On some reversibility properties of Hamilton-Jacobi equations

Carlos Esteve Yagüe

September 2019

Universidad Autónoma de Madrid

DeustoTech

DyCon DYNAMIC CONTROL

European Research Council
Established by the European Commission

We consider the following Cauchy problem

$$\begin{cases} u_t + H(Du) = 0, & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = g(x), & x \in \mathbb{R}^n, \end{cases} \tag{HJ}$$

where $g \in W^{1,\infty}_{loc}(\mathbb{R}^n)$ is the initial condition and

$$H : \mathbb{R}^n \longrightarrow \mathbb{R}$$

is a Hamiltonian satisfying

$$H \text{ is continuous, } \underline{\text{strictly convex}} \text{ and } \frac{H(|x|)}{|x|} \underset{|x|\to\infty}{\longrightarrow} +\infty. \tag{H}$$

Optimal control problem

For a fixed $T > 0$, consider the following optimal control problem:

$$\begin{cases} \dot{\mathbf{x}}(s) = \alpha(s), & 0 \leq s \leq T \\ \mathbf{x}(0) = x \end{cases}$$

where the initial position $x \in \mathbb{R}^n$ is given and $\alpha : [0, T] \longrightarrow \mathbb{R}^n$ is any measurable function, that we call **control**.

Payoff functions: Running payoff: $L : \mathbb{R}^n \longrightarrow \mathbb{R}$ Final payoff: $g : \mathbb{R}^n \longrightarrow \mathbb{R}$

For fixed $x \in \mathbb{R}^n$ and $t \in (0, T]$, we define

$$u(t, x) = \inf_{\alpha(\cdot)} \left\{ \int_0^t L(\alpha(s))ds + g(\mathbf{x}(t)) \right\}.$$

We can prove that $u(t, x)$ is a viscosity solution of the dynamic programming equation

$$u_t + \sup_{z \in \mathbb{R}^n} \{ z \cdot Du - L(z) \} = 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^n.$$

We define, for $v \in \mathbb{R}^n$

$$H(v) = \sup_{z \in \mathbb{R}^n} \{ z \cdot v - L(z) \}.$$

Optimal control problem

For a fixed $T > 0$, consider the following optimal control problem:

$$\begin{cases} \dot{\mathbf{x}}(s) = \alpha(s), & 0 \leq s \leq T \\ \mathbf{x}(0) = x \end{cases}$$

where the initial position $x \in \mathbb{R}^n$ is given and $\alpha : [0, T] \longrightarrow \mathbb{R}^n$ is any measurable function, that we call **control**.

Payoff functions: Running payoff: $L : \mathbb{R}^n \longrightarrow \mathbb{R}$ Final payoff: $g : \mathbb{R}^n \longrightarrow \mathbb{R}$

For fixed $x \in \mathbb{R}^n$ and $t \in (0, T]$, we define

$$u(t, x) = \inf_{\alpha(\cdot)} \left\{ \int_0^t L(\alpha(s))ds + g(\mathbf{x}(t)) \right\}.$$

We can prove that $u(t, x)$ is a viscosity solution of the dynamic programming equation

$$u_t + \sup_{z \in \mathbb{R}^n} \{z \cdot Du - L(z)\} = 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^n.$$

We define, for $v \in \mathbb{R}^n$

$$H(v) = \sup_{z \in \mathbb{R}^n} \{z \cdot v - L(z)\}.$$

Optimal control problem

For a fixed $T > 0$, consider the following optimal control problem:

$$\begin{cases} \dot{\mathbf{x}}(s) = \alpha(s), & 0 \leq s \leq T \\ \mathbf{x}(0) = x \end{cases}$$

where the initial position $x \in \mathbb{R}^n$ is given and $\alpha : [0, T] \longrightarrow \mathbb{R}^n$ is any measurable function, that we call **control**.

