Obstacle problems: theory and numerics

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- $f \in L^2(\Omega)$ is a given source term,
- $\psi \in H^2(\Omega) \cap C^0(\overline{\Omega})$ is a given **obstacle**,
- $\Omega \subset \mathbb{R}^n$ is bounded and regular.

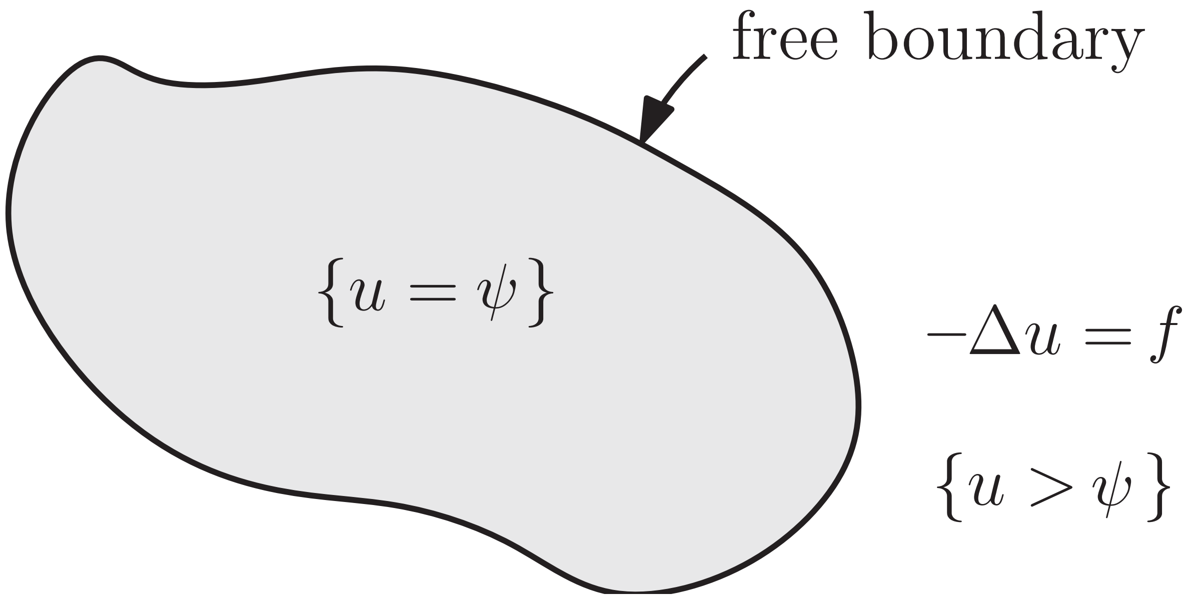

Figure: The contact set $\{u = \psi\}$ and the free boundary $\partial\{u > \psi\}$ in the classical obstacle problem.

Elasticity

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Mathematically,

$$\int_\Omega |\nabla u|^2 dx \to \min$$

for $u \geq \psi$, where u is the vertical displacement of the membrane.

The Stefan problem

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Example: **Melting ice submerged in liquid water.**

In the simplest case, $\theta_t - \Delta\theta = 0$ in $\{\theta > 0\}$ and $\theta = 0$ elswhere.

It can be shown that this is a special case of the parabolic obstacle problem.

Optimal stopping

Let $\{X_t\}$ be a stochastic process in $\mathbb{R}^n$ (price of stocks), and ψ a given payoff function.

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It turns out that the maximum expected payoff u solves

$$\begin{cases} u \geq \psi & \text{in } \mathbb{R}^n \\ Lu \geq 0 & \text{in } \mathbb{R}^n \\ Lu = 0 & \text{in } \{u > \psi\}, \end{cases}$$

where L is the infinitesimal generator of $\{X_t\}$.

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If $\{X_t\}$ is the Brownian motion, then $L = \frac{1}{2}\Delta$.

Interacting populations

Let $W \in L^1_{loc}(\mathbb{R}^2)$ be an interaction potential: repulsive when particles are near (to avoid colision) and attractive when they are far (to make a structure or a group).

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over Borel measures μ with mass 1.

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Result: $\mu_{\text{min}} := (-\Delta)^s u$ where u satisifes

$$\min\{(-\Delta)^s u, u-\psi\} = 0$$

for some obstacle ψ depending on W.

