displaystyle\int_{\mathbb{T}} F_0 d\theta = g(\nu), \\ \displaystyle\int_{\mathbb{T}\times\mathbb{R}} F_0(\theta, \nu) d\theta d\nu = 1, \quad F_0 \ge 0, \end{cases} \tag{13}$$

where the distribution of natural frequencies are given by $g(\nu)$ and the convective velocity $V[F]$ and degenerate diffusion coefficient $\mu(F)$ are given by the following relations:

$$V[F](t,\theta,\nu) := \nu + \kappa \int_{\mathbb{T}\times\mathbb{R}} \sin(\theta_* - \theta) F(t,\theta_*,\nu_*) d\theta_* d\nu_*,$$

$$\mu(F)(t,\theta,\nu) := \sigma \left(\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta_* - \theta) F(t,\theta_*,\nu_*) d\theta_* d\nu_* \right)^2, \quad (t,\theta,\nu) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}.$$

A rigorous analysis between two models (11) and (13) should suggest a convergence of F^N to F as $N \to \infty$. This will be discussed later in Section 2.3. Before we see this, we review the concept of a classical solution to (13).

Definition 2.1. Suppose that initial distribution F_0 is in $H^s(\mathbb{T}\times\mathbb{R})$, $s > \frac{1}{2}d + 3$. Then, $F = F(t,\theta,\nu)$ is a *classical solution* to (13) if the following conditions hold.

(1) $F \in C^1([0,T]; L^2(\mathbb{T}\times\mathbb{R}))$,
(2) $F(t,\cdot,\nu)$ is in $C^2(\mathbb{T})$ for all time t and a.e. on $\nu \in \mathbb{R}$,
(3) F satisfies the equation (13) pointwise.

Remark 1. Note that the equation (13) has no drift or diffusion on ν. This represents that there is no dynamics on ν in (11). Hence, in Definition 2.1, the regularity on ν is trivial and does not need to be stated further.

Proposition 2. *Suppose that initial datum F_0 is nonnegative, and has unit mass:*

$$\int_{\mathbb{T}\times\mathbb{R}} F_0(\theta,\nu) d\theta d\nu = 1 \quad \textit{and} \quad F_0 \geq 0,$$

and let F be a classical solution to (13) with the initial datum F_0. Then, we have

$$F(t,\cdot,\cdot) \geq 0 \ \textit{ for each } t \geq 0 \quad \textit{and} \quad \int_{\mathbb{T}\times\mathbb{R}} F(\theta,\nu) d\theta d\nu = 1.$$

Proof. Note that the equation (13) has a divergence form, hence the conservation of mass is obvious. □

As for the stochastic system (10) in the previous subsection, we may restate the equation (13) in terms of the mean-field quantities. Similar to the order parameters (8), we define real-valued functions $R^\infty(t)$ and $\psi^\infty(t)$ for (13):

$$R^\infty e^{\mathrm{i}\psi^\infty} := \int_{\mathbb{T}\times\mathbb{R}} F(t,\theta,\nu) e^{\mathrm{i}\theta} d\theta d\nu. \tag{14}$$

Again, a direct calculation shows that

$$R^\infty = \int_{\mathbb{T}\times\mathbb{R}} F \cos(\theta - \psi^\infty) d\theta d\nu \quad \text{and} \quad 0 = \int_{\mathbb{T}\times\mathbb{R}} F \sin(\theta - \psi^\infty) d\theta d\nu.$$

Then, we can rewrite (13) in a mean-field form:

$$\begin{cases} \partial_t F + \partial_\theta (V[F]F) = \partial_\theta^2 (\mu[F]F), \\ V[F](t,\theta,\nu)] = \nu - \kappa R^\infty \sin(\theta - \psi^\infty), \quad \mu[F](t,\theta) = \sigma (R^\infty)^2 \sin^2(\theta - \psi^\infty), \\ F(0,\cdot,\cdot) = F_0, \quad \displaystyle\int_{\mathbb{T}} F_0 d\theta = g(\nu), \quad \int_{\mathbb{T}\times\mathbb{R}} F_0 d\theta = 1, \quad F_0 \geq 0. \end{cases} \tag{15}$$

Note that the model (15) has a bounded and smooth vector field $V[F]$ from sinusoidal functions. Therefore, we may use a standard iteration method and a priori estimate on energies used in [15, 21].

Theorem 2.2. [15, 21] (Global existence of classical solution) *Suppose that initial datum F_0 is in L^1 as a function of θ and ν, and also in L^2 as a function of ν such that $F_0(\cdot,\nu) \in H^3(\mathbb{T})$:*

$$F_0 \in L^1(\mathbb{T}\times\mathbb{R}) \quad \text{and} \quad L^2(\mathbb{R}; H^3(\mathbb{T})).$$

Then, there exists the unique classical solution F to (15). In particular,

$$F \in C(0,\infty; L^2(\mathbb{R}; H^3(\mathbb{T}))).$$

2.3. **Stochastic mean-field limit.** In order to discuss the relationship between the stochastic system (11) and KS-FP equation (13), we need to compare solutions from these two different models. In literature [5, 6, 15, 21], this comparison procedure has been done with various approaches, where there are two main obstacles. First, the initial data on a stochastic system is given by random variables on a sample space, while that of a kinetic equation is a distribution function, hence we need a transformation between two data in different systems. Second, we should suggest a common model containing two systems in order to measure the distance between two data along the timeline. In this section, we present the mean-field limit theory in detail to supplement explanations in [5, 15], which uses the Sznitman's arguments [28].

For a deterministic model, for example, if $\sigma = 0$ in (11) and (13), an empirical measure (12) deduces (11) from (13). Then, the dynamics of finite oscillators can be interpreted as a solution of the kinetic model, and hence, it only remains to define a proper distance between two measure-valued solutions. However, this is not true for the stochastic systems. Between a stochastic system and a Fokker-Planck equation, McKean suggested an intermediate model called *Vlasov-McKean process* for the Vlasov equation under some regularity conditions. In the same way, we may define Kuramoto-McKean process $(\bar\theta^i_t, \bar\nu^i_t)$:

$$\begin{cases} d\bar\theta^i_t = (\bar\nu^i_t - \kappa R^\infty_t \sin(\bar\theta^i_t - \psi^\infty_t))dt - \sqrt{2\sigma}R^\infty_t \sin(\bar\theta^i_t - \psi^\infty_t)dB^i_t, \\ d\bar\nu^i_t = 0, \quad t>0,\ i=1,\cdots,N, \\ (\bar\theta^i_0, \bar\nu^i_0) = (\theta^i_0, \nu^i_0), \end{cases} \tag{16}$$

where the order parameters R^∞ and ψ^∞ are given in (14) from the distribution $F(t) = F(t,\cdot,\cdot)$. Then, from the following lemma, the law of $(\bar\theta^i_t, \bar\nu^i_t)$ is exactly the same as $F(t)$ for every t when B^i_t of (11) are independent and identically distributed Brownian motions.

Proposition 3. *Let $(\bar\theta^i_t, \bar\nu^i_t)$ be a solution of (16) with the initial data $(\bar\theta^i_0, \bar\nu^i_0)$ satisfying law$(\bar\theta^i_0, \bar\nu^i_0) = F_0$ for all i, where F_0 and $F(t)$ is given from (15). Then, the law of $(\bar\theta^i_t, \bar\nu^i_t)$ is $F(t)$.*

Proof. Let μ_t be the law of $(\bar\theta^i_t, \bar\nu^i_t)$ for a fixed i. Then, it suffices to show that

$$d\mu_t = F(t,\theta,\nu)d\theta d\nu, \quad t \ge 0. \tag{17}$$

Let h be a smooth test function on $\mathbb{T}\times\mathbb{R}$, and F be the classical solution of the KS-FP equation (15).