Payoff functions: Running payoff: $L : \mathbb{R}^n \longrightarrow \mathbb{R}$ Final payoff: $g : \mathbb{R}^n \longrightarrow \mathbb{R}$

For fixed $x \in \mathbb{R}^n$ and $t \in (0, T]$, we define

$$u(t, x) = \inf_{\alpha(\cdot)} \left\{ \int_0^t L(\alpha(s))ds + g(\mathbf{x}(t)) \right\}.$$

We can prove that $u(t, x)$ is a viscosity solution of the dynamic programming equation

$$u_t + \sup_{z \in \mathbb{R}^n} \{z \cdot Du - L(z)\} = 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^n.$$

We define, for $v \in \mathbb{R}^n$

$$H(v) = \sup_{z \in \mathbb{R}^n} \{z \cdot v - L(z)\}.$$

$$\begin{cases} u_t + H(Du) = 0, & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = g(x), & x \in \mathbb{R}^n, \end{cases} \tag{HJ}$$

In general we cannot expect to have C^1 solutions. Therefore, we need to consider **generalized solutions**:

$$u \in W^{1,\infty}_{\text{loc}}, \text{ satisfying (HJ) a.e.}$$

We have no uniqueness.

$$\begin{cases} u_t + H(Du) = 0, & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = g(x), & x \in \mathbb{R}^n, \end{cases} \tag{HJ}$$

Definition

We say $u \in C([0,T] \times \mathbb{R}^n)$ is a viscosity solution if

$$u(0,x) = g(x)$$

and for any $(t,x) \in (0,T) \times \mathbb{R}^n$ we have

$$\begin{array}{ll} p_t + H(p_x) \leq 0 & \text{for all } (p_t, p_x) \in D^+u(t,x) \\ p_t + H(p_x) \geq 0 & \text{for all } (p_t, p_x) \in D^-u(t,x) \end{array}$$

where the super- and sub-differentials are defined by

$$\begin{aligned} D^+u(t,x) = \{(p_t,p_x) \ : \ p_t = \varphi_t(t,x),\ p_x = D\varphi(t,x), \quad \exists \varphi \in C^1, \\ u - \varphi \leq 0,\ (u-\varphi)(t,x) = 0\}, \\ D^-u(t,x) = \{(p_t,p_x) \ : \ p_t = \varphi_t(t,x),\ p_x = D\varphi(t,x), \quad \exists \varphi \in C^1, \\ u - \varphi \geq 0,\ (u-\varphi)(t,x) = 0\}. \end{aligned}$$

For the problem (HJ) we have existence and uniqueness of a viscosity solution *[Crandall-Lions, 83]*. In addition, since H is convex, the solution is semiconcave.

Definition: A function u is semiconcave in $\mathbb{R}^n$ if for any compact set $K \in \mathbb{R}^n$, there exists $C_K > 0$ such that

$$\lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda)y) \leq \lambda(1-\lambda)C_K|x-y|^2,$$

for any $x, y \in K$ and $\lambda \in [0,1]$.

Some properties:

- If u is semiconcave then u is locally Lipschitz continuous and its super-differential D^+u is nonempty everywhere.
- If u is semiconcave and $D^+u(x)$ is a singleton $\forall x \in \mathbb{R}^n$, then $u \in C^1(\mathbb{R}^n)$.

Definition
A function $u \in C([0,T] \times \mathbb{R}^n)$ is called a backward (viscosity) solution of (HJ) if $v(t,x) := u(T-t,x)$ is a viscosity solution of

$$v_t - H(Dv) = 0 \qquad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Proposition: The following properties are equivalent:

- u is a backward solution of $u_t + H(Du) = 0$.
- u is a forward solution of $-u_t - H(Du) = 0$.
- The function $w(t,x) = -u(T-t,x)$ is a forward solution of $w_t + H(-Du) = 0$.
- For any $(t,x) \in (0,T) \times \mathbb{R}^n$ we have

$$\begin{aligned} p_t + H(p_x) &\geq 0 \quad \text{for all } (p_t,p_x) \in D^+u(t,x) \\ p_t + H(p_x) &\leq 0 \quad \text{for all } (p_t,p_x) \in D^-u(t,x). \end{aligned}$$

Corollary: If H is strictly convex, then any backward solution is semiconvex.

