MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Robust control of incompressible flows

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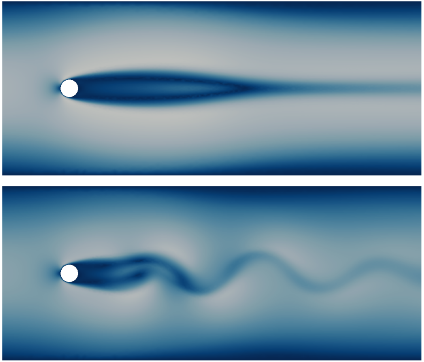
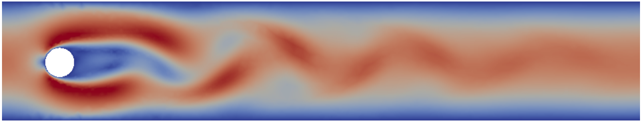
Seminar talk at University of Deusto (CMC), Bilbao



Outline

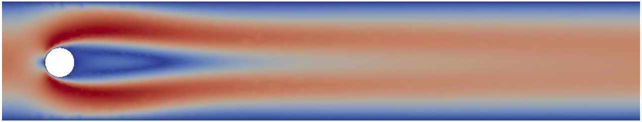


Feedback Control

Problem: The steady state is unstable: any perturbation – no matter how small – will trigger a transition into a periodic regime.

Goal: Stabilizing feedback controller that can handle:

- limited measurements,
- short evaluation time,
- system uncertainties.



Idea: Linearization-based feedback control for stabilization of the steady state.

[Raymond'05/'06, PB&JH'15, Breiten&Kunisch'14]

$$\begin{aligned} \dot{v} + (v \cdot \nabla)v - \nu\Delta v + \nabla p &= Bu, \\ \nabla \cdot v &= 0 \end{aligned}$$

Linearization & Semi-Discretization →

$$\begin{aligned} \dot{v} - Av - J^{\mathsf{T}} p &= Bu, \\ Jv &= 0 \end{aligned}$$

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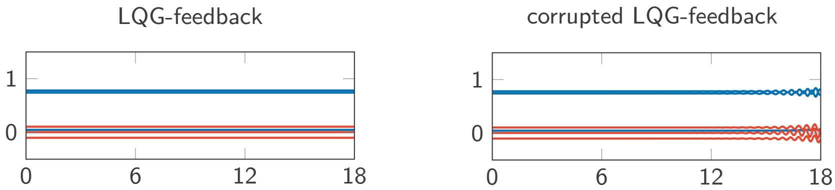
Fragility of Observer-Based Controllers

LQG controllers have no guaranteed robustness margins and will likely fail in the presence of system uncertainties.



Introduction

In fact: [IEEE Transaction on Automatic Control ('78)]:

> **Guaranteed Margins for LQG Regulators**
>
> JOHN C. DOYLE
>
> ***Abstract*—There are none.**

Good news: Uncertainties that come from

- [Curtain'03]: Galerkin approximations of evolution systems,
- [PB&JH'17]: stable mixed-FEM approximation of the flow equations,
- [this talk, PB&JH'16]: errors in the linearization point,

can be qualified as a coprime factor perturbation of the associated transfer function.

Even better news:

- [this talk]: We can employ robust observer/controller design.

Uncertain Linearization Points are Coprime Factor Uncertainties

Transfer functions

Mapping of inputs (controls) to outputs (measurements) in frequency domain, i.e., after Laplace transform of the system.

$$\begin{aligned} \dot x &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{\mathcal{L}(s)} \quad \begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) \end{aligned}$$

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1. A *nominal* system has the transfer function

$$G(s) = C(sI - A)^{-1}B \in \mathbb{C}^{q,r}.$$

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1. A *nominal* system has the transfer function

$$G(s) = C(sI - A)^{-1}B \in \mathbb{C}^{q,r}.$$

2. But uncertainty in the operator gives another transfer function

$$G_\Delta(s) = C(sI - A - A_\Delta)^{-1}B \in \mathbb{C}^{q,r}.$$

Uncertain Linearization Points are Coprime Factor Uncertainties

Coprime Factorization

Given a transfer function $G(s)$ of a linear system,

$$G(s) = M^{-1}(s)N(s)$$

is a **(left) coprime factorization** if there exist $X(s)$, $Y(s)$ such that the Bezout identity

$$M(s)X(s) + N(s)Y(s) = I$$

holds. Here, N, M, X, Y are all rational matrix functions with all poles in the open left half of the complex plane, i.e., they all represent stable linear systems.