- (Dynamics of $h(\bar\theta^i_t, \nu^i_t)$): It follows from Itô's formula that

$$\begin{aligned} dh(\bar\theta^i_t, \nu^i_t) &= \Big[(\nu^i_t - \kappa R^\infty_t \sin(\bar\theta^i_t - \psi^\infty_t))\partial_\theta h + \sigma (R^\infty_t)^2 \sin^2(\bar\theta^i_t - \psi^\infty_t)\partial^2_\theta h\Big] dt \\ &\quad - \sqrt{2\sigma} R^\infty_t \sin(\bar\theta^i_t - \psi^\infty_t)\partial_\theta h dB^i_t. \end{aligned} \tag{18}$$

We integrate the relation (18) in t and take expectation of the resulting relation to get

$$\frac{d}{dt}\mathbb{E}[h(\bar\theta_t^i,\nu_t^i)] = \mathbb{E}\Big[(\nu_t^i - \kappa R_t^\infty \sin(\bar\theta_t^i - \psi_t^\infty))\partial_\theta h + \sigma(R_t^\infty)^2\sin^2(\bar\theta_t^i-\psi_t^\infty)\partial_\theta^2 h\Big],$$

or equivalently,

$$\begin{aligned}&\frac{d}{dt}\int h(\bar\theta_t^i,\nu_t^i)d\mu_t\\ &\quad= \int \partial_\theta h(\nu_t^i - \kappa R_t^\infty \sin(\bar\theta_t^i - \psi_t^\infty))d\mu_t + \int \sigma(R_t^\infty)^2\sin^2(\bar\theta_t^i-\psi_t^\infty)\partial_\theta^2 h d\mu_t.\end{aligned}$$

Since h was arbitrary, the law μ_t is a weak solution of the following equation on G:

$$\begin{cases}\partial_t G + \partial_\theta(V[F]G) = \sigma\partial_\theta^2\Big((R_t^\infty)^2\sin^2(\theta-\psi_t^\infty)G\Big),\\ V[F](t,\theta,\nu) = \nu - \kappa R_t^\infty\sin(\theta-\psi_t^\infty),\end{cases}\tag{19}$$

where F is given by (13). This is a linear equation on G, which has the unique weak solution (we may use energy estimates as in Theorem 2.2), and the initial datum is F_0. From (13) with the same initial condition F_0, F is also a solution of (19). Hence, one has the desired equality (17). □

Therefore, instead of interpreting a system of stochastic equations into a Fokker-Planck equation, we add a system of stochastic equation (16) to the Fokker-Planck equation which represents oscillators. Then, we may use these oscillator representations for the comparison of F^N and F as follows.

Definition 2.3. Let x and y be random variables on $\mathbb{T}\times\mathbb{R}$ with probability laws μ_1 and μ_2, respectively. Then, the Wasserstein metric $W_2(\mu_1,\mu_2)$ between μ_1 and μ_2 is defined as follows:

$$W_2(\mu_1,\mu_2) := \inf_{(x,y)}(\mathbb{E}[\tilde d(x,y)^2])^{\frac12},$$

where $\tilde d(\cdot,\cdot)$ is a distance function on $\mathbb{T}\times\mathbb{R}$ and the infimum runs over all random variables x and y.

Remark 2. For a fixed t, let (θ_t^i,ν_t^i) and $(\bar\theta_t^i,\bar\nu_t^i)$ be solutions of (11) and (16), where their corresponding distributions are F^N and F, respectively. Then, we have

$$W_2(F^N,F)^2 \le \mathbb{E}[d(\theta_t^i,\bar\theta_t^i)^2],$$

for a distance function $d(\cdot,\cdot)$ in $\mathbb{T}$.

Finally, from Remark 2, direct calculations on θ_t^i and $\bar\theta_t^i$ lead to the mean-field limit from (11) to (13). In this process, we need to use boundedness and smoothness on the drift and noise strength in the system (11).

Theorem 2.4. [28] *(Finite-time mean-field limit) The mean-field limit from (11) to (13) holds in a finite time interval when we have independent and identically distributed Brownian motions B_t^i. In particular, we have the mean-field limit in the following sense:*

$$\lim_{N\to\infty}\sup_{0\le t\le T} W_2(F, law(\theta_t^1,\nu_t^1)) = 0,$$

where T is a finite constant.

Remark 3. In [3], the stochastic noise is not independent on each phase. For such cases, we cannot guarantee the finite-time mean-field limit since the limit equation is still a stochastic equation and the variance will not vanish. However, we still have identically distributed $law(\theta_t^i, \nu_t^i)$ when the initial data satisfy

$$law(\theta_0^i, \nu_0^i) = F_0,$$

where F_0 is the initial data of F in (3). In this sense, we can compare F and $F^N := law(\theta_t^1, \nu_t^1)$ to prove the mean-field limit. In [15], they presented that an asymptotic-type mean-field limit holds for the whole time interval, by clarifying emergent behaviors of these two distributions F and F^N have the same characters.

2.4. **Presentation of main results.** In this subsection, we briefly summarize our main results on the stochastic stability of phase-locked states. From the idea of Remark 3, we focused on the emergent behaviors of the stochastic particle system (2) and the KS-FP equation (3) separately.

2.4.1. *Stochastic particle system.* Consider the situation where all random sources B_t^i are uniform over all i and natural frequencies are all zeros:

$$B_t^i = B_t, \quad \nu^i = 0, \quad 1 \le i \le N.$$

Hence, (2) becomes

$$d\theta_t^i = \frac{\kappa}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^i)dt + \frac{\sqrt{2\sigma}}{N}\sum_{k=1}^N \sin(\theta_t^k - \theta_t^i)dB_t, \quad i = 1, \cdots, N. \tag{20}$$

As in the deterministic model ($\sigma = 0$), the accumulation of phases at a constant point,

$$\theta_t^i \equiv \theta_\infty \in \mathbb{R}, \quad i = 1, \cdots, N,$$

is clearly a solution to (20). In the following theorem, we study good properties of the system (20).

Theorem 2.5. *Let $\Theta_t = (\theta_t^1, \cdots, \theta_t^N) \in \mathbb{R}^N$ be a solution of (20) with a bounded initial datum. Then, the phase process Θ_t is uniformly bounded along time,*

$$\sup_{0 \le t \le \infty} |\theta_t^i| < \infty, \quad i = 1, \cdots, N.$$

Moreover, if $\kappa > 2\sigma$, we have

$$\mathbb{E}[(R_t)^2] \ge \mathbb{E}[(R_0)^2], \quad t \ge 0.$$

For a detailed behavior of phases, we introduce two functionals $D(\Theta_t)$ and $V(\Theta_t)$:

$$D(\Theta_t) := \max_{i,j} |\theta_t^i - \theta_t^j| \quad \text{and} \quad V(\Theta_t) := \max_{1 \le i,j \le N} 2\sin^2\left(\frac{\theta_t^i - \theta_t^j}{2}\right).$$

In addition, for $D_\infty \in (0, \pi/2)$ and $\varepsilon \in \left(0, \frac{1}{N}\right)$, we define $D_0 = D_0(D_\infty, \varepsilon, \sigma)$ as a solution to the following trigonometric equation:

$$\sin\frac{x}{2} := \frac{\varepsilon^{2\sigma}}{2\sqrt{e}}\sin\frac{D_\infty}{2}, \quad x \in (0, \pi). \tag{21}$$