The classical obstacle problem

The problem: Find $u \in \mathcal{K}(\psi)$ s.t.

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Theorem

There exists a unique solution $u \in \mathcal{K}(\psi)$ to Problem (1).

Variational inequality

Proposition

Problem (1) is equivalent to: Find $u \in \mathcal{K}(\psi)$ s.t.

$$\int_\Omega \nabla u \cdot \nabla(v-u)dx \geq \int_\Omega f(v-u)dx$$

for all $v \in \mathcal{K}(\psi)$.

Euler-Lagrange equations

Choosing "appropriate" variations v and integrating by parts, the VI implies that the minimizer u solves

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Problem: We need $H^2(\Omega)$ regularity to integrate by parts and to have $=$ a.e.

$H^2(\Omega)$-regularity

Penalization method: Fix $\varepsilon > 0$. Find $u^\varepsilon \in H_0^1(\Omega)$ weak solution to

$$-\Delta u^\varepsilon = \frac{1}{\varepsilon}\beta(u^\varepsilon - \psi) + f \quad \text{in } \Omega. \tag{2}$$

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Problem: No Lax-Milgram because no linearity!
Solution: Monotone operator theory.

Proposition

There exists a unique solution $u^\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)$ to equation (2) and

$$\|u^\varepsilon\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

for some $C = C(\Omega, n)$ independent of ε.

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If $f, \max\{-\Delta\psi - f, 0\} \in L^p(\Omega)$ *for some* $p \in [2, \infty)$, *then* $u \in W^{2,p}(\Omega)$,

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If $f, \max\{-\Delta\psi - f, 0\} \in L^p(\Omega)$ for some $p \in [2, \infty)$, then $u \in W^{2,p}(\Omega)$, and if $p > n$, then $u \in C^{1,\alpha}(\overline{\Omega})$ for $\alpha \leq 1 - \frac{n}{p} < 1$.

$C^{1,1}_{loc}(\Omega)$-regularity

Assume $f \in L^\infty(\Omega)$, $\psi \in C^{1,1}(\Omega)$ and set $\mathbf{u} := u - \psi$, $\mathbf{f} := f - \Delta\psi$.

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Theorem

$\mathbf{u} \in C^{1,1}(\omega)$ *for any* $\omega \Subset \Omega$.

Idea: Use the FEM to solve the penalized problem and take ε small to have a good approximation.