Fact: N, M are coprime $\iff$ N, M have no common zeros in the right half plane.

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Coprime Factor Perturbation

$$G_\Delta(s) = [N(s) + N_\Delta(s)]\,[M(s) + M_\Delta(s)]^{-1}(s) \approx G(s) = N(s)M^{-1}(s),$$

where $N + N_\Delta$, $M + M_\Delta$ are stable.

Uncertain Linearization Points are Coprime Factor Uncertainties

Consider a state linear system (A, B, C) with

- $A\colon \mathcal{D}(A) \subset Z \to Z$ a generator of a C_0-semigroup
- $B\colon U \to Z$ bounded and $C\colon Z \to Y$ bounded
- U, Y, Z Hilbert spaces, U, Y finite dimensional

and

$$(A_\Delta, B, C) \sim G_\Delta \approx G \sim (A, B, C)$$

with a certain difference in the dynamics which is caused, say, by an inexact linearization.

Theorem (PB&JH '16)

If (A_Δ, B, C) and (A, B, C) are jointly stabilizable (or detectable), i.e., there exists a state feedback K (or L) that stabilizes (or makes detectable) both (A_Δ, B, C) (or (A, B, C))[1], then G differs from G_Δ through a coprime factor perturbation

$$\Delta = \begin{bmatrix} M_\Delta & N_\Delta \end{bmatrix}$$

with $\|\Delta\|_\infty \to 0$ as $A \to A_\Delta$ in the operator norm.

[1] That is, $A + BK$ and $A_\Delta + BK$ or $A + LC$ and $A_\Delta + LC$ are all stable.

Robust controllers for coprime factor uncertainty

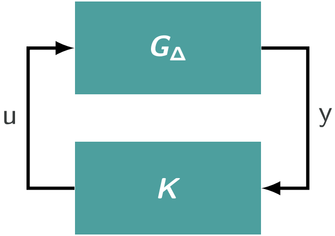
An admissable controller K stabilizes

$$G_\Delta = (M + M_\Delta)^{-1}(N + N_\Delta)$$

for all $\|\Delta\|_\infty = \|[M_\Delta \ N_\Delta]\|_\infty < \epsilon$,

if and only if[2]

- K stabilizes $G = M^{-1}N$ and
- $\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1}M^{-1}\|_\infty \leq \epsilon^{-1}$.



Design of robust controllers

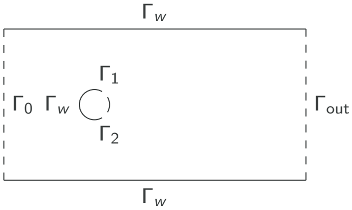
In finite dimensions, the Riccati based $\mathcal{H}_\infty$-controller with parameter γ is robustly stabilizing with $\epsilon = \gamma^{-1}$; see Cor. 3.9 in [McFarlane&Glover'90].

[2] See, e.g. [McFarlane&Glover'90] for the finite dimensional case and [Curtain&Zwart'95] for the infinite dimensional case

Outline

Application to Incompressible Flows

We consider



$$\dot{V} + (V \cdot \nabla)V + \nabla P - \nu \Delta V = 0,$$
$$\operatorname{div} V = 0, \quad \text{in } \Omega,$$

$$\nu \frac{\partial V}{\partial n} - nP = 0 \text{ on } \Gamma_{out},$$
$$V = 0 \text{ on } \Gamma_w,$$
$$V = ng_0 \cdot \alpha \text{ on } \Gamma_0,$$
$$V = ng_1 \cdot u_1 \text{ on } \Gamma_1,$$
$$V = ng_2 \cdot u_2 \text{ on } \Gamma_2,$$

where

- V ... velocity,
- P ... pressure,
- ν ... diffusion parameter,
- g_0, g_1, g_2 ... spatial shape functions,
- u_1, u_2 ... scalar input functions,
- α ... magnitude of the inflow velocity,
- n ... normal vector at the boundaries.