Theorem 2.6. *Suppose that parameters and the initial data satisfy*

$$0 < D_\infty < \frac{\pi}{2}, \quad 0 < \varepsilon < \frac{1}{N}, \quad \max_{1\le i,j\le N} |\theta_0^i - \theta_0^j| < D_0(D_\infty, \varepsilon, \sigma),$$

and let Θ_t be a solution process of (20) with initial random variable Θ_0. Then, one has a stochastic stability:

$$\mathbb{P}\left\{\Theta_t \in \mathbb{T}^N : \sup_{0\le t<\infty} D(\Theta_t) < D_\infty \quad \text{and} \quad \limsup_{t\to\infty} V(\Theta_t) = 0\right\} \ge 1 - N\varepsilon.$$

Proof. The proof will be given in Section 3.3. □

Remark 4. Note that the above theorem has two restrictions: the initial data is close enough to the one-point distribution, and the probability of synchronization is not 1. These limitations are due to the linear approximation. The multiplicative noise guarantees that an oscillator θ_t^i cannot exceed the values $\psi_t^N + \pi$ or $\psi_t^N - \pi$. However, the Lyapunov functional, $2\sin^2((\theta_t^i - \theta^j)/2)$, is monotonically decreasing only if all oscillators gather within a quarter of the circle. We will see detailed arguments in Section 3.

As a direct application of Theorem 2.6, we have the following probabilistic estimate for the convergence of average phase ψ_t^N as follows.

Corollary 1. *Under the same assumptions of Theorem 2.2, one has*

$$\mathbb{P}\{\Theta_t \in \mathbb{T}^N : \exists\ \psi_\infty^N := \lim_{t\to\infty} \psi_t^N\} \ge 1 - N\varepsilon.$$

2.4.2. *The KS-FP equation.* For the KS-FP equation dealing with infinitely many oscillators, we also assume that the natural frequencies are all zero. Under this condition, we derive the following synchronization estimate:

Theorem 2.7. *Suppose that the coupling strength, diffusion coefficient and initial datum satisfy*

$$\kappa > \sigma > 0, \quad R_0^\infty > 0, \quad F_0(\theta, \nu) = f_0(\theta)\delta_\nu(0),$$

and let F be a solution to (15). Then, R_t^∞ does not decrease and we have an emergent behavior of F,

$$\lim_{t\to\infty} \int_{\mathbb{T}\times\mathbb{R}} F \sin^2(\theta - \psi_t^\infty) d\theta d\nu = 0.$$

Proof. The proof will be given in Section 4. □

Remark 5. The result of Theorem 2.7 implies that, if the coupling strength is bigger than the noise strength, the distribution can aggregate at most two points (ψ_t^∞ and $\psi_t^\infty + \pi_t$) as time elapses. However, we can not guarantee the convergence of ψ_t^∞ under the effect of multiplicative noise, though one can see that the time derivative of ψ_t^∞ converges to zero.

3. Stochastic aggregation of identical Kuramoto oscillators

In this section, we study a stochastic aggregation estimate for the identical Kuramoto ensemble which provides a proof of Theorem 2.2.

3.1. **Uniform boundedness of phase diameter.** In order to see the emergent behavior, we will study the dynamics of the phase diameter. For deterministic case, $\sigma = 0$, the boundedness of phase diameter yields the complete synchronization of the deterministic Kuramoto model due to the gradient flow structure (See [12] for a detailed discussion). In this section, we are interested in the Kuramoto ensemble with the same natural frequencies. With common random source B_t, system (2) has translational symmetry. Therefore, without loss of generality, we may assume that

$$\nu^i = 0, \quad 1 \le i \le N.$$

For notational simplicity, from now on, we suppress N and t dependence in order parameters R_t^N and ψ_t^N:

$$R := R_t := R_t^N, \quad \psi := \psi_t := \psi_t^N.$$

From (6) and (10), the phase process θ_t^i satisfies

$$d\theta_t^i = -\kappa R \sin(\theta_t^i - \psi)dt - \sqrt{2\sigma} R \sin(\theta_t^i - \psi)dB_t, \quad t > 0,\ 1 \le i \le N. \tag{22}$$

Lemma 3.1. *Let Θ_t be a solution to (22). Then, the relative phase difference $\theta_t^i - \theta_t^j$ satisfies*

$$\begin{aligned} d(\theta_t^i - \theta_t^j) &= -2\kappa R \cos\Big(\frac{\theta_t^i + \theta_t^j}{2} - \psi\Big) \sin\Big(\frac{\theta_t^i - \theta_t^j}{2}\Big)dt \\ &\quad - 2\sqrt{2\sigma} R \cos\Big(\frac{\theta_t^i + \theta_t^j}{2} - \psi\Big) \sin\Big(\frac{\theta_t^i - \theta_t^j}{2}\Big)dB_t. \end{aligned} \tag{23}$$

Proof. Note that phase processes θ_t^i and θ_t^j satisfy

$$\begin{aligned} d\theta_t^i &= -\kappa R \sin(\theta_t^i - \psi)dt - \sqrt{2\sigma} R \sin(\theta_t^i - \psi)dB_t, \\ d\theta_t^j &= -\kappa R \sin(\theta_t^j - \psi)dt - \sqrt{2\sigma} R \sin(\theta_t^j - \psi)dB_t,\ t > 0. \end{aligned}$$

Then, one has

$$\begin{aligned} &d(\theta_t^i - \theta_t^j) \\ &= -\kappa R\Big(\sin(\theta_t^i - \psi) - \sin(\theta_t^j - \psi)\Big)dt - \sqrt{2\sigma} R\Big(\sin(\theta_t^i - \psi) - \sin(\theta_t^j - \psi)\Big)dB_t. \end{aligned}$$

Now we use the above relation and the identity

$$\sin(\theta_t^i - \psi) - \sin(\theta_t^j - \psi) = 2\cos\Big(\frac{\theta_t^i + \theta_t^j}{2} - \psi\Big) \sin\Big(\frac{\theta_t^i - \theta_t^j}{2}\Big)$$

to get the desired result. □

Remark 6. Note that two phase processes θ_t^i and θ_t^j are influenced by the same random source. This fact is crucially used in the stochastic stability of relative phases [22].

Next, we consider the uniform boundedness of solution processes. Since we consider the periodic domain $\mathbb{T}$, the processes θ_t^i are trivially bounded in the sense of $\mathbb{T}$. However, we may lift the system (2) on $\mathbb{T}^N$ to the system (2) on $\mathbb{R}^N$ and make their initial distribution to be bounded, for example, within $[0, 2\pi]$. From this point of view, we can measure how many rotations they have. The following lemma suggests that the phases rotate only a finite number of times from the dynamics of relative phases, $\theta_t^i - \theta_t^j$.