Forward-backward solutions

If $u \in C^1([0, T] \times \mathbb{R}^n)$, clearly we have

u is a solution of (HJ) if and only if u is a backward solution of (HJ).

Theorem*[Barron-Cannarsa-Jensen-Sinestrari,1999]*:
Let $H(\cdot)$ be strictly convex and superlinear. If u is a forward and backward solution of

$$u_t + H(Du) = 0, \qquad (t, x) \in (0, T) \times \mathbb{R}^n,$$

then $u \in C^1((0, T) \times \mathbb{R}^n)$.

Goals: For a given $T > 0$

- Determine all the solutions which are both forward and backward in $(0, T) \times \mathbb{R}^n$ (then C^1).
- Given a C^1 solution $u(t, x)$ in $]0, T[\times\mathbb{R}^n$, determine all the initial conditions $g(x)$ such that $u_g(T, x) = u(T, x)$.

If $u \in C^1([0,T] \times \mathbb{R}^n)$, clearly we have

u is a solution of (HJ) if and only if u is a backward solution of (HJ).

Theorem*[Barron-Cannarsa-Jensen-Sinestrari,1999]*:
Let $H(\cdot)$ be strictly convex and superlinear. If u is a forward and backward solution of

$$u_t + H(Du) = 0, \qquad (t,x) \in (0,T) \times \mathbb{R}^n,$$

then $u \in C^1((0,T) \times \mathbb{R}^n)$.

Goals: For a given $T > 0$

- Determine all the solutions which are both forward and backward in $(0,T) \times \mathbb{R}^n$ (then C^1).
- Given a C^1 solution $u(t,x)$ in $]0,T[\times\mathbb{R}^n$, determine all the initial conditions $g(x)$ such that $u_g(T,x) = u(T,x)$.

For fixed $T > 0$ and $w \in W^{1,\infty}(\mathbb{R};\mathbb{R})$. We introduce the following multivalued function

$$\begin{array}{rccl} p & : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & \{x - T\,H'(z) \ : \ z \in D^+ w(x)\} \end{array}$$

We also define the following set

$$I_T^{HJ}(w) := \{u_0 \in W^{1,\infty}(\mathbb{R};\mathbb{R}) : \ S_T^{HJ} u_0 = w\},$$

where $S_T^{HJ} u_0$ is $u(T,x)$, where u is the viscosity solution of (HJ) with initial condition u_0.

Theorem *[Colombo-Perrolaz, 2019]*: Let H be C^2 and uniformly convex and $T > 0$. Fix $w \in W^{1,\infty}(\mathbb{R};\mathbb{R})$. The following statements are equivalent:

(a) $I_T^{HJ}(w) \neq \emptyset$;

(b) The multivalued function p defined above satisfies

$$\sup p(x) \leq \inf p(x+y), \qquad \text{for all } x \in \mathbb{R} \text{ and } y \in \mathbb{R}^+ \setminus \{0\}. \qquad \text{(O)}$$

Construction of C^1 solutions

For any $w \in W^{1,\infty}(\mathbb{R};\mathbb{R})$, the unique backward solution is given by Hopf-Lax formula as follows:

$$v(t,x) = \max_{y\in\mathbb{R}^n} \left\{ w(y) - (T-t)H^* \left(\frac{y-x}{T-t} \right) \right\},$$

where $H^*(x) = \sup\{x \cdot y - H(y) \ : \ y \in \mathbb{R}^n\}$.

Using again Hopf-Lax formula, we obtain the viscosity (forward) solution with initial data $v(0,x)$:

$$u(t,x) = \min_{z\in\mathbb{R}^n} \left\{ w(0,z) + t\, H^* \left(\frac{x-z}{t} \right) \right\}.$$

If w satisfies (O), we can prove

$$u(T,x) = w(x), \quad \text{and then} \quad u = v, \quad \text{in } [0,T] \times \mathbb{R}^n.$$

This implies u is the unique $C^1(]0,T[\times\mathbb{R}^n)$ solution of (HJ) such that $u(T,\cdot) = w(\cdot)$.