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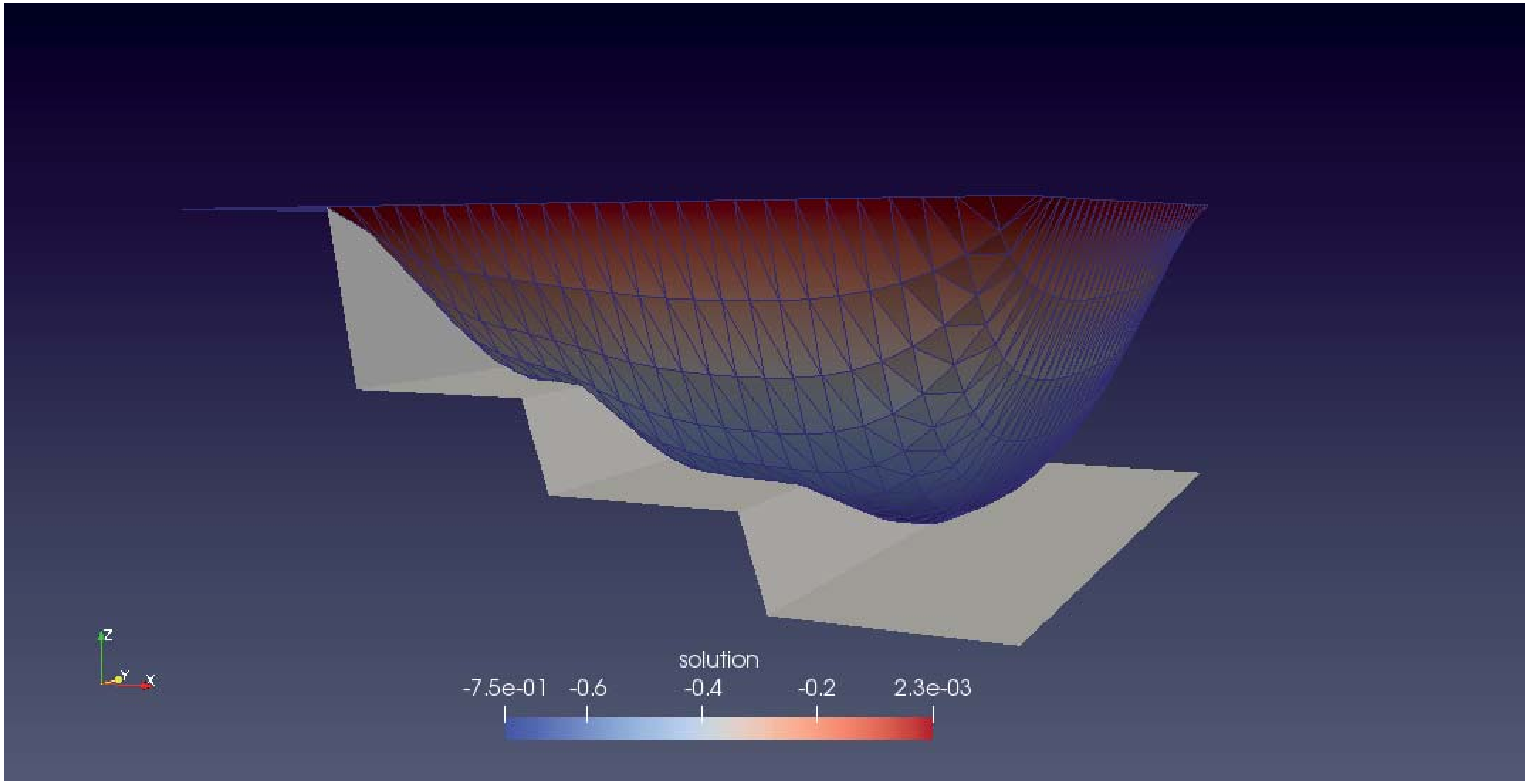
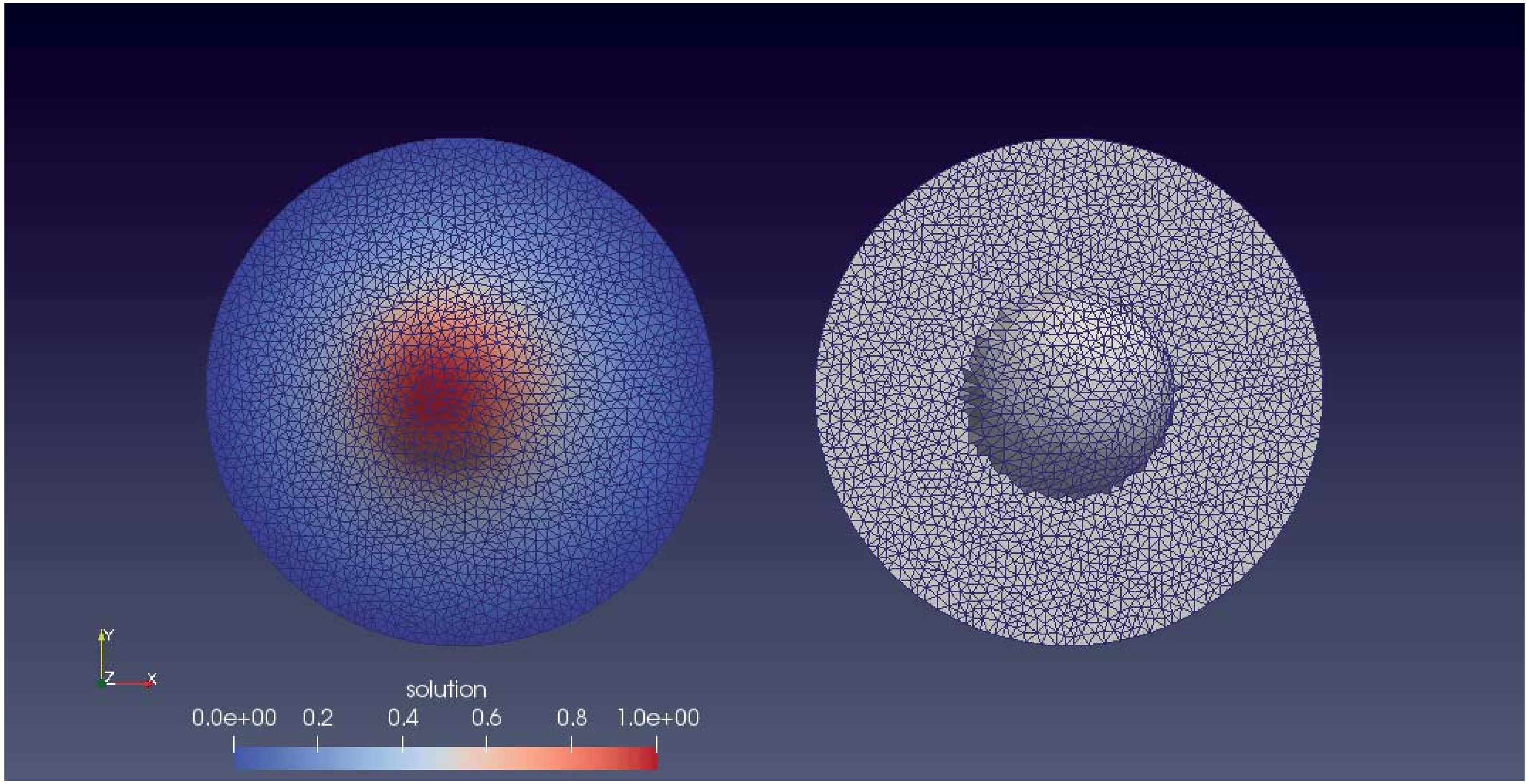
Example 1: $\Omega = [-1, 1]^2$, $f = -10$, $\psi =$ "stairs", $\varepsilon = 10^{-6}$, $N = 64$:



Example 2: $\Omega = B(0, 2.5)$, $f = 0$, $\psi = \sqrt{1 - |x|^2}\chi_{\{|x|<1\}}$, $\varepsilon = 10^{-6}$, $N = 64$:



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with $u_0 \in H_0^1(\Omega)$, $f \in L^2(\mathcal{Q})$ and $\psi \in L^2(0,T;H^2 \cap H_0^1(\Omega))$,
$\psi_t \in L^2(\mathcal{Q})$, $\psi(0) = 0$.

Theorem

There exists a unique solution $u \in \mathcal{K}(\psi) \cap L^2(0, T; H^2 \cap H_0^1(\Omega))$ *to the parabolic obstacle problem* (3).

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Idea: For fixed $\varepsilon > 0$, find $u^\varepsilon \in L^2(0, T; H^2 \cap H_0^1(\Omega))$, $u_t \in L^2(\mathcal{Q})$, $u(0) = u_0$ a weak solution of

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If β is uniformly Lipschitz, by a classical result this problem has a solution for fixed ε.

Lemma

There exists $C > 0$ s.t.

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\nabla u^\varepsilon(t)\|_{L^2(\Omega)} + \|u_t^\varepsilon\|_{L^2(\mathcal{Q})} + \|\Delta u^\varepsilon\|_{L^2(\mathcal{Q})}$$
$$\leq C\Big(\|\Delta\psi\|_{L^2(\mathcal{Q})} + \|\psi_t\|_{L^2(\mathcal{Q})} + \|f\|_{L^2(\mathcal{Q})} + \|\nabla u_0\|_{L^2(\Omega)}\Big)$$

for all $\varepsilon > 0$.

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for all $\varepsilon > 0$.

This implies that $\{u^\varepsilon\}$, $\{u^\varepsilon_t\}$ and $\{\Delta u^\varepsilon\}$ are bounded in $L^2(0,T;H^1_0(\Omega))$ and $L^2(\mathcal{Q})$ respectively. Use a compactness argument to prove the Theorem.