Lemma 3.2. *Let $\Theta_t \in \mathbb{R}^N$ be phase processes of (22) with bounded initial phases Θ_0. Then, the phase process Θ_t is uniformly bounded:*

$$\sup_{0\le t<\infty} |\theta_t^i| < \infty, \quad i = 1, \cdots, N.$$

Proof. Suppose that $\bar{n}$ is a positive integer such that

$$\max_{i,j} |\theta_0^i - \theta_0^j| < 2\bar{n}\pi. \tag{24}$$

Note that for each i and j, we may use the equation (23). Let the relative phase

$$\theta_t^{ij} := \theta_t^i - \theta_t^j.$$

Then, from (23), the Itô process θ_t^{ij} satisfies that

$$d(\theta_t^{ij}) = 0, \quad \text{whenever} \quad \theta_t^{ij} = 2n\pi \quad \text{for some } n \in \mathbb{Z}. \tag{25}$$

From this property and the initial assumption (24), we claim:

$$|\theta_t^{ij}| \le 2\bar{n}\pi.$$

We use the uniqueness of the solution as in the deterministic differential equations. We define the following stopping time for each i and j,

$$\tau_{ij} := \inf\{t > 0 : \theta_t^{ij} > 2\bar{n}\pi\}.$$

Then, we have

$$\theta_{\tau_{ij}}^{ij} = 2\bar{n}\pi \quad \text{if } \tau_{ij} < \infty, \quad \text{and} \quad \theta_{\tau_{ij}}^{ij} < 2\bar{n}\pi \quad \text{if } \tau_{ij} = \infty.$$

Note that from the time τ_{ij}, (25) guarantees that the process

$$\theta_t^{ij} = 2\bar{n}\pi, \quad t \ge \tau_{ij}$$

satisfies the dynamics (23) for $t \ge \tau_{ij}$. Hence the process $\theta_{t\wedge\tau_{ij}}^{ij}$ is also a solution of (23). From the uniqueness of Itô process, this is the only solution, so that we have, almost surely,

$$\theta_t^{ij} \le 2\bar{n}\pi, \quad t \ge 0. \tag{26}$$

We may prove the lower bound in the same way.

On the other hand, it follows from Proposition 1 that

$$\frac{1}{N}\sum_{i=1}^N \theta_t^i = \frac{1}{N}\sum_{i=1}^N \theta_0^i, \quad t > 0. \tag{27}$$

We now use (26) and (27) to get

$$\left|\theta_t^i - \frac{1}{N}\sum_{j=1}^N \theta_0^j\right| = \left|\theta_t^i - \frac{1}{N}\sum_{j=1}^N \theta_t^j\right| = \left|\frac{1}{N}\sum_{j=1}^N (\theta_t^i - \theta_t^j)\right| \le 2\bar{n}\pi.$$

Thisimplies the uniform boundedness for θ_t^i.

□

Remark 7. The result of Lemma 3.2 implies that the rotation numbers,

$$\rho_i := \lim_{t\to\infty} \frac{\theta_t^i}{t}, \quad i = 1, \cdots, N,$$

are identically zero. The coincidence in rotation numbers gives evidence of frequency synchronization among θ_t^i.

3.2. **Dynamics of order parameters.** In this part, we derive a dynamical system for R and ψ.

Lemma 3.3. *Let Θ_t be a solution to (22). Then the order parameters R and ψ satisfy*

$$\begin{aligned} dR = \Big[& \frac{\kappa R}{N}\sum_{i=1}^N \sin^2(\theta_t^i-\psi) - \frac{\sigma R^2}{N}\sum_{i=1}^N \sin^2(\theta_t^i-\psi)\cos(\theta_t^i-\psi) \\ & + \frac{\sigma R}{N^2}\sum_{i=1}^N\sum_{j=1}^N \sin(\theta_t^i-\psi)\sin(\theta_t^j-\psi)\cos(\theta_t^i-\psi)\cos(\theta_t^j-\psi)\Big]dt \\ & - \frac{\sqrt{2\sigma}R_t}{N}\sum_{i=1}^N \sin^2(\theta_t^i-\psi)dB_t \end{aligned}$$

and

$$\begin{aligned} d\psi = \Big[& -\frac{\kappa}{N}\sum_{i=1}^N \sin(\theta_t^i-\psi)\cos(\theta_t^i-\psi) - \frac{\sigma R_t}{N}\sum_{i=1}^N \sin^3(\theta_t^i-\psi) \\ & + \frac{\sigma}{N^2}\sum_{i=1}^N\sum_{j=1}^N \sin(\theta_t^i-\psi)\sin(\theta_t^j-\psi)\sin(\theta_t^i+\theta_t^j-2\psi)\Big]dt \\ & + \frac{\sqrt{2\sigma}}{N}\sum_{i=1}^N \sin(\theta_t^i-\psi)\cos(\theta_t^i-\psi)dB_t. \end{aligned}$$

Proof. For notational simplicity, in the sequel we use handy notation:

$$\theta^i := \theta_t^i \quad \text{and} \quad \partial_j := \partial_{\theta^j}.$$

Then, it follows from Itô's formula for R and ψ that we have

$$\begin{aligned} dR &= \sum_{i=1}^N \partial_i R d\theta^i + \frac{1}{2}\sum_{i,j=1}^N \partial_i\partial_j R d\theta^i d\theta^j, \\ d\psi &= \sum_{i=1}^N \partial_i \psi d\theta^i + \frac{1}{2}\sum_{i,j=1}^N \partial_i\partial_j \psi d\theta^i d\theta^j. \end{aligned} \tag{28}$$

Thus, we need the quantities $\partial_i R$, $\partial_i\psi$, $\partial_i\partial_j R$ and $\partial_i\partial_j\psi$ below.

• Step A (Derivations of $\partial_j R$ and $\partial_j\psi$): Recall that R and ψ satisfy

$$R = \frac{1}{N}\sum_{i=1}^N \cos(\theta^i-\psi), \quad 0 = \frac{1}{N}\sum_{i=1}^N \sin(\theta^i-\psi). \tag{29}$$

Then, we differentiate $(29)_2$ with respect to θ^j to get

$$\begin{aligned} 0 &= \frac{1}{N}\sum_{i=1}^N \cos(\theta^i-\psi)(\delta_{ji}-\partial_j\psi) \\ &= -\frac{1}{N}\Big(\sum_{i=1}^N \cos(\theta^i-\psi)\Big)\partial_j\psi + \frac{1}{N}\cos(\theta^j-\psi) = -R\partial_j\psi + \frac{1}{N}\cos(\theta^j-\psi). \end{aligned}$$

This yields

$$\partial_j \psi = \frac{1}{RN} \cos(\theta^j - \psi). \tag{30}$$

Similarly, we differentiate $(29)_1$ with respect to θ^j and use $(29)_2$ to get

$$\begin{aligned} \partial_j R &= -\frac{1}{N} \sum_{i=1}^N \sin(\theta^i - \psi)(\delta_{ji} - \partial_j \psi) \\ &= \Big[\frac{1}{N} \sum_{i=1}^N \sin(\theta^i - \psi) \Big] \partial_j \psi - \frac{1}{N} \sin(\theta^j - \psi) = -\frac{1}{N} \sin(\theta^j - \psi), \end{aligned} \tag{31}$$

where δ_{ij} is the Kronecker delta.