Application to Incompressible Flows

To design a controller, we proceed as follows

1. We relax the Dirichlet control $V\big|_{\Gamma_1} = g_1 u - \varepsilon(\nu \frac{\partial V}{\partial n} - Pn)$
2. Let v_α be the steady state solution for zero inputs, and let $v_\delta(t) = V(t) - v_\alpha$ the deviation.
3. We consider the linearization

$$\dot{v}_\delta + (v_\delta \cdot \nabla)v_\alpha + (v_\alpha \cdot \nabla)v_\delta + \nabla p_\delta - \nu\Delta v_\delta = 0$$

   that is a valid approximation as long as v_δ is small.

$$\dot v_\delta + (v_\delta \cdot \nabla) v_\alpha + (v_\alpha \cdot \nabla) v_\delta + \nabla p_\delta - \nu \Delta v_\delta = 0$$
$$\operatorname{div} v_\delta = 0$$

Then, with

$$\mathcal{H}_{div} := \{ v \in L^2(\Omega) : \operatorname{div} v = 0, v \cdot n = 0 \text{ on } \Gamma_w \cap \Gamma_{\mathrm{out}} \}$$

as the state space, the (orthogonal) *Leray*-projector

$$\Pi \in \mathcal{L}(L^2(\Omega)) \colon L^2(\Omega) \mapsto \mathcal{H}_{div},$$

and $x := \Pi v_\delta$ the model reads[3]

$$\begin{aligned} \dot x &= A_\alpha x + \Pi B u \quad \text{in } \mathcal{H}_{div}, \\ y &= Cx \end{aligned}$$

where

- $A_\alpha \colon \mathcal{D}(A_\alpha) \subset \mathcal{H}_{div} \to \mathcal{H}_{div}$ is the *Oseen* operator
- $\Pi B \colon \mathbb{R}^2 \to \mathcal{H}_{div}$ is the input operator
- $C \colon \mathcal{H}_{div} \to \mathbb{R}^q$ is the output operator

[3] The pressure p_δ is gone, since Π maps along the orthogonal complement of the gradient

Application to Incompressible Flows

✓ The linearized model is a standard (A, B, C) system

- we know: A_α is the generator of a C_0-semi group [Raymond'06]
- we choose: C to be bounded
- we show below: ΠB is bounded.

➙ The theory for robust stabilization of linearization errors applies.

Application to Incompressible Flows

As for the input on Γ_1 (and similarly for Γ_2):

- The input operator is defined via

$$\langle B_1 u, w \rangle = -\frac{1}{\varepsilon} \int_{\Gamma_1} n g_1 w \, \mathrm{ds} \cdot u, \quad w \in \mathcal{H}_{div},$$

  - as it comes from the integration by parts of $\langle -\nu \Delta V + \nabla P, w \rangle$
  - and the definition of the Robin conditions.

✓ This operator is bounded as a map $B_1 \colon \mathbb{R} \to \mathcal{H}_{div}$, since:

  - $\sup_{u \in \mathbb{R}, |u|=1} \|B_1 u\|_{X^*} < \infty$ if $\sup_{w \in X, \|w\|=1} |\frac{1}{\varepsilon} \int_{\Gamma_1} n g_1 w \, \mathrm{ds}| < \infty$,
  - the *trace operator* $w \mapsto nw\big|_{\Gamma_1}$ is bounded for $X = \mathcal{H}_{div}$,
  - and since $\mathcal{H}_{div}$ is a closed subspace of $L^2(\Omega)$ so that $\mathcal{H}_{div} \simeq (\mathcal{H}_{div})^*$,

➙ provided that the shape function g_1 is sufficiently smooth.

- Interestingly, $B \colon U \to L^2(\Omega)$ is not bounded, but $\Pi B \colon U \to L^2(\Omega)$ is.

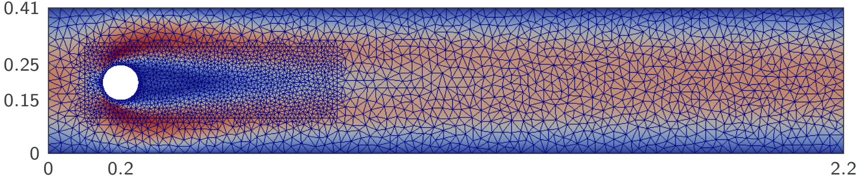


Fig.: 2D cylinder wake, discretized by Taylor-Hood (P_2/P_1) finite elements.