Construction of C^1 solutions

For any $w \in W^{1,\infty}(\mathbb{R};\mathbb{R})$, the unique backward solution is given by Hopf-Lax formula as follows:

$$v(t,x) = \max_{y\in\mathbb{R}^n} \left\{ w(y) - (T-t)H^*\left(\frac{y-x}{T-t}\right)\right\},$$

where $H^*(x) = \sup\{x \cdot y - H(y) \ : \ y \in \mathbb{R}^n\}$.

Using again Hopf-Lax formula, we obtain the viscosity (forward) solution with initial data $v(0,x)$:

$$u(t,x) = \min_{z\in\mathbb{R}^n} \left\{ w(0,z) + t\, H^*\left(\frac{x-z}{t}\right)\right\}.$$

If w satisfies (O), we can prove

$$u(T,x) = w(x), \quad \text{and then} \quad u = v, \quad \text{in } [0,T]\times\mathbb{R}^n.$$

This implies u is the unique $C^1(]0,T[\times\mathbb{R}^n)$ solution of (HJ) such that $u(T,\cdot) = w(\cdot)$.

Construction of C^1 solutions

For any $w \in W^{1,\infty}(\mathbb{R};\mathbb{R})$, the unique backward solution is given by Hopf-Lax formula as follows:

$$v(t,x) = \max_{y\in\mathbb{R}^n} \left\{ w(y) - (T-t)H^*\left(\frac{y-x}{T-t}\right)\right\},$$

where $H^*(x) = \sup\{x\cdot y - H(y) \;:\; y \in \mathbb{R}^n\}$.

Using again Hopf-Lax formula, we obtain the viscosity (forward) solution with initial data $v(0,x)$:

$$u(t,x) = \min_{z\in\mathbb{R}^n} \left\{ w(0,z) + t\, H^*\left(\frac{x-z}{t}\right)\right\}.$$

If w satisfies (O), we can prove

$$u(T,x) = w(x), \quad \text{and then} \quad u = v, \quad \text{in } [0,T]\times\mathbb{R}^n.$$

This implies u is the unique $C^1(]0,T[\times\mathbb{R}^n)$ solution of (HJ) such that $u(T,\cdot) = w(\cdot)$.

Theorem *[Colombo-Perrolaz, 2019]*: Let H be C^2 and uniformly convex and $T > 0$. Fix $w \in W^{1,\infty}(\mathbb{R};\mathbb{R})$ such that $I_T^{HJ}(w) \neq \emptyset$ and $\partial_x w \in \mathrm{SBV}_{loc}(\mathbb{R};\mathbb{R})$. Let p be the function defined above. Then, $u_0 \in I_T^{HJ}(w)$ if and only if the following two conditions hold:

- for all $x \in \mathbb{R}$ such that p is differentiable at x and $p'(x) \neq 0$,

$$\lim_{y\to x} \frac{u_0(p(y)) - u_0(p(x))}{p(y) - p(x)} = \partial_x w(x);$$

- for all $x \in \mathbb{R}$ such that $\partial_x w(x-) \neq \partial_x w(x+)$,

$$\begin{array}{ll} \forall y \in]p(x-), p(x+)[& u_0(y) + T\, H^*\left(\frac{x-y}{T}\right) \geq w(x), \\ \forall y \in \{p(x-), p(x+)\} & u_0(y) + T\, H^*\left(\frac{x-y}{T}\right) = w(x), \end{array}$$

Our goal is to obtain similar results for the *n*-dimensional case:

1. State an Oleinik condition to characterize all the C^1 solutions in a certain time interval $[0, T]$.

   We can use Hopf-Lax formula (which is implicit):

$$w(x) = \min_{z \in \mathbb{R}^n} \max_{y \mathbb{R}^n} \left\{ w(y) - T\, H^* \left(\frac{y - z}{T} \right) + T\, H^* \left(\frac{x - z}{T} \right) \right\}.$$

2. For a given profile w satisfying the Oleinik condition, characterize all the initial data u_0 such that $S_T^{HJ} u_0 = w$.

   This seems to be much more technically difficult, compared with the 1-dimensional case.

3. Do the numerics for the case in $\mathbb{R}^2$.

Thanks for the attention!