• Step B (Derivations of $\partial_i \partial_j R$ and $\partial_i \partial_j \psi$): We use (30) to find

$$\partial_i \partial_j R = -\frac{1}{N} \cos(\theta^j - \psi)(\delta_{ij} - \partial_i \psi) = -\frac{1}{N} \cos(\theta_t^j - \psi) \Big(\delta_{ij} - \frac{1}{RN} \cos(\theta^i - \psi) \Big).$$

This yields

$$\begin{aligned} \partial_j^2 R &= -\frac{1}{N} \cos(\theta^j - \psi) + \frac{1}{RN^2} \cos^2(\theta^j - \psi), \\ \partial_i \partial_j R &= \frac{1}{RN^2} \cos(\theta^i - \psi) \cos(\theta^j - \psi), \quad \text{for} \quad i \neq j. \end{aligned} \tag{32}$$

Similarly, we differentiate (30) with respect to θ^i using (30) to get

$$\begin{aligned} \partial_i \partial_j \psi &= -\frac{1}{RN} \sin(\theta^j - \psi)(\delta_{ij} - \partial_i \psi) - \frac{1}{R^2 N} \partial_i R \cos(\theta^j - \psi) \\ &= -\frac{1}{RN} \sin(\theta^j - \psi) \Big(\delta_{ij} - \frac{1}{RN} \cos(\theta^i - \psi) \Big) + \frac{1}{R^2 N^2} \sin(\theta^i - \psi) \cos(\theta^j - \psi). \end{aligned}$$

This implies

$$\begin{aligned} \partial_j^2 \psi &= -\frac{1}{RN} \sin(\theta^j - \psi) + \frac{1}{R^2 N^2} \sin(2(\theta^j - \psi)), \\ \partial_i \partial_j \psi &= \frac{1}{R^2 N^2} \sin(\theta^i + \theta^j - 2\psi), \quad \text{for} \quad i \neq j. \end{aligned}$$

• Step C (Derivations of dR and $d\psi$): We use (22) to see

$$d\theta^i d\theta^j = 2\sigma R^2 \sin(\theta^i - \psi) \sin(\theta^j - \psi) dt. \tag{33}$$

In (28), we use (22), (31), (32) and (33) to obtain the desired estimate. Similarly, we get the relation for $d\psi$. □

Remark 8. It is well known that the deterministic Kuramoto model with $\sigma = 0$ can be written as a gradient flow with potential $V(\Theta) = -\frac{\kappa |R(\Theta)|^2}{2}$:

$$\dot{\Theta} = -\nabla_\Theta V(\Theta).$$

This implies that $|R(\Theta)|^2$ is nondecreasing.

From these equations, we have basic synchronization properties.

Lemma 3.4. *Suppose that $\kappa > 2\sigma$, and let Θ_t be a solution to (22). Then, we have*

$$\mathbb{E}[R_t^2] \geq \mathbb{E}[R_0^2], \quad t \geq 0.$$

Proof. First, note that the equation of R in Lemma 3.3 yields

$$dR_t dR_t = \frac{2\sigma R_t^2}{N^2}\sum_{i,j=1}^N \sin^2(\theta_t^i-\psi_t)\sin^2(\theta_t^j-\psi_t)dt.$$

Then, it follows from Itô's formula that

$$\begin{aligned}
dR_t^2 &= 2R_t dR_t + dR_t dR_t \\
&= \Big[\frac{2\kappa R_t^2}{N}\sum_{i=1}^N \sin^2(\theta_t^i-\psi_t) - \underbrace{\frac{2\sigma R_t^3}{N}\sum_{i=1}^N \sin^2(\theta_t^i-\psi_t)\cos(\theta_t^i-\psi_t)}_{=:\mathcal{I}_{11}} \\
&\quad + \underbrace{\frac{2\sigma R_t^2}{N^2}\sum_{i=1}^N\sum_{j=1}^N \sin(\theta_t^i-\psi_t)\sin(\theta_t^j-\psi_t)\cos(\theta_t^i-\psi_t)\cos(\theta_t^j-\psi_t)}_{=:\mathcal{I}_{12}} \\
&\quad + \frac{2\sigma R_t^2}{N^2}\sum_{i=1}^N\sum_{j=1}^N \sin^2(\theta_t^i-\psi_t)\sin^2(\theta_t^j-\psi_t)\Big]dt \\
&\quad - \frac{2\sqrt{2\sigma}}{N}R_t^2\sum_{i=1}^N \sin^2(\theta_t^i-\psi_t)dB_t.
\end{aligned} \tag{34}$$

• (Estimate of $\mathcal{I}_{11}$): We use $R_t \le 1$ and $|\cos(\theta_t^i-\psi_t)|\le 1$ to see

$$|\mathcal{I}_{11}| \le \frac{2\sigma R_t^2}{N}\sum_{i=1}^N \sin^2(\theta_t^i-\psi_t). \tag{35}$$

• (Estimate of $\mathcal{I}_{12}$): We use $|\cos(\theta_t^i-\psi_t)\cos(\theta_t^j-\psi_t)|\le 1$ and

$$|\sin(\theta_t^i-\psi_t)\sin(\theta_t^j-\psi_t)| \le \frac{1}{2}\Big(\sin^2(\theta_t^i-\psi_t)+\sin^2(\theta_t^j-\psi_t)\Big),$$

to get

$$|\mathcal{I}_{12}| \le \frac{2\sigma R_t^2}{N}\sum_{i=1}^N \sin^2(\theta_t^i-\psi_t). \tag{36}$$

In (34), we combine all estimates (35) and (36) to obtain

$$\begin{aligned}
dR_t^2 &\ge \Big[\frac{2(\kappa-2\sigma)R_t^2}{N}\sum_{i=1}^N \sin^2(\theta_t^i-\psi_t) + \frac{2\sigma R_t^2}{N^2}\sum_{i=1}^N\sum_{j=1}^N \sin^2(\theta_t^i-\psi_t)\sin^2(\theta_t^j-\psi_t)\Big]dt \\
&\quad - \frac{2\sqrt{2\sigma}}{N}R_t^2\sum_{i=1}^N \sin^2(\theta_t^i-\psi_t)dB_t.
\end{aligned}$$

We need a generalized comparison lemma [11] to realize the above inequality. Since the processes ψ_t and θ_t^i are all predictable, we may compare R_t^2 with other processes. We integrate the above relation using the nonnegativity of the coefficients in the drift term to get

$$R_t^2 \ge R_0^2 - \frac{2\sqrt{2\sigma}}{N}\sum_{i=1}^N\int_0^t R_s^2\sin^2(\theta_s^i-\psi_s)dB_s. \tag{37}$$

We now take expectations to both sides of (37) to find

$$\mathbb{E}[R_t^2] \geq \mathbb{E}[R_0^2], \quad t \geq 0.$$

□

Note that Lemma 3.2, Remark 7 and Lemma 3.4 conclude the frequency synchronization result and Theorem 2.5. For the final ingredient of Theorem 2.6, we state Doob's exponential martingale inequality. We can roughly estimate redundant random terms in terms of their noise strength.

Lemma 3.5. *(Doob's exponential martingale inequality, [24]) Let M_t be a martingale where its quadratic variation $\langle M, M\rangle_t$ satisfying*

$$M_0 = 0 \quad \textit{and} \quad \sup_{0<t<T} |\langle M, M\rangle_t| < C.$$

Then, for $h > 1$ and $T \in (0, \infty)$, one has

$$\mathbb{P}\Big\{ \sup_{0<t<T} M_t > \sqrt{\log h} \Big\} \leq h^{-1/2C}.$$

3.3. **Proof of Theorem 2.6.** We split its proof into two parts: First, we introduce a nonlinear functional measuring the transversal phase differences and derive its Itô derivatives. Second, we apply Lemma 3.5 for the nonlinear functional to get a lower bound for the probability where oscillators are confined in a small neighborhood of the average phase.