Euler's method

At least formally, we can show existence by discretizing u_t via the implicit Euler scheme: for $i \in \{1, \ldots, N\}$ find u_i s.t.

$$\int_\Omega \left(\frac{u_i - u_{i-1}}{h}\right)(v - u_i) + \nabla u_i \cdot \nabla(v - u_i) \geq \int_\Omega f(v - u_i).$$

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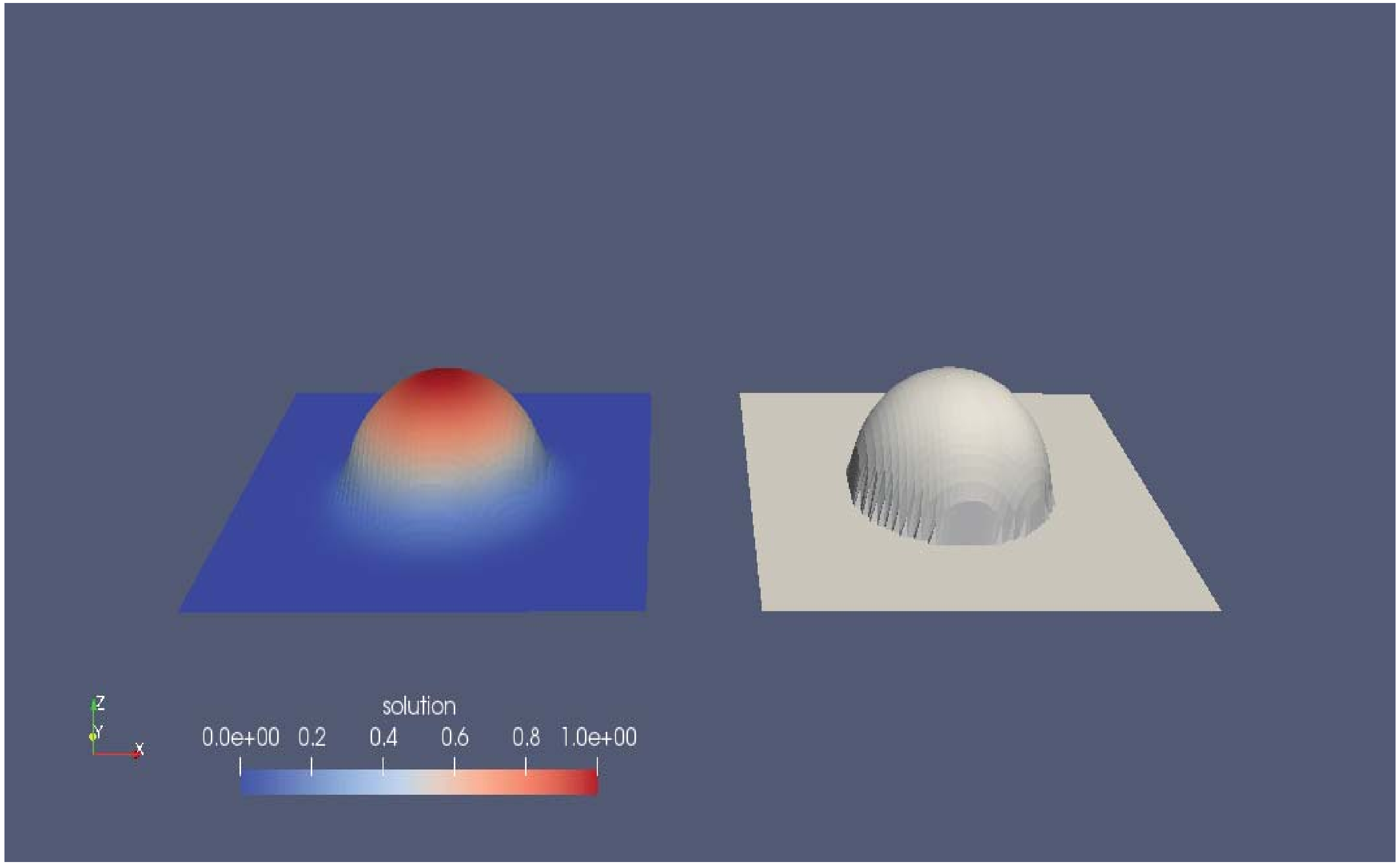
We then "glue" the solutions $\{u_i\}_{i=1}^N$ to create a suitable approximation, show estimates and conclude by compactness.