- Navier-Stokes equation
- Reynolds 90
- 9,843 velocity nodes
- distributed observations:
  - 3 sensors in the wake
  - measuring both v components

Target

Stabilize the steady-state and compensate perturbations to suppress vertex shedding.

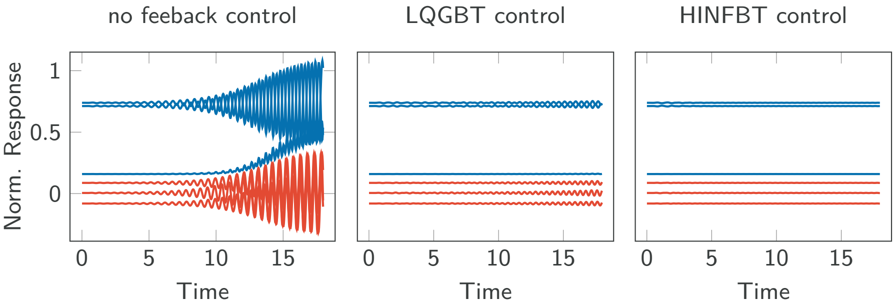
- boundary control:
  - 2 outlets at the cylinder periphery
  - control by injection and suction

Numerical Example

ℓ	$\frac{\|v_\infty - v_\ell\|}{\|v_\infty\|}$	$\|\Delta_\ell\|_{\mathcal{H}_\infty}$	γ_ℓ^{-1}
3	0.094	2.323	0.103
5	0.030	0.579	0.204
6	0.018	0.168	0.233
7	0.011	0.226	0.237
8	0.006	0.123	0.240
10	0.002	0.028	0.242

Test setup:

- v_∞ the (exact) steady state
- $v_\ell \approx v_\infty$ computed by ℓ *Picard* steps starting from the Stokes-solution
- $A := A(v_\infty)$ the exact linearization
- $A + A_\Delta := A_\ell$ the inexact linearization about v_ℓ
- Δ_ℓ the difference in the coprime factorizations
- see [PB&JH&SW'19] for how to compute the norms.



- error in linearization: 8%
- reduced-order controller dimension: 7
- trigger of instabilities by input disturbance on time interval $[0, 1]$:

$$u_\delta(t) = \begin{bmatrix} 0.01\sin(2t\pi) \\ -0.01\sin(2t\pi) \end{bmatrix}$$

Outline

Conclusions

Summary

Robust controller

- that compensate linearization errors
- can be analysed via coprime factorizations and
- can be designed with Riccati-based $\mathcal{H}_\infty$-theory.

The general ∞-dimensional theory

- applies to control of incompressible flows
- if Dirichlet control is relaxed as Robin control.

In finite dimensional simulations, we can

- compute the errors in the factorization
- and provide controller with guaranteed robustness.

Outlook

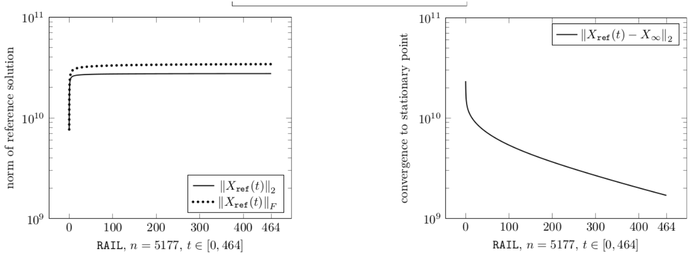
- Quantify the error in the factorizations.
- Incorporate the discretization error in the controller design.

Misc and Bilbao

- We design feedback controls:

$$u = -B^* X x \tag{1}$$

- For the presented stabilization we base on the steady state Riccati solution X_∞.
- For finite time horizons we would consider the differential Riccati equation
- Well known and well observed $X(t) \to X_\infty$ as $t \to \infty$ (turnpike)
- My plan for Bilbao:
  - Theory – combine the Riccati results and turn pike results.
  - Practice: exploit the turnpike for efficient solvers for the differential Riccati equation.



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