• Step A (Design of a suitable discrepancy functional): The phase difference between the i-th and j-th oscillators is often measured by the Lipschitz functional $|\theta_t^i - \theta_t^j|$ for the deterministic setting. For the stochastic processes, however, we employ a differentiable nonlinear functional $V_{ij}(t)$:

$$V_{ij}(t) := (1 - \cos(\theta_t^i - \theta_t^j)) = 2\sin^2\left(\frac{\theta_t^i - \theta_t^j}{2}\right),$$
$$V_i(t) := \max_j V_{ij}(t), \quad V(t) = \max_i V_i(t).$$

Then, it is easy to see that

$$V_{ij}(t) \approx \frac{|\theta_t^i - \theta_t^j|^2}{2} \quad \text{if } |\theta_t^i - \theta_t^j| \ll 1, \quad \text{and} \quad V_{ij}(t) = 0 \iff \theta_t^i - \theta_t^j = 0.$$

Since $D(\Theta_t)$ or $V(t)$ is hard to be estimated with their definition, we instead use $V_i(t)$. For $D_\infty \in [0, \pi/2)$, we set $R_\infty := \cos D_\infty$ so that

$$V(t) < 1 - \cos D_\infty = 1 - R_\infty \quad \text{if and only if} \quad D(\Theta_t) < D_\infty.$$

From definition of $V(t)$, we have

$$\begin{aligned} V(t) &= 2\max_{j,k} \sin^2\left(\frac{\theta_t^j - \theta_t^k}{2}\right) \leq 2\max_{j,k}\sin^2\left(\frac{\theta_t^j - \theta_t^i}{2} + \frac{\theta_t^i - \theta_t^k}{2}\right) \\ &\leq 4\max_j \sin^2\left(\frac{\theta_t^j - \theta_t^i}{2}\right) + 4\max_k \sin^2\left(\frac{\theta_t^i - \theta_t^k}{2}\right) \\ &\leq 4V_i(t). \end{aligned}$$

Hence, if $4V_i(t) < 1 - R_\infty$, then we have

$$V(t) < 1 - R_\infty \quad \text{and} \quad D(\Theta_t) < D_\infty. \tag{38}$$

The differentiability plays a key role in the estimation of $V(t)$. If we adopt the diameter function,

$$\max_{i,j} |\theta_t^i - \theta_t^j|,$$

then we need to consider the maximal and minimal oscillators:

$$\theta_t^M := \max_i \theta_t^i, \quad \theta_t^m := \min_i \theta_t^i.$$

For a deterministic model with analytic vector fields, their trajectories are piecewise smooth since the collisions between oscillators are finite in a finite time. For a stochastic system, it is usually not true since the Brownian motion is not differentiable.

Let i, j be indices in $\{1, \cdots, N\}$. Next, we study dV_{ij} and $d\log V_{ij}$ as follows. First, we recall

$$\begin{aligned} d(\theta_t^i - \theta_t^j) &= -\kappa R_t \Big(\sin(\theta_t^i - \psi_t) - \sin(\theta_t^j - \psi_t) \Big) dt \\ &\quad - \sqrt{2\sigma} R_t \Big(\sin(\theta_t^i - \psi_t) - \sin(\theta_t^j - \psi_t) \Big) dB_t, \quad \text{or} \\ d(\theta_t^i - \theta_t^j) &= -2\kappa R_t \cos\Big(\frac{\theta_t^i + \theta_t^j}{2} - \psi_t \Big) \sin\Big(\frac{\theta_t^i - \theta_t^j}{2} \Big) dt \\ &\quad - 2\sqrt{2\sigma} R_t \cos\Big(\frac{\theta_t^i + \theta_t^j}{2} - \psi_t \Big) \sin\Big(\frac{\theta_t^i - \theta_t^j}{2} \Big) dB_t. \end{aligned} \tag{39}$$

⋄ (Derivation of $dV_{ij}(t)$): We apply Itô's lemma and (39) to see

$$\begin{aligned} dV_{ij}(t) &= \Big[-\kappa R_t (\sin(\theta_t^i - \psi) - \sin(\theta_t^j - \psi)) \sin(\theta_t^i - \theta_t^j) \\ &\qquad + \sigma R_t^2 (\sin(\theta_t^i - \psi) - \sin(\theta_t^j - \psi))^2 \cos(\theta_t^i - \theta_t^j) \Big] dt \\ &\quad - \sqrt{2\sigma} R_t (\sin(\theta_t^i - \psi) - \sin(\theta_t^j - \psi)) \sin(\theta_t^i - \theta_t^j) dB_t \\ &= \Big[-4\kappa R_t \cos\left(\frac{\theta_t^i + \theta_t^j}{2} - \psi \right) \cos\left(\frac{\theta_t^i - \theta_t^j}{2} \right) \\ &\qquad + 4\sigma R_t^2 \cos^2\left(\frac{\theta_t^i + \theta_t^j}{2} - \psi \right) \cos\left(\theta_t^i - \theta_t^j \right) \Big] \sin^2\left(\frac{\theta_t^i - \theta_t^j}{2} \right) dt \\ &\quad - 4\sqrt{2\sigma} R_t \cos\left(\frac{\theta_t^i + \theta_t^j}{2} - \psi \right) \cos\left(\frac{\theta_t^i - \theta_t^j}{2} \right) \sin^2\left(\frac{\theta_t^i - \theta_t^j}{2} \right) dB_t \\ &= \Big[-2\kappa R_t \cos\left(\frac{\theta_t^i + \theta_t^j}{2} - \psi \right) \cos\left(\frac{\theta_t^i - \theta_t^j}{2} \right) \\ &\qquad + 2\sigma R_t^2 \cos^2\left(\frac{\theta_t^i + \theta_t^j}{2} - \psi \right) \cos\left(\theta_t^i - \theta_t^j \right) \Big] V_{ij}(t) dt \\ &\quad - 2\sqrt{2\sigma} R_t \cos\left(\frac{\theta_t^i + \theta_t^j}{2} - \psi \right) \cos\left(\frac{\theta_t^i - \theta_t^j}{2} \right) V_{ij}(t) dB_t. \end{aligned}$$

⋄ (Derivation of $d\log V_{ij}(t)$): If we expand it for $\log V_{ij}(t)$, then we have

$$
\begin{aligned}
d(\log V_{ij}(t)) &= \frac{dV_{ij}}{V_{ij}} - \frac{1}{2V_{ij}^2}dV_{ij}dV_{ij} \\
&= -2\Big[\kappa R_t \cos\left(\frac{\theta_t^i+\theta_t^j}{2}-\psi\right)\cos\left(\frac{\theta_t^i-\theta_t^j}{2}\right) \\
&\quad - \sigma R_t^2\cos^2\left(\frac{\theta_t^i+\theta_t^j}{2}-\psi\right)\cos\left(\theta_t^i-\theta_t^j\right) \\
&\quad + 2\sigma R_t^2\cos^2\left(\frac{\theta_t^i+\theta_t^j}{2}-\psi\right)\cos^2\left(\frac{\theta_t^i-\theta_t^j}{2}\right)\Big]dt \\
&\quad - 2\sqrt{2\sigma}R_t\cos\left(\frac{\theta_t^i+\theta_t^j}{2}-\psi\right)\cos\left(\frac{\theta_t^i-\theta_t^j}{2}\right)dB_t.
\end{aligned} \tag{40}
$$

• Step B (Synchronization estimates): We next use the linearization approach. We choose

$$D_\infty \in (0,\pi/2) \quad \text{and} \quad D(\Theta) := \sup_{i,j}|\theta_t^i-\theta_t^j|,$$

and introduce the first escape time τ of the diameter process $D(\Theta_t)$ from the interval $[0,D_\infty]$:

$$\tau := \inf\{t>0 : D(\Theta_t) > D_\infty\} \ \text{ and } \ R_\infty := \cos(D_\infty) > 0,$$