We again solve the penalized problem by using the FEM.
Example: $\Omega = [-2, 2]^2$, $f = 0$, $\psi = \sqrt{1 - |x|^2}\chi_{\{|x|<1\}}$, $\varepsilon = 10^{-6}$, $n = 64$, $T = 5$:



Minimize over $\psi \in \mathcal{U}_{ad}$ the cost

$$\mathcal{J}(\psi, u) := \|u - u_d\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega)}$$

s.t.

$$u \in \mathcal{K}(\psi),$$
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This problem fits in the framework of *mathematical programs with equilibrium constraints (MPEC)*.

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We write

$$\mathcal{J}(\psi) := \|\sigma(\psi) - u_d\|^2_{L^2(\mathcal{Q})} + \|\psi_t\|^2_{L^2(\mathcal{Q})} + \|\Delta\psi\|^2_{L^2(\mathcal{Q})},$$

where $\sigma : \psi \mapsto u$ is the solution map.

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There exists a minimizer $\psi \in \mathcal{U}_{ad}$ *to* $\mathcal{J}$.

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We now want to deduce an optimality system. But we don't know the regularity of σ.

Idea: Consider the penalized problem, show conditions for the approximations, deduce estimates, conclude by compactness.

For fixed $\varepsilon > 0$, consider the approximate cost functional

$$\mathfrak{J}_\varepsilon(\psi) := \|\sigma_\varepsilon(\psi) - u_d\|^2_{L^2(\mathcal{Q})} + \|\psi_t\|^2_{L^2(\mathcal{Q})} + \|\Delta\psi\|^2_{L^2(\mathcal{Q})},$$

where $\sigma_\varepsilon : \psi \mapsto u^\varepsilon$ is the solution map for the penalized problem.

Theorem

For fixed $\varepsilon > 0$, σ_ε has a weak Gâteaux derivative ξ^ε in $L^2(0,T;H_0^1(\Omega))$. Moreover, ξ^ε satisfies $\xi_t^\varepsilon \in L^2(0,T;H^{-1}(\Omega))$ and

$$\begin{cases} \xi_t^\varepsilon - \Delta\xi^\varepsilon + \frac{1}{\varepsilon}\beta'(u^\varepsilon - \psi)(\xi^\varepsilon - v) = 0 & \text{in } \mathcal{Q} \\ \xi^\varepsilon(0) = 0 & \text{in } \Omega. \end{cases}$$

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Lemma

Fix $\varepsilon > 0$. *Given a minimizer* ψ^ε *of* $\mathcal{J}_\varepsilon$, *there exists an adjoint state* $p^\varepsilon \in L^2(0,T;H_0^1(\Omega))$, $p_t^\varepsilon \in L^2(0,T;H^{-1}(\Omega))$ *s.t.*

$$\begin{cases} -p_t^\varepsilon - \Delta p^\varepsilon + \frac{1}{\varepsilon}\beta'(u^\varepsilon - \psi^\varepsilon)p^\varepsilon = u^\varepsilon - u_d & \text{in } \mathcal{Q} \\ p^\varepsilon(T) = 0 & \text{in } \Omega. \end{cases}$$

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For fixed $\varepsilon > 0$, the minimizer ψ^ε satisfies $\psi^\varepsilon \in L^2(0, T; H^3 \cap H_0^1(\Omega))$, $\psi_t^\varepsilon \in L^2(\mathcal{Q})$, $\psi_{tt}^\varepsilon \in \mathcal{W}'$ and

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in the sense of $\mathcal{W}'$.

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We furthermore deduce adequate estimates on ψ^ε and p^ε that allow us to pass to the limit in the above by compactness.

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- Geometry of the contact set? (Nirenberg et al. 70s)
- Regularity of the solution map $\sigma : \psi \mapsto u$? (Kinderlehrer et al. 70s)
- Convergence rates for the FEM approximation, error estimates for the solution and the free boundary (Nocchetto et al. 2010s).

More questions and topics

- Regularity of the free boundary? (Caffarelli et al. 70s)
- Geometry of the contact set? (Nirenberg et al. 70s)
- Regularity of the solution map $\sigma : \psi \mapsto u$? (Kinderlehrer et al. 70s)
- Convergence rates for the FEM approximation, error estimates for the solution and the free boundary (Nocchetto et al. 2010s).
- Redo everything with $(-\Delta)^s$ (Caffarelli, Figalli, Ros-Oton et al. 2010s).

Thank you for your attention.