Then, for $t\in[0,\tau)$, one has

$$\cos(\theta_t^i-\theta_t^j) > R_\infty.$$

Next, we estimate the noise term in (40) using Doob's martingale inequality. For this, we consider a new process M_t^{ij} whose dynamics is given by the following equation:

$$dM_t^{ij} = -2\sqrt{2\sigma}R\cos\left(\frac{\theta_t^i+\theta_t^j}{2}-\psi\right)\cos\left(\frac{\theta_t^i-\theta_t^j}{2}\right)dB_t, \quad t>0, \quad M_0^{ij}=0.$$

Since there is no drift, we know that M_t^{ij} is a centered martingale. Now apply Lemma 3.5 with $C=8\sigma$ to find

$$\mathbb{P}\{\sup_{0<t<T} M_t^{ij} > \sqrt{\log h}\} \le h^{(-1/16\sigma)}.$$

For the maximum of all M_t^{ij}, we get

$$\mathbb{P}\{\sup_{0<t<T}\max_j M_t^{ij} > \sqrt{\log h}\} \le \sum_{j=1}^N \mathbb{P}\{\sup_{0<t<T} M_t^{ij} > \sqrt{\log h}\} \le Nh^{(-1/16\sigma)}.$$

For a fixed i, let τ_M be a stopping time of the above probability,

$$\tau_M := \inf\{t>0 : \sup_j M_t^{ij} > \sqrt{\log h}\}.$$

By definition of τ_M, one has

$$\mathbb{P}\{\tau_M=\infty\} \ge 1 - Nh^{(-1/16\sigma)}.$$

We now use linearization technique to bound cosine functions: for $t < \tau$, we have

$$\cos^2\left(\frac{\theta_t^i - \theta_t^j}{2}\right) \geq \cos\left(\theta_t^i - \theta_t^j\right) > R_\infty, \quad \text{and} \quad \cos\left(\frac{\theta_t^i + \theta_t^j}{2} - \psi\right) > R_\infty.$$

Moreover, we know that $R > 1/\sqrt{2}$ from $D(\Theta_t) \leq \pi/2$. We apply these bounds to the equation (40). Then, for $t < \tau_M \wedge \tau$, we have

$$\begin{aligned} d(\log V_{ij}(t)) \leq -2R_t\Big[&\kappa \cos\left(\frac{\theta_t^i + \theta_t^j}{2} - \psi\right)\cos\left(\frac{\theta_t^i - \theta_t^j}{2}\right) \\ &+ \sigma R_t \cos^2\left(\frac{\theta_t^i + \theta_t^j}{2} - \psi\right)\cos^2\left(\frac{\theta_t^i - \theta_t^j}{2}\right)\Big]dt + dM_t^{ij} \\ \leq& -(\sqrt{2}\kappa R_\infty^2 + \sigma R_\infty^3)dt + dM_t^{ij}. \end{aligned}$$

Next, we use the comparison lemma [11] and the bound

$$\max_j M_t^{ij} \leq \sqrt{\log h} \quad \text{for} \;\; t < \tau_M \wedge \tau,$$

to see that for $t < \tau_M \wedge \tau$,

$$\log V_i(t) \leq \log V_i(0) + \sqrt{\log h} - (\sqrt{2}\kappa R_\infty^2 + \sigma R_\infty^3)t. \tag{41}$$

From the assumptions on the initial data (21), if we let $h = \varepsilon^{-16\sigma}$, then we obtain

$$\begin{aligned} V_i(t) &\leq V_i(0)\exp(\sqrt{-16\sigma \log \varepsilon}) \\ &< 2\sin^2\frac{D_0}{2}\exp(\sqrt{-16\sigma\log\varepsilon}) \\ &= \frac{1}{2e}\varepsilon^{4\sigma}\exp(\sqrt{-16\sigma\log\varepsilon})\sin^2\frac{D_\infty}{2} \\ &= \frac{1}{2e}\exp(4\sigma\log\varepsilon + 2\sqrt{-4\sigma\log\varepsilon})\sin^2\frac{D_\infty}{2} \\ &= \frac{1}{2e}\exp(1 - (\sqrt{-4\sigma\log\varepsilon} - 1)^2)\sin^2\frac{D_\infty}{2} \\ &\leq \frac{1}{2}\sin^2\frac{D_\infty}{2} = \frac{(1 - R_\infty)}{4}, \quad \text{for} \;\; t < \tau_M \wedge \tau. \end{aligned}$$

Thus, according to the relationship (38) and the continuity argument, there exists some constant $\bar{D}_\infty < D_\infty$ such that

$$D(\Theta_t) < \bar{D}_\infty < D_\infty, \quad \text{for} \;\; t \leq \tau_M \wedge \tau. \tag{42}$$

From the continuity argument, this implies that $\tau \geq \tau_M$. If not, the definition of τ implies that $D(\Theta_{\tau_M \wedge \tau}) = D(\Theta_\tau) = D_\infty$, which is contradictory to (42) since Θ_t is continuous.

Finally, we apply $V(t) \leq 4V_i(t)$ to (41), then we get the probability of decaying $V(t)$,

$$\mathbb{P}\left\{V(t) \leq (1 - R_\infty)\exp(-(\sqrt{2}\kappa R_\infty^2 + \sigma R_\infty^3)t)\right\} \geq \mathbb{P}\{\tau_M = \infty\} \geq 1 - N\varepsilon.$$

Therefore, we have exponentially decreasing $V(t)$ except for a positive probability $N\varepsilon$, where the smallness of ε depends on $D(\Theta_0)$.

4. Synchronization estimate for the KS-FP equation

Next, we study emergent behavior of the KS-FP equation (3). We use a classical approach which arise from [21], and recently used in [16] to show the synchronization of phases. In this section, we focus on differences to the dynamics which comes from the multiplicative noise. As for the stochastic model in Section 3, we assume the natural frequencies are identical to zero, hence, $g(\nu)$ is a Dirac delta function:

$$g(\nu) = \delta(\nu).$$

Then, we may set

$$F(t,\theta,\nu) =: f(t,\theta)\delta(\nu),$$

where $f = f(t,\theta)$ is a density function on the phase space $\mathbb{T}$ at time t:

$$f(t,\theta) := \int_{\mathbb{R}} F(t,\theta,\nu)d\nu.$$

For notational simplicity, as in Section 3, we suppress ∞ and t dependence in order parameters $R^\infty(t)$ and $\psi^\infty(t)$:

$$R := R(t) := R^\infty(t), \quad \psi := \psi(t) := \psi^\infty(t).$$

Then, it follows from the equation (15) that f satisfies

$$\begin{cases} \partial_t f = \kappa R \partial_\theta \left(f \sin(\theta-\psi)\right) + \sigma R^2 \partial_\theta^2 (f \sin^2(\theta-\psi)), \\ f(0) = f_0, \quad Re^{\mathrm{i}\psi} = \displaystyle\int_{\mathbb{T}} f(t,\theta) e^{\mathrm{i}\theta} d\theta, \\ \int_{\mathbb{T}} f_0 d\theta = 1, \quad f_0 \geq 0, \end{cases} \tag{43}$$

where the order parameters satisfy

$$R = \int_{\mathbb{T}} f \cos(\theta-\psi)d\theta \quad \text{and} \quad 0 = \int_{\mathbb{T}} f\sin(\theta-\psi)d\theta. \tag{44}$$

By direct calculations, we get the dynamics of R and ψ:

$$\begin{aligned} \frac{dR}{dt} &= \kappa R \int_{\mathbb{T}} f \sin^2(\theta-\psi)d\theta - \sigma R^2 \int_{\mathbb{T}} f \sin^2(\theta-\psi)\cos(\theta-\psi)d\theta, \\ R\frac{d\psi}{dt} &= -\kappa R \int_{\mathbb{T}} f \sin(\theta-\psi)\cos(\theta-\psi)d\theta - \sigma R^2 \int_{\mathbb{T}} f\sin^3(\theta-\psi)d\theta. \end{aligned} \tag{45}$$

Remark 9. As in other models with noise effect, $R(t)$ can decrease in the presence of multiplicative noise. However, contrary to additive noise, $R(t)$ cannot be zero from nonzero initial condition. Suppose that $R(0) > 0$. Then, it follows from (45) that

$$\frac{dR}{dt} \geq -\sigma R^2, \quad t > 0,$$

since

$$|\sin^2(\theta-\psi)\cos(\theta-\psi)| \leq 1 \quad \text{and} \quad \int_{\mathbb{T}} |f| = 1.$$

Moreover, if the coupling strength κ is larger than the noise strength σ, $R(t)$ does not decrease.

Lemma 4.1. *Suppose that κ and σ satisfy $\kappa > \sigma$, and let f be a solution to (43) with the initial datum f_0 satisfying $R(0) =: R_0 > 0$. Then, $R(t)$ is nondecreasing:*

$$\frac{dR}{dt} \geq 0 \quad \text{and} \quad \int_0^\infty \int_{\mathbb{T}} f \sin^2(\theta-\psi)d\theta dt < \infty.$$

Proof. We use the boundedness of $\cos(\theta-\psi)$ and R from (45) to see

$$
\begin{aligned}
\frac{dR}{dt} &= \kappa R \int_{\mathbb{T}} f \sin^2(\theta-\psi)d\theta - \sigma R^2 \int_{\mathbb{T}} f \sin^2(\theta-\psi)\cos(\theta-\psi)d\theta \\
&\ge (\kappa R - \sigma R^2)\int_{\mathbb{T}} f\sin^2(\theta-\psi)d\theta \ge (\kappa-\sigma)R\int_{\mathbb{T}} f\sin^2(\theta-\psi)d\theta \ge 0.
\end{aligned} \tag{46}
$$

Thus, R is non-decreasing, and has a limit value as $t\to\infty$:

$$
\exists\ R_\infty \quad \text{such that } \lim_{t\to\infty} R(t) \quad \text{and} \quad R_0 \le R(t) \le R_\infty, \quad t\ge 0.
$$

We use the above relation and (46) to obtain

$$
\frac{dR}{dt} \ge (\kappa-\sigma)R_0 \int_{\mathbb{T}} f\sin^2(\theta-\psi)d\theta \ge 0.
$$

This implies the boundedness of the integration:

$$
\int_0^\infty \int_{\mathbb{T}} f\sin^2(\theta-\psi)d\theta dt \le \frac{R_\infty - R_0}{R_0(\kappa-\sigma)} < \infty.
$$

□

From the monotonicity, we may conclude the convergence of f.

Lemma 4.2. *Let f be a solution to (43) with initial datum f_0 satisfying $R(0) =: R_0 > 0$. If $\kappa > \sigma$, we have*

$$
\lim_{t\to\infty} \int_{\mathbb{T}} f\sin^2(\theta-\psi)d\theta = 0.
$$

Proof. It follows from Lemma 4 that

$$
\int_0^\infty \Big[\int_{\mathbb{T}} f\sin^2(\theta-\psi)d\theta\Big]dt < \infty.
$$

From Barbalat's lemma, we only need to show that the integrand $\int_{\mathbb{T}} f\sin^2(\theta-\psi)d\theta$ is uniformly continuous on t. We may calculate the time derivative as follows,

$$
\begin{aligned}
&\frac{d}{dt}\left[\int_{\mathbb{T}} f\sin^2(\theta-\psi)d\theta\right] \\
&= \int_{\mathbb{T}} (\partial_t f)\sin^2(\theta-\psi)d\theta - 2\frac{d\psi}{dt}\int_{\mathbb{T}} f\sin(\theta-\psi)\cos(\theta-\psi)d\theta \\
&= \mathcal{I}_{21} + \mathcal{I}_{22}.
\end{aligned}
$$

- (Estimate on $\mathcal{I}_{21}$): We use $\partial_t f$ of (43) and integral by parts to get

$$
\begin{aligned}
\mathcal{I}_{21} &= \int_{\mathbb{T}} (\partial_t f)\sin^2(\theta-\psi)d\theta \\
&= \kappa R \int_{\mathbb{T}} \partial_\theta\left(f\sin(\theta-\psi)\right)\sin^2(\theta-\psi)d\theta \\
&\quad + \sigma R^2 \int_{\mathbb{T}} \partial_\theta^2 (f\sin^2(\theta-\psi))\sin^2(\theta-\psi)d\theta \\
&= -\kappa R\int_{\mathbb{T}} f\sin(\theta-\psi)(2\sin(\theta-\psi)\cos(\theta-\psi))d\theta \\
&\quad + \sigma R^2 \int_{\mathbb{T}} f\sin^2(\theta-\psi)(2\cos(2(\theta-\psi)))d\theta \\
&\le 2\kappa + 2\sigma,
\end{aligned}
$$

where we used the boundedness of sinusoidal functions and R.

• (Estimate on $\mathcal{I}_{22}$): In the same way as for $\mathcal{I}_{21}$, from (45), we get

$$\left|\frac{d\psi}{dt}\right| \le \kappa + \sigma.$$

This implies that we have

$$|\mathcal{I}_{22}| = 2\left|\frac{d\psi}{dt}\right| \int_{\mathbb{T}} f\,|\sin(\theta-\psi)\cos(\theta-\psi)|\,d\theta \le 2(\kappa+\sigma).$$

Hence, we conclude that the derivative of the integrand is bounded and it is uniformly continuous. □

Lemma and Lemma 4.2 concludes the convergence result, Theorem 2.7.

5. Conclusion

In this paper, we discussed the synchronization phenomena of two Kuramoto models under random environment: the stochastic system and the kinetic equation. Between these two models, Sznitman's theory [28] can be applied to show the mean-field limit in a finite time interval. For the whole time interval, we need a good understanding on the emergent behaviors of these models.

With these theoretical basis, we showed that some weak concepts of synchronization hold for each model, when their natural frequencies are identical. First, for the stochastic system, we assumed that the same Brownian motion affects on each oscillator, and showed that the phases of oscillators tend to have the same value under a positive probability when they initially close to each other. This is a phenomenon called stochastic stability. Unlike other collective models such as the Cucker-Smale model, this technique can not guarantee the stability with a probability 1, since the domain $\mathbb{T}$ is periodic. This probability of phase synchronization depends on the accumulation of initial data.

Second, in the KS-FP model, we used a classical approach to get the synchronization estimate. We showed that the monotonicity of R holds for large coupling strength, and the distribution of phases converges to a one- or two-point distribution. These convergence results coincide with the stochastic system, and they are exclusive properties of multiplicative noise, which cannot be expected for other types of randomness such as additive noise.

There are still remaining open problems on the emergent behavior to conclude the mean-field limit. On the one hand, the emergent dynamics of the KS-FP model needs more detailed analysis. For example, we still do not know the condition of convergence of ψ, and also we need to determine whether F converges to a one-point distribution or not. On the other hand, the stochastic system with independent Brownian motions is expected to show the similar behavior to our model, however, we could not suggest the common features or differences in this paper. Emergent behavior of stochastic systems have many interesting open problems which can be applied to various interactions and types of noise